University of Ottawa

Department of Mathematics and Statistics

Math 1302A, Winter 2009

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Assignment 3 – Due March 13, 2009

For all of the questions below, you must show your work and justify your answers. In particular, you must show each step in any row reduction and state what operation you are performing at each step.

1. **(6 points)** Suppose that

$$A = \begin{bmatrix} -8 & 0 & -24 & -8 & -9 \\ 2 & 1 & 8 & 4 & 1 \\ 1 & 0 & 3 & 1 & 1 \\ -1 & 1 & -1 & 1 & 0 \end{bmatrix}.$$

a) Find a basis for $ColA$.
b) Find a basis for $NulA$.
c) Determine $\dim(ColA)$ and $\dim(NulA)$.

Solution: First, reduce the matrix to its equivalent row reduced echelon form

$$A \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 3 & 1 & 1 \\ 2 & 1 & 8 & 4 & 1 \\ -8 & 0 & -24 & -8 & -9 \\ -1 & 1 & -1 & 1 & 0 \end{bmatrix} \xrightarrow[\substack{-2R_1+R_2 \to R_2 \\ R_1+R_4 \to R_4}]{8R_1+R_3 \to R_3} \begin{bmatrix} 1 & 0 & 3 & 1 & 1 \\ 0 & 1 & 2 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 2 & 2 & 1 \end{bmatrix} \xrightarrow{-R_2+R_4 \to R_4} \begin{bmatrix} 1 & 0 & 3 & 1 & 1 \\ 0 & 1 & 2 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow{-1 \times R_3} \begin{bmatrix} 1 & 0 & 3 & 1 & 1 \\ 0 & 1 & 2 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow[\substack{R_3+R_2 \to R_2 \\ -2R_3+R_4 \to R_4}]{-R_3+R_1 \to R_1} \begin{bmatrix} 1 & 0 & 3 & 1 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

a) The first, second and last columns are pivot columns, hence a basis for the $ColA$ is

$$\left\{ \begin{bmatrix} -8 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -9 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

b) To find a basis for the $NulA$, we need to solve $A\vec{x} = \vec{0}$. Using the above row reduced echelon form, the general solution is

$$\begin{cases} x_1 & = -3x_3 - x_4 \\ x_2 & = -2x_3 - 2x_4 \\ x_3 & \text{is free} \\ x_4 & \text{is free} \\ x_5 & = 0 \end{cases}$$

Let $x_3 = s$ and $x_4 = t$, and write the general solution in vector parametric form:

$$\begin{bmatrix} -3s - t \\ -2s - 2t \\ s \\ t \\ 0 \end{bmatrix} = s \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore, a basis for the null space of A is

$$\left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

c) For the dimensions, count the number of elements in each basis:

$$\dim(ColA) = 3, \quad \dim(NulA) = 2.$$

2. **(4 points)** What is the dimension of the subspace spanned by the following vectors?

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 6 \\ 7 \end{bmatrix}, \begin{bmatrix} 8 \\ 5 \\ 9 \\ 11 \end{bmatrix}.$$

Solution: The subspace spanned by these vectors is the same as the column space of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 5 & 8 \\ 2 & 3 & 0 & 5 \\ 3 & 0 & 6 & 9 \\ 0 & 4 & 7 & 11 \end{bmatrix}.$$

To find the dimension of the column space, count the number of pivot columns:

$$A \xrightarrow[-3R_1+R_3\to R_3]{-2R_1+R_2\to R_2} \begin{bmatrix} 1 & 2 & 5 & 8 \\ 0 & -1 & -10 & -11 \\ 0 & -6 & -9 & -15 \\ 0 & 4 & 7 & 11 \end{bmatrix} \xrightarrow{-1\times R_2} \begin{bmatrix} 1 & 2 & 5 & 8 \\ 0 & 1 & 10 & 11 \\ 0 & -6 & -9 & -15 \\ 0 & 4 & 7 & 11 \end{bmatrix}$$

$$\xrightarrow[-4R_2+R_4\to R_4]{6R_2+R_3\to R_3} \begin{bmatrix} 1 & 2 & 5 & 8 \\ 0 & 1 & 10 & 11 \\ 0 & 0 & 51 & 51 \\ 0 & 0 & -33 & -33 \end{bmatrix} \xrightarrow{\frac{33}{51}R_3+R_4\to R_4} \begin{bmatrix} 1 & 2 & 5 & 8 \\ 0 & 1 & 10 & 11 \\ 0 & 0 & 51 & 51 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

the first, second and third columns are pivot, hence the dimension of the subspace generated by the columns is 3.

3. **(5 points)** Consider the production model $x = Cx + d$ for an economy with three sectors, where

$$C = \begin{bmatrix} .4 & .6 & .0 \\ .4 & .3 & .2 \\ .1 & 0 & .4 \end{bmatrix}, \quad d = \begin{bmatrix} 72 \\ 150 \\ 58 \end{bmatrix}.$$

Determine the production level necessary to satisfy the final demand.

Solution: We need to solve $\overrightarrow{x} = C\overrightarrow{x} + \overrightarrow{d}$, or equivalently

$$(I - C)\overrightarrow{x} = \overrightarrow{d}.$$

Write the augmented matrix:

$$[I - C|\overrightarrow{d}\,] = \left[\begin{array}{ccc|c} .6 & -0.6 & 0 & 72 \\ -.4 & 0.7 & -.2 & 150 \\ -.1 & 0 & 0.6 & 58 \end{array}\right] \xrightarrow[10\times R_3]{\substack{10\times R_1 \\ 10\times R_2}} \left[\begin{array}{ccc|c} 6 & -6 & 0 & 720 \\ -4 & 7 & -2 & 1500 \\ -1 & 0 & 6 & 580 \end{array}\right]$$

$$\xrightarrow{\frac{1}{6}R_1} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 120 \\ -4 & 7 & -2 & 1500 \\ -1 & 0 & 6 & 580 \end{array}\right] \xrightarrow[R_1+R_3\to R_3]{4R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 120 \\ 0 & 3 & -2 & 1980 \\ 0 & -1 & 6 & 700 \end{array}\right] \xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 120 \\ 0 & 1 & -2/3 & 660 \\ 0 & -1 & 6 & 700 \end{array}\right]$$

$$\xrightarrow[R_2+R_3\to R_3]{R_2+R_1\to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 780 \\ 0 & 1 & -2/3 & 660 \\ 0 & 0 & 16/3 & 1360 \end{array}\right] \xrightarrow{\frac{3}{16}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 780 \\ 0 & 1 & -2/3 & 660 \\ 0 & 0 & 1 & 255 \end{array}\right] \xrightarrow[\frac{2}{3}R_3+R_2\to R_2]{\frac{2}{3}R_3+R_1\to R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 950 \\ 0 & 1 & 0 & 830 \\ 0 & 0 & 1 & 255 \end{array}\right]$$

The production level necessary to overcome the demand is

$$\overrightarrow{x} = \begin{bmatrix} 950 \\ 830 \\ 255 \end{bmatrix}.$$