Lecture : TUE - 17-MARCH-2009

CH. 5: Eigenvalues and eigenvectors

§5.1 Eigenvectors and eigenvalues

Def: • An eigenvector of an $\boxed{n \times n}$ mx. A is a non-zero vector x such that: $Ax = \lambda x$ for some scalar λ.

• A scalar λ is called an eigenvalue of A if there is a non-trivial solution x of $Ax = \lambda x$; such an x is called an eigenvector corresponding to λ.

STORY: - - - - -

2/308 We need to solve $Ax = \lambda x$; i.e. $Ax = (-2)x$; i.e. $Ax + 2x = 0$.

$(A+2I)x = 0 \Rightarrow$ $A + 2I = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}$ and then $\left(\begin{array}{cc|c} 9 & 3 & 0 \\ 3 & 1 & 0 \end{array}\right) \xrightarrow{R_2 - \frac{1}{3}R_1}$

$\left(\begin{array}{cc|c} 9 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{9}R_1} \left(\begin{array}{cc|c} 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{array}\right)$. How many solutions? ∞-ly many.

So non-trivial solutions. Let us find one: $x_2 = t$, t scalar, x_2 free

Then $x_1 = -\frac{1}{3}t \Rightarrow x = \begin{pmatrix} -1/3\, t \\ t \end{pmatrix} = t \begin{pmatrix} -1/3 \\ 1 \end{pmatrix}$. Choose $t = 1 \Rightarrow x = \begin{pmatrix} -1/3 \\ 1 \end{pmatrix}$.

(it is NOT 0).

6/308 $Ax = \begin{pmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = -2 \cdot x$

So: YES! $\lambda = -2$

5/308 $Ax = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \cdot x \Rightarrow$ YES $\Rightarrow \lambda = 0$

3/308 NO!!

• NOTE: The equation from the definition can be written as: $(A - \lambda I)x = 0$!!!

Def: The set of all solutions of $(A-\lambda I)x=0$ is called the EIGENSPACE of A corresponding to λ.

NOTE: it is a null space: $\text{null}(A-\lambda I)$.

$\boxed{14/308}$ $A-\lambda I_3 = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & -13 & 1 \end{pmatrix} - (-2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & -1 \\ 1 & -1 & 0 \\ 4 & -13 & 3 \end{pmatrix}$

NOW: $\left(\begin{array}{ccc|c} 3 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 4 & -13 & 3 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 3 & 0 & -1 & 0 \\ 4 & -13 & 3 & 0 \end{array}\right) \xrightarrow[R_3-4R_1]{R_2-3R_1} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & -9 & 3 & 0 \end{array}\right)$

$\rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{3}R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow x_3 = t,\ x_2 = \frac{1}{3}t;\ x_1 = \frac{1}{3}t$

$\Rightarrow x = \begin{pmatrix} 1/3\, t \\ 1/3\, t \\ t \end{pmatrix} = t\begin{pmatrix} 1/3 \\ 1/3 \\ 1 \end{pmatrix}$. A BASIS is: $\left\{\begin{bmatrix} 1/3 \\ 1/3 \\ 1 \end{bmatrix}\right\}$.

Def A triangular matrix is a mx. whose entries below (or Above) the main Diagonal are zero: Upper / Lower

THM The eigenvalues of a triangular mx are the entries on its main Diagonal.

Pf: Row Reduction is the key!!

$\boxed{18/308}$ 4, 0, −3

THM: 0 is an eigenvalue of A $\Leftrightarrow$ A is not invertible.

Pf: 0 is an eigenvalue $\Leftrightarrow$ $Ax = 0\cdot x$ has a nontrivial solution $\Leftrightarrow$ $Ax=0$ has a non-trivial solution $\overset{\text{I.M.T}}{\Leftrightarrow}$ A is not inv.

Thm: If $v_1, v_2, \ldots, v_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ mx A, then the set $\{v_1, v_2, \ldots, v_r\}$ is LIN. IND

LATER USE

25/309 • A inv. $\Rightarrow \lambda \neq 0$; so λ^{-1} exists.

• there is a non-zero x s.t. $Ax = \lambda x$. So $A^{-1}Ax = A^{-1}\lambda x$

$\Rightarrow I_n x = \lambda A^{-1}x \Rightarrow x = \lambda A^{-1}x \Rightarrow \lambda^{-1}x = A^{-1}x \Rightarrow$

$A^{-1}x = \lambda^{-1}x \underset{x \neq 0}{\Rightarrow} \lambda^{-1}$ is an eigenvalue of A^{-1}!

26/309 Let λ be an eigenvalue of A. Then there is a non-zero x s.t. $Ax = \lambda x \Rightarrow AAx = A\lambda x \Rightarrow 0x = \lambda Ax$

$\Rightarrow 0 = \lambda\lambda x \Rightarrow 0 = \lambda^2 x \underset{x \neq 0}{\Rightarrow} \lambda^2 = 0 \Rightarrow \lambda = 0$

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