University of Ottawa

Department of Mathematics and Statistics

Math 1302A, Winter 2009

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Assignment 2 – Due February 27, 2009

For all of the questions below, you must show your work and justify your answers. In particular, you must show each step in any row reduction and state what operation you are performing at each step.

1. **(5 points)** If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $AB = \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}$ determine the first and second column of B.

Solution: Denote the columns of B by $\vec{b}_1, \vec{b}_2, \vec{b}_3$, so $B = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix}$. If $\vec{b}_1 = \begin{bmatrix} x \\ y \end{bmatrix}$, then

$$\begin{aligned} x + 2y &= 5 \\ 3x + 4y &= 8 \end{aligned}$$

We row reduce the corresponding augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & 4 & 8 \end{array}\right] \xrightarrow{R_2 - 3R_1 \to R_2} \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -2 & -7 \end{array}\right] \xrightarrow{\frac{-1}{2}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 7/2 \end{array}\right] \xrightarrow{R_1 - 2R_2 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 7/2 \end{array}\right]$$

Thus $y = \frac{7}{2}$, and $x = -2$. Therefore, $\vec{b}_1 = \begin{bmatrix} -2 \\ \frac{7}{2} \end{bmatrix}$.

Similarly, if $b_2 = \begin{bmatrix} a \\ b \end{bmatrix}$, then

$$\begin{aligned} a + 2b &= 6 \\ 3a + 4b &= 9 \end{aligned}$$

We row reduce the corresponding matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 6 \\ 3 & 4 & 9 \end{array}\right] \xrightarrow{R_2 - 3R_1 \to R_2} \left[\begin{array}{cc|c} 1 & 2 & 6 \\ 0 & -2 & -9 \end{array}\right] \xrightarrow{\frac{-1}{2}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 2 & 6 \\ 0 & 1 & 9/2 \end{array}\right] \xrightarrow{R_1 - 2R_2 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 9/2 \end{array}\right]$$

Thus $b = \frac{9}{2}$, and $a = -3$. Therefore, $\vec{b}_2 = \begin{bmatrix} -3 \\ \frac{9}{2} \end{bmatrix}$.

2. **(4 points)** If A, B and C are $n \times n$ invertible matrices, is there a matrix X that satisfies the matrix equation $AC^{-1}(2X + A)AB^{-1} = I_n$? If so, find X.

Solution: Multiply on the left by A^{-1}, and on the right by B:

$$C^{-1}(2X + A)A = A^{-1}B$$

Multiply on the left by C, and on the right by A^{-1}:

$$2X + A = CA^{-1}BA^{-1}$$

Subtracting A from both sides gives:

$$2X = CA^{-1}BA^{-1} - A$$

Hence $X = \frac{1}{2}(CA^{-1}BA^{-1} - A)$.

3. **(6 points)** a) Is the following matrix invertible? If so, find its inverse.

$$A = \begin{bmatrix} 1 & 3 & 0 & -1 \\ -3 & -5 & -8 & 3 \\ -4 & -12 & 6 & 4 \\ 0 & -1 & 2 & 1 \end{bmatrix}$$

Solution:

$$\left[\begin{array}{cccc|cccc} 1 & 3 & 0 & -1 & 1 & 0 & 0 & 0 \\ -3 & -5 & -8 & 3 & 0 & 1 & 0 & 0 \\ -4 & -12 & 6 & 4 & 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3+4R_1 \to R_3]{R_2+3R_1 \to R_2} \left[\begin{array}{cccc|cccc} 1 & 3 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 4 & -8 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 6 & 0 & 4 & 0 & 1 & 0 \\ 0 & -1 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[(1/6)R_3 \to R_3]{R_4+(1/4)R_2 \to R_4} \left[\begin{array}{cccc|cccc} 1 & 3 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 4 & -8 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2/3 & 0 & 1/6 & 0 \\ 0 & 0 & 0 & 1 & 3/4 & 1/4 & 0 & 1 \end{array}\right] \xrightarrow{(1/4)R_2 \to R_2} \left[\begin{array}{cccc|cccc} 1 & 3 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 3/4 & 1/4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2/3 & 0 & 1/6 & 0 \\ 0 & 0 & 0 & 1 & 3/4 & 1/4 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1+R_4 \to R_1} \left[\begin{array}{cccc|cccc} 1 & 3 & 0 & 0 & 7/4 & 1/4 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3/4 & 1/4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2/3 & 0 & 1/6 & 0 \\ 0 & 0 & 0 & 1 & 3/4 & 1/4 & 0 & 1 \end{array}\right] \xrightarrow{R_2+2R_3 \to R_2} \left[\begin{array}{cccc|cccc} 1 & 3 & 0 & 0 & 7/4 & 1/4 & 0 & 1 \\ 0 & 1 & 0 & 0 & 25/12 & 1/4 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 2/3 & 0 & 1/6 & 0 \\ 0 & 0 & 0 & 1 & 3/4 & 1/4 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1-3R_2 \to R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -9/2 & -1/2 & -1 & 1 \\ 0 & 1 & 0 & 0 & 25/12 & 1/4 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 2/3 & 0 & 1/6 & 0 \\ 0 & 0 & 0 & 1 & 3/4 & 1/4 & 0 & 1 \end{array}\right].$$

Since we reduced A to I_4, it follows that A is invertible and

$$A^{-1} = \begin{bmatrix} -9/2 & -1/2 & -1 & 1 \\ 25/12 & 1/4 & 1/3 & 0 \\ 2/3 & 0 & 1/6 & 0 \\ 3/4 & 1/4 & 0 & 1 \end{bmatrix}.$$

b) Find the inverse of the matrix $B + C$, where

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 6 & 4 \\ 2 & -1 \end{bmatrix}.$$

Solution: Note that $B + C = \begin{bmatrix} 7 & 6 \\ 5 & 3 \end{bmatrix}$. Then the inverse is given by: $\frac{-1}{9} \begin{bmatrix} 3 & -6 \\ -5 & 7 \end{bmatrix}$.