4.9 MARKOV CHAINS

STORY: Markov Chains are used to describe an experiment or measurement that is performed many times (in the same way), where the OUTCOME of each trial of the experiment/measurement will be one of the several specified possible outcomes, AND where the outcome of one trial depends only on the immediately preceding trial.

Recall (1.10)
- A Region is divided < City / Country Side
- $x_0 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$ indicates: < 60 % lives in City / 40 % lives in CS
- THE SUM is 1.

Def: 1) A vector with nonnegative entries that ADD UP to 1 is called a PROBABILITY VECTOR.

2) A STOCHASTIC mx. is a square mx. whose columns are probability vectors.

3) A Markov Chain is a sequence of probability vectors $x_0, x_1, x_2, \ldots$ together with a STOCHASTIC mx P, such that: $x_1 = Px_0,\ x_2 = Px_1, \ldots,\ x_{k+1} = Px_k;\ k \geq 0$

4) The vector x_k is called A STATE VECTOR:
(The entries in x_k list, respectively, the probabilities that the system/exp./measurement is in some state)

(Recall $x_0 = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$, etc $\to 1.10$)

EXP: Suppose our migration mx. is

$$M = \begin{array}{cc} & \begin{array}{cc} C & C\text{-}S \end{array} \\ & \left[\begin{array}{cc} 0.95 & 0.03 \\ 0.05 & 0.97 \end{array}\right] \begin{array}{c} C \\ C\text{-}S \end{array} \end{array}$$

(i.e., each year 5% of City moves to C-S, and 3% of C-S population moves to City).

▸M is a stochastic mx. Say, initially : $\begin{cases} 600{,}000 \text{ in City} \\ 400{,}000 \text{ in C-Side} \end{cases}$

So $x_0 = \begin{pmatrix} 600{,}000 \\ 400{,}000 \end{pmatrix} \begin{array}{c} C \\ C\text{-}S \end{array}$. After 1 year: $x_1 = Mx_0 =$

$$= \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \begin{bmatrix} 600{,}000 \\ 400{,}000 \end{bmatrix} = \begin{bmatrix} 582{,}000 \\ 418{,}000 \end{bmatrix} \begin{array}{c} C \\ C\text{-}S \end{array}$$

After 2 years:

$$x_2 = Mx_1 = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \begin{bmatrix} 582{,}000 \\ 418{,}000 \end{bmatrix} = \begin{bmatrix} 565{,}000 \\ 435{,}000 \end{bmatrix}$$

GOAL: We want to know what is the distribution in the LONG RUN!!

Def: if P is a stochastic mx., then a steady-state vector (or equilibrium vector) for P is a probability vector q such that: $Pq = q$.

IDEA: if we apply $P, P, P \ldots$ we get only: q, q, q;

EXP: $q = \begin{pmatrix} 0.375 \\ 0.625 \end{pmatrix}$ is a steady-state vector for our M:

$$Mq = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \cdot \begin{bmatrix} 0.375 \\ 0.625 \end{bmatrix} \underset{\uparrow}{=} \begin{bmatrix} 0.375 \\ 0.625 \end{bmatrix} = q$$

check it!

Q: Does such a vector always exist? Is it unique?

DEF: • A stochastic mx. P is called <u>regular</u> if some power of it (P^k) contains ONLY strictly positive entries.

• A sequence $\{x_k\}_{k=0,1,2,\dots}$ (of vectors) <u>CONVERGES</u> to q as $k \to \infty$ IF THE ENTRIES of x_k <u>CAN BE MADE</u> arbitrarly CLOSE TO the entries of q, by taking k sufficiently large.

THM: if P is an $n \times n$ regular stochastic mx, then P has a UNIQUE steady-state vector q

<u>If</u> x_0 is <u>any</u> initial state, and $x_{k+1} = Px_k$; $k \geq 0$, <u>then</u> the Markov chain $\{x_k\}_k$ converges to q as $k \to \infty$.

DO: 3/296 a)

$$M = \begin{array}{c} \text{FROM} \\ \begin{array}{cc|c} H & i & \\ \end{array} \\ \end{array}$$

$$M = \begin{array}{cc} H & i \\ \end{array}$$

$$\begin{array}{c|cc|c} & H & i & \\ M = & 0.95 & 0.45 & H \\ & 0.05 & 0.55 & i \end{array}$$

b) $x_0 = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} \begin{matrix} H \\ i \end{matrix}$; $x_1 = M x_0 = \begin{pmatrix} 0.95 & 0.45 \\ 0.05 & 0.55 \end{pmatrix} \begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 0.85 \\ 0.15 \end{pmatrix} \begin{matrix} H \\ i \end{matrix}$

$i \Rightarrow$ 15% ILL ON TUE!

$$x_2 = M x_1 = \begin{pmatrix} 0.95 & 0.45 \\ 0.05 & 0.55 \end{pmatrix} \begin{pmatrix} 0.85 \\ 0.15 \end{pmatrix} = \begin{pmatrix} 0.875 \\ 0.125 \end{pmatrix} \begin{matrix} H \\ i \end{matrix}$$

SO ON WED 12.5% are <u>ILL</u>

c) TODAY: $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{matrix} H \\ i \end{matrix}$; TOMORROW: $x_1 = M x_0 =$

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$$\begin{pmatrix} 0.95 & 0.45 \\ 0.05 & 0.55 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.95 \\ 0.05 \end{pmatrix}$$ ← tomorrow, As we said i-

In 2 days: $x_2 = Mx_1 = \begin{pmatrix} 0.95 & 0.45 \\ 0.05 & 0.55 \end{pmatrix}\begin{pmatrix} 0.95 \\ 0.05 \end{pmatrix} = \begin{pmatrix} 0.925 \\ 0.075 \end{pmatrix}\begin{matrix} H \\ ill \end{matrix}$

so: the probability is 92.5% (or you may just say 0.925 !!!) to be Healthy

(d) what happens in the long-Run? We need to find the steady-state vector q.

$Mq = q \Rightarrow (M-I)q = 0$; $M - I = \begin{pmatrix} 0.95 & 0.45 \\ 0.05 & 0.55 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -0.05 & 0.45 \\ 0.05 & -0.45 \end{pmatrix}$. Solve $\left(\begin{array}{cc|c} -0.05 & 0.45 & 0 \\ 0.05 & -0.45 & 0 \end{array}\right)$

$\xrightarrow{R_2 + R_1}$ $\left(\begin{array}{cc|c} -0.05 & 0.45 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\Rightarrow -0.05q_1 + 0.45q_2 = 0$

$\quad\;\; q_1 \qquad q_2$

$\Rightarrow q_1 = \dfrac{0.45}{0.05} q_2$. BUT q is a prob. vector: $q_1 + q_2 = 1 \Rightarrow q_1 = 1 - q_2 \Rightarrow \dfrac{0.45}{0.05} q_2 = 1 - q_2$

$1 = q_2\left(\dfrac{0.45}{0.05} + 1\right) \Rightarrow 1 = q_2 \dfrac{0.50}{0.05} \Rightarrow q_2 = \dfrac{0.05}{0.50} =$

$= \dfrac{1}{10} \Rightarrow q_1 = \dfrac{9}{10} \Rightarrow q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0.9 \\ 0.1 \end{pmatrix}\begin{matrix} H \\ i \end{matrix}$

long Run distribution