Lecture : TUE 31st of March

12/326 $A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$; $A - \lambda I_3 = \begin{pmatrix} 4-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{pmatrix}$.

$$\det \begin{pmatrix} 4-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{pmatrix} = (4-\lambda) \det \begin{pmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix} + 2(-1) \det \begin{pmatrix} 2 & \\ 2 & 4 \end{pmatrix}$$

$$+ 2 \det \begin{pmatrix} 2 & 4-\lambda \\ 2 & 2 \end{pmatrix} = (4-\lambda)\left[(4-\lambda)^2 - 2^2\right] - 2\left[8 - 2\lambda - 4\right]$$

$$+2\left[4 - 8 + 2\lambda\right] = (4-\lambda)(4-\lambda-2)(4-\lambda+2) - 2\left[4-2\lambda\right] +$$

$$+2\left[2\lambda - 4\right] = (4-\lambda)(2-\lambda)(6-\lambda) - 4(2-\lambda) + 4(\lambda-2)$$

$$= (2-\lambda)\left[24 - 4\lambda - 6\lambda + \lambda^2 - 4 - 4\right] =$$

$$= (2-\lambda)\left[\lambda^2 - 10\lambda + 16\right] = (2-\lambda)(\lambda-2)(\lambda-8).$$

Solve: $(2-\lambda)(\lambda-2)(\lambda-8) = 0 \Rightarrow \lambda_1 = \lambda_2 = 2; \quad \lambda_3 = 8$

Construct $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}$.

TRY to get P: FOR $\lambda_1 = 2$, solve $(A - \lambda_1 I_3)x = 0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} x_2 = t, \ x_3 = s \ ; t, s \text{ scalar} \\ x_1 = -t-s \end{cases}$$

So $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -t-s \\ t \\ s \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. Basis for the eigenspace: $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$; dimension ②

$\boxed{\text{FOR } \lambda_3 = 8}$, solve $(A - \lambda_3 I_3)X = 0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

$$\left(\begin{array}{ccc|c} -4 & 2 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right) \xrightarrow[\frac{1}{-2}R_1]{R_3-R_2} \left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 2 & -4 & 2 & 0 \\ 0 & 6 & -6 & 0 \end{array}\right) \xrightarrow{R_2-R_1} \left(\begin{array}{ccc|c} 2 & -1 & -1 & \\ 0 & -3 & 3 & \\ 0 & 6 & -6 & \end{array}\right.$$

$$\longrightarrow \left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \begin{cases} x_3 = t, \ t \text{ scalar}; \\ 2x_1 = x_2 + x_3 \end{cases} \quad \begin{array}{l} x_2 = t \\ x_1 = t \end{array}$$

$\rightarrow$ $x = t\begin{pmatrix}1\\1\\1\end{pmatrix}$; t#. BASIS for eigenspace: $\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$;

dimension is ①. Since $2+1=3$ $\Rightarrow$ diagonalizable.

$$P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

§ $\boxed{1.10}$ DIFFERENCE EQUATIONS

Def: An equation of the form: $x_{k+1} = A x_k$; for $k = 0, 1, 2, \ldots$ is called a LINEAR DIFFERENCE EQUATION (or RECURRENCE relation)

POINT: if x_0 is known, and A is given, one may compute $x_1 = A x_0$; and then $x_2 = A x_1$, and then $x_3 = A x_2$, etc...

Q: Where a difference equation may appear??

A: In the following model:

FIX a year — SAY 2000 — and a Region:

$\rightarrow$ The Population of CITY is r_0;

$\rightarrow$ The population of COUNTRY SIDE is s_0.

$\rightarrow$ Let $x_0 = \begin{bmatrix} r_0 \\ s_0 \end{bmatrix}$ $\begin{array}{l} \leftarrow \text{CITY pop. in 2000} \\ \leftarrow \text{COUNTRISIDE pop. in 2000.} \end{array}$

→ For 2001 and subsequent years, the population of CITY and COUNTRY SIDE is given by vectors:

$$x_1 = \begin{bmatrix} r_1 \\ s_1 \end{bmatrix}; \quad x_2 = \begin{bmatrix} r_2 \\ s_2 \end{bmatrix}, \ \ldots\ldots$$

Q: HOW THESE VECTORS might be related?

Suppose that some demographic studies show that EACH YEAR 5% of city pop moves to the COUNTRY side (and 95 remains in CITY), while 3% of COUNTRY SIDE pop. move to city (and 97% remains in the COUNTRY SIDE);

SO: AFTER 1 years:

$$\begin{array}{l} \text{CITY} \\ \text{COUNTRY SIDE} \end{array} \begin{bmatrix} r_1 \\ s_1 \end{bmatrix} = \begin{bmatrix} 0.95 \cdot r_0 + 0.03 \cdot s_0 \\ 0.05\, r_0 + 0.97\, s_0 \end{bmatrix} = r_0 \begin{bmatrix} 0.95 \\ 0.05 \end{bmatrix} + s_0 \begin{bmatrix} 0.03 \\ 0.97 \end{bmatrix} = \underbrace{\begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}}_{=M} \begin{bmatrix} r_0 \\ s_0 \end{bmatrix}$$

; Hence: $x_1 = M x_0$;

→ M is called the migration matrix.

→ The eq. $x_1 = M x_0$ describes the changes (in population) between 2000 and 2001.

→ if the Migration percentages remain the same (constant), then the change from 2001 to 2002 is given by: $x_2 = M x_1$

→ We get: $x_{k+1} = M x_k$, $k = 0, 1, 2, \ldots$

EXC: Suppose in 2000 there were 600,000 in CITY; and 400,000 in the COUNTRY SIDE. Compute the populations in 2002!

SOL:

$$X_0 = \begin{array}{c|c} \begin{bmatrix} 600{,}000 \\ 400{,}000 \end{bmatrix} & \begin{array}{c} C \\ C\text{-}S \end{array} \end{array} \quad \text{in } 2000.$$

$$X_1 = \begin{pmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{pmatrix} \begin{pmatrix} 600{,}000 \\ 400{,}000 \end{pmatrix} = \begin{pmatrix} 582{,}000 \\ 418{,}000 \end{pmatrix} \begin{array}{c} C \\ C\text{-}S \end{array}$$

$$X_2 = \begin{pmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{pmatrix} \begin{pmatrix} 582{,}000 \\ 418{,}000 \end{pmatrix} = \begin{pmatrix} 565{,}440 \\ 434{,}560 \end{pmatrix} \begin{array}{c} C \\ C\text{-}S \end{array}$$

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$$M = \begin{array}{cccc} A & E & W & \\ \hline 0.97 & 0.05 & 0.10 & A \\ 0.00 & 0.80 & 0.05 & E \\ 0.03 & 0.05 & 0.85 & W \end{array}$$

SOL:

$$X_0 = \begin{pmatrix} 305 \\ 48 \\ 98 \end{pmatrix};$$

Monday

$$\underset{\text{TUE}}{X_1} = M X_1 = \begin{pmatrix} 0.97 & 0.05 & 0.10 \\ 0.00 & 0.80 & 0.05 \\ 0.03 & 0.05 & 0.85 \end{pmatrix} \begin{pmatrix} 305 \\ 48 \\ 98 \end{pmatrix} \simeq \begin{pmatrix} 308 \\ 48 \\ 95 \end{pmatrix}$$

$$X_2 = \begin{pmatrix} 0.97 & 0.05 & 0.10 \\ 0.00 & 0.80 & 0.05 \\ 0.03 & 0.05 & 0.85 \end{pmatrix} \begin{pmatrix} 308 \\ 48 \\ 95 \end{pmatrix} \simeq \begin{pmatrix} 311 \\ \boxed{48} \\ 92 \end{pmatrix} \longleftarrow \quad \Longleftarrow \underline{\text{WED}}$$