MATH 1302, WINTER 2009
ASSIGNMENT 1 – SOLUTIONS

For all of the questions below, you must show each step in any row reduction and state what operation you are performing at each step.

1. **(5 points)** Solve the following system. Check your answer.

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & & & = & 0 \\ 6x_1 & + & 2x_2 & - & x_3 & = & 2 \\ & & -3x_2 & + & x_3 & = & 5 \end{array}$$

Solution: We form the augmented matrix and row reduce.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 6 & 2 & -1 & 2 \\ 0 & -3 & 1 & 5 \end{array}\right] \xrightarrow{-6R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -4 & -1 & 2 \\ 0 & -3 & 1 & 5 \end{array}\right] \xrightarrow{\frac{-1}{4}R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{4} & \frac{-1}{2} \\ 0 & -3 & 1 & 5 \end{array}\right]$$

$$\xrightarrow{3R_2+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{4} & \frac{-1}{2} \\ 0 & 0 & \frac{7}{4} & \frac{7}{2} \end{array}\right] \xrightarrow{\frac{4}{7}R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{4} & \frac{-1}{2} \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{\frac{-1}{4}R_3+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{-R_2+R_1\to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Switching back to equation notation gives

$$x_1 = 1$$
$$x_2 = -1$$
$$x_3 = 2$$

and so the system has the unique solution $(1, -1, 2)$. To check our answer, we substitute this solution back into the original equations:

$$\begin{array}{rcrcrcl} 1 & + & (-1) & & & = & 0 \\ 6(1) & + & 2(-1) & - & 2 & = & 2 \\ & & -3(-1) & + & 2 & = & 5 \end{array}$$

Since each substitution yields a true statement, we have verified that $(1, -1, 2)$ is indeed a solution.

Marking scheme:
- 2 points for correct row reduction
- 1 point for indicating row operations
- 1 point for writing down the correct solution from their (R)REF
- 1 point for checking the answer

2. **(4 points)** Determine if the system corresponding to the following augmented matrix is consistent or inconsistent (you do not need to completely solve the system if it is consistent).

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ -1 & 1 & 1 & -1 \\ 2 & 1 & 1 & 1 \end{array}\right]$$

Solution: We row reduce.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ -1 & 1 & 1 & -1 \\ 2 & 1 & 1 & 1 \end{array}\right] \xrightarrow[\substack{R_1+R_2\to R_2 \\ -2R_1+R_3\to R_3}]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & -3 \end{array}\right] \xrightarrow{-R_2+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -4 \end{array}\right]$$

Since the last column is a pivot column, the system is inconsistent.

Marking scheme:

- 2 point for correct row reduction
- 1 point for indicating row operations
- 1 point for stating the system is inconsistent

3. **(6 points)** Find the general solution of the following system. Indicate which variables are basic and which are free. Check your answer.

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & - & x_3 & + & 2x_4 & = & 1 \\ x_1 & + & 2x_2 & + & x_3 & & & = & 5 \\ -2x_1 & - & 4x_2 & + & x_3 & - & 3x_4 & = & -4 \end{array}$$

Solution: We find the augmented matrix and row reduce.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 1 \\ 1 & 2 & 1 & 0 & 5 \\ -2 & -4 & 1 & -3 & -4 \end{array}\right] \xrightarrow[\substack{-R_1+R_2\to R_2 \\ 2R_1+R_3\to R_3}]{} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 2 & -2 & 4 \\ 0 & 0 & -1 & 1 & -2 \end{array}\right] \xrightarrow{\frac{1}{2}R_2\to R_2} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & -1 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_2+R_3\to R_3} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2+R_1\to R_1} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Having reduced to row reduced echelon form, we return to equation notation.

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & & & & x_4 & = & 3 \\ & & & & x_3 & - & x_4 & = & 2 \end{array}$$

The variables x_1 and x_3 are basic and the variables x_2 and x_4 are free. Solving for the basic variables in terms of the free ones, we obtain the general solution of the system:

$$\begin{aligned} &x_1 = 3 - 2x_2 - x_4 \\ &x_2 \quad \text{free} \\ &x_3 = 2 + x_4 \\ &x_4 \quad \text{free} \end{aligned}$$

We check our answer by substituting our expressions into the original equations (leaving in the free variables).

$$\begin{array}{rcrcrcrcr}
(3-2x_2-x_4) & + & 2x_2 & - & (2+x_4) & + & 2x_4 & = & 1 \\
(3-2x_2-x_4) & + & 2x_2 & + & (2+x_4) & & & = & 5 \\
-2(3-2x_2-x_4) & - & 4x_2 & + & (2+x_4) & - & 3x_4 & = & -4
\end{array}$$

Since each substitution yields a true statement, we have verified our solution.

Marking scheme:

- 2 points for correct row reduction
- 1 point for indicating row operations
- 1 point for indicating which variables are free and which are basic
- 1 point for correct general solution
- 1 point for checking answer