Chapter 3 Determinants

3.1. Intro to Determinants

Recall (2.2) that a 2×2 mx. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if its determinant $\det A = ad - bc$ is not zero

- We want to extend this fact to larger matrices.
- We shall give a RECURSIVE definition of a determinant

Def: Let $A = (a_{ij})_{i,j}$ be an $n\times n$ mx, $n \geq 2$.

- For each i and each j let A_{ij} be the submatrix of A obtained (from A) by deleting the i^{th} Row and the j^{th} Column from A.

- Then The Determinant of A is: $\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} \det A_{1n}$.

NOTATION: $\det A = |A|$. IT IS a #, not a mx.

Do EXC 2/190 $\det \begin{bmatrix} 0 & 5 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & 1 \end{bmatrix} = 0 \cdot \det \begin{bmatrix} -3 & 0 \\ 4 & 1 \end{bmatrix} + 5(-1)^{1+2} \cdot \det \begin{pmatrix} 4 & 0 \\ 2 & 1 \end{pmatrix} + 1 \cdot (-1)^{1+3} \det \begin{pmatrix} 4 & -3 \\ 2 & 4 \end{pmatrix} = -5(4-0) + 1(16+6$

$= -20 + 22 = \boxed{2}$

Def: Given a mx. $A = (a_{ij})_{i,j}$ the (i,j)-cofactor of A is the # $C_{ij} = (-1)^{i+j} \det A_{ij}$

So $\det A = a_{11} \cdot C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}$

It is called a COFACTOR EXPANSION ACROSS the first ROW.

NOTE: OUR definition of det matches that one of a 2x2 mx: $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \cdot d - b \cdot c.$

It turns out that we may use expansion across Any Row, or Column, and we still get the same #: det A

THM

$$\begin{cases} \det A = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in} \\ \det A = a_{1j} C_{1j} + a_{2j} \cdot C_{2j} + \cdots + a_{nj} \cdot C_{nj} \end{cases}$$

Expansion across the i^{th} Row

expansion across the j^{th} Column

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$\boxed{\text{Do: } 10, 13 / 190;}$ 37, 38 / 191 ; 24, 20 / 191

(3.2) PROPERTIES OF DETERMINANTS

Q: HOW determinants change when Row Operations are performed

A:

(THM) Let A be a square mx.

a) if a multiple of one Row of A is added to another Row to produce a mx. B, then $\det B = \det A$

b) if 2 Rows of A are interchanged to produce B, then $\det B = -\det A$.

c) if one row of A is multiplied by k ($\neq 0$!!!!) to produce B, then $\det B = k \cdot \det A$.

DO: 11/199

$$A = \begin{bmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 4 & 10 & -4 & -1 \end{bmatrix} \xrightarrow[R_3 + 2R_2]{R_4 - 2R_1} \begin{bmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

So, $\det A = \det B = 5 \cdot (-1)^{1+2} \det \begin{bmatrix} 3 & 1 & -3 \\ 0 & -2 & 3 \\ 0 & 2 & 1 \end{bmatrix} =$

$= -5 \left[3 \cdot (-1)^{1+1} \det \begin{bmatrix} -2 & 3 \\ 2 & 1 \end{bmatrix} \right] = -15 \left[-2 - 6 \right] = +8 \cdot 15 = \boxed{120}$

DO: 15, 18, 19/199

THM: A square mx. A is invertible if and only if:

$\det A \neq 0$

DO: 23/199

THM: a) $\det(A^T) = \det A$; b) $\det(A \cdot B) = (\det A) \cdot (\det B)$.

DO: 31/200; 32/200; 33/200; 34/200; 35/200, 36/200

if TIME: 26/199 !!