(5.2) Lecture TUE: 24 MARCH 2009

Recall (from 5.1) 0 is an eigenvalue of $A \Leftrightarrow A$ is not invertible.

THM: (INVERTIBLE MX. THM) Let A be an $n \times n$ mx. TFASE:

1. A is invertible
2. 0 is NOT an eigenvalue of A
3. $\det A \neq 0$.

Reasoning: λ is an eigenvalue of an $n \times n$ mx. $A \Leftrightarrow$ there is a nonzero x st. $(A-\lambda I)x = 0 \Leftrightarrow \mathrm{Nul}\,(A-\lambda I) \neq \{0\} \Leftrightarrow A-\lambda I$ is NOT invertible $\Leftrightarrow \det(A-\lambda I) = 0$

FACT: λ is an eigenvalue of A $\boxed{\text{IF AND ONLY IF}}$ $\det(A-\lambda I) = 0$.

DEF: $\det(A-\lambda I) = 0$ is called the characteristic equation of A.

DEF: $\det(A-\lambda I)$ is called the characteristic polynomial of A.

EXP: 10/317 $A = \begin{pmatrix} 0 & 3 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$; $A - \lambda I_3 = \begin{pmatrix} 0 & 3 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} =$

$= \begin{pmatrix} -\lambda & 3 & 1 \\ 3 & -\lambda & 2 \\ 1 & 2 & -\lambda \end{pmatrix}$; $\det(A-\lambda I_3) = -\lambda \det\begin{pmatrix} -\lambda & 2 \\ 2 & -\lambda \end{pmatrix} + 3(-1) \cdot \det\begin{pmatrix} 3 & 2 \\ 1 & -\lambda \end{pmatrix} + 1 \cdot \det\begin{pmatrix} 3 & -\lambda \\ 1 & 2 \end{pmatrix} = -\lambda(\lambda^2 - 4) + (-3)(-3\lambda - 2) + (6 + \lambda) = -\lambda^3 + 14\lambda + 12$

8/317 $A-\lambda I_2 = \begin{pmatrix} 7 & -2 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 7-\lambda & -2 \\ 2 & 3-\lambda \end{pmatrix}$

$\det \begin{pmatrix} 7-\lambda & -2 \\ 2 & 3-\lambda \end{pmatrix} = (7-\lambda)(3-\lambda) + 4 =$

$= \lambda^2 - 10\lambda + 25 \Rightarrow \lambda_{1,2} = \begin{cases} \frac{10-\sqrt{100-100}}{2} \\ \frac{10+\sqrt{100+100}}{2} \end{cases}$

$\Rightarrow \boxed{\lambda_{1,2} = 5}$.

16/318 $A-\lambda I = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 8 & -4 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 1 & -5 & 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} =$

$= \begin{pmatrix} 5-\lambda & 0 & 0 & 0 \\ 8 & -4-\lambda & 0 & 0 \\ 0 & 7 & 1-\lambda & 0 \\ 1 & -5 & 2 & 1-\lambda \end{pmatrix} \Rightarrow \det(A-\lambda I) = 0$ has roots:

5, -4, 1

$\boxed{m_1}$ $\boxed{m_1}$ $\boxed{m_2}$

(WHY?! expansion across 1st Row!!)

EXC: if the characteristic eq. of a 6×6 mx. is $\lambda^6 - 4\lambda^5 - 12\lambda^4 = 0$, find all eigenvalues.

SOL: $\lambda^4(\lambda^2 - 4\lambda - 12) = 0 \Rightarrow \lambda^4(\lambda - 6)(\lambda + 2) = 0$

$\Rightarrow \begin{cases} 0, 0, 0, 0 \\ 6 \\ -2 \end{cases}$

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§ SIMILARITY

DEF: 2 $n\times n$ matrices A, B are called SIMILAR if there is an invertible mx. P such that $P^{-1}AP = B$.

THM: if A and B are similar, then they have the same characteristic polynomial.

SOL: $\det(A-\lambda I) = \det(PBP^{-1}-\lambda I) =$
$= \det(PBP^{-1}-\lambda PP^{-1}) = \det(P(B-\lambda I)P^{-1}) =$
$= (\det P)(\det B-\lambda I)\det P^{-1} = \det(B-\lambda I)$

Done

5.3 DIAGONALIZATION

Def: A square mx. D is called diagonal if its off main diagonal entries are zero!!

EXP: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$

FACT: POWERS of such D are easy to compute.

SAY $D = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow D^2 = D\cdot D = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 5^2 & 0 \\ 0 & 2^2 \end{pmatrix}.$

$D^3 = D^2 D = \begin{pmatrix} 5^2 & 0 \\ 0 & 2^2 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 5^3 & 0 \\ 0 & 2^3 \end{pmatrix}, \ldots$

$D^n = \begin{pmatrix} 5^n & 0 \\ 0 & 2^n \end{pmatrix}, \boxed{n \geq 1}$

Def: A square mx. A is diagonalizable if A is SIMILAR to a diagonal mx., i.e.

there is a $\left\{ \begin{array}{l} \text{diagonal mx } D \\ \text{invertible mx. } P \end{array} \right.$ such that

$A = PDP^{-1}$

<u>FACT:</u> POWERS of such A are easy to compute

$A^2 = PDP^{-1} \cdot PDP^{-1} = PD^2P^{-1}$, $A^3 = \underbrace{PD^2P^{-1}}_{A^2} \underbrace{PDP^{-1}}_{A} = PD^3P^{-1}$, -

- - , $A^n = P \cdot D^n \cdot P^{-1}$; - - -

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