University of Ottawa

Department of Mathematics and Statistics

Math 1302A, Winter 2009
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Assignment 4 – Due April 3, 2009

For all of the questions below, you must show your work and justify your answers. In particular, you must show each step in any row reduction and state what operation you are performing at each step.

1. **(7 points)** If possible, diagonalize the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. That is, find a diagonal matrix D and invertible matrix P such that $D = P^{-1}AP$.

Solution: We compute the characteristic polynomial (by expanding across the first row) of A as follows:

$$
\begin{aligned}
\det(xI_3 - A) &= \begin{vmatrix} x & -1 & -1 \\ -1 & x & -1 \\ -1 & -1 & x \end{vmatrix} \\
&= x \begin{vmatrix} x & -1 \\ -1 & x \end{vmatrix} - (-1) \begin{vmatrix} -1 & -1 \\ -1 & x \end{vmatrix} + (-1) \begin{vmatrix} -1 & x \\ -1 & -1 \end{vmatrix} \\
&= x(x^2 - 1) + (-x - 1) - (1 + x) \\
&= x(x+1)(x-1) - (x+1) - (x+1) \\
&= (x+1)(x(x-1) - 1 - 1) = (x+1)(x^2 - x - 2) \\
&= (x+1)(x+1)(x-2)
\end{aligned}
$$

It follows that the eigenvalues of A are: $x_1 = 2$ and $x_2 = -1$.

To find the eigenspace corresponding to the eigenvalue 2 we solve $(2I_3 - A)\vec{x} = \vec{0}$ as follows.

$$
\left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{ccc|c} -1 & -1 & 2 & 0 \\ -1 & 2 & -1 & 0 \\ 2 & -1 & -1 & 0 \end{array}\right] \xrightarrow[R_3 + 2R_1 \to R_3]{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|c} -1 & -1 & 2 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right]
$$

$$
\xrightarrow[\frac{-1}{3}R_3 \to R_3]{\substack{-R_1 \to R_1 \\ \frac{1}{3}R_2 \to R_2}} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{R_3 - R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 - 2R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Thus, the eigenspace corresponding to the eigenvalue 2 is Span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ and a basis is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

To find the eigenspace corresponding to the eigenvalue -1 we solve $(-I_3 - A)\vec{x} = \vec{0}$ as follows:

$$\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right] \xrightarrow[R_3 - R_1 \to R_3]{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus the eigenspace corresponding to the eigenvalue -1 is Span $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ and a basis is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Therefore $D = P^{-1}AP$ where $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

2. **(4 points)** a) If $z = 3 - 2i$ and $w = -5 + 7i$, compute the following.

(a) $\bar{z} + \bar{w}$
(b) $|z| + |w|$
(c) $\frac{z}{w}$
(d) wz

Solution:

(a) $\bar{z} + \bar{w} = 3 + 2i + (-5 - 7i) = -2 - 5i$
(b) $|z| + |w| = \sqrt{9+4} + \sqrt{25+49} = \sqrt{13} + \sqrt{74}$
(c)

$$\frac{z}{w} = \frac{3-2i}{-5+7i} = \left(\frac{3-2i}{-5+7i}\right)\left(\frac{-5-7i}{-5-7i}\right) = \frac{(-15-14) + i(-21+10)}{(-5)^2 + 7^2} = \frac{-29}{74} - \frac{11}{74}i$$

(d) $wz = (-5+7i)(3-2i) = (-15+14) + i(21+10) = -1 + 31i$

3. **(4 points)** Is $\lambda = 3$ an eigenvalue of $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & -3 & 0 \end{bmatrix}$? If so, find the corresponding eigenspace. Do **not** use the characteristic equation.

Solution: We solve the system $(3I_3 - A)\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right] \xrightarrow[R_1+R_2\to R_1]{R_3-3R_2\to R_3} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_1\to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since the system has nontrivial solutions, 3 is an eigenvalue of A. The corresponding eigenspace is the solution set of this system and therefore is

$$\text{Span}\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$