- Lecture - FRI -

(5.3) DIAGONALIZATION

Def A square mx. D is called DIAGONAL if its off main diagonal entries are zero!

EXP: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$

FACT: Powers of such D are easy to compute:

SAY $D = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow D^2 = D \cdot D = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 5^2 & 0 \\ 0 & 2^2 \end{pmatrix}$;

$D^3 = \begin{pmatrix} 5^2 & 0 \\ 0 & 2^2 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 5^3 & 0 \\ 0 & 2^3 \end{pmatrix}$, ..., $D^n = \begin{pmatrix} 5^n & 0 \\ 0 & 2^n \end{pmatrix}$; $n \geq 1$

Def: A square mx. A is diagonalizable if there is a diagonal mx D, and an invertible mx. P such that:

$$A = PDP^{-1} \quad (\text{or} \quad P^{-1}AP = D).$$

FACT: Powers of such A are easy to compute:

$A^2 = A \cdot A = PD\underbrace{P^{-1} \cdot P}_{I}DP^{-1} = PD^2P^{-1}$; $A^3 = A^2 \cdot A = PD^2\underbrace{P^{-1} \cdot P}_{I}DP^{-1} = P \cdot D^3 \cdot P^{-1}$; ...

$A^m = P \cdot D^m \cdot P^{-1}$; $m \geq 1$.

DO 1, 3/325 : EASY !!  ?!?!

THM(The diagonalization Theorem)
An $n \times n$ mx. A is diagonalizable IF AND ONLY IF A has $\boxed{n}$ linearly Independent eigenvectors.

in fact: $A = PDP^{-1}$, with D diagonal mx, if and only if the columns of P are $\boxed{n}$ linearly indep. eigenvectors of A. In this case, the diagonal entries of D are the eigenvalues of A that correspond, respectively, to the eigenvectors in P

DO 11/326 Diagonalize (if possible) $A = \begin{pmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{pmatrix}$.

SOL: 1) Find eigenvalues!!

- $\det(A-\lambda I_3) = \det\begin{pmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{pmatrix} = (-1-\lambda)\cdot\det\begin{pmatrix} 4-\lambda & 0 \\ 1 & 3-\lambda \end{pmatrix}$

$+ 4(-1)\det\begin{pmatrix} -3 & 0 \\ -3 & 3-\lambda \end{pmatrix} + (-2)\det\begin{pmatrix} -3 & 4-\lambda \\ -3 & 1 \end{pmatrix} =$

$= (-1-\lambda)\left[(4-\lambda)(3-\lambda)\right] - 4(-3)(3-\lambda) + (-2)\left[-3 + 12 - 3\lambda\right]$

$= (-1-\lambda)(4-\lambda)(3-\lambda) + 12(3-\lambda) - 2\left[9 - 3\lambda\right] =$

$= (3-\lambda)\left[(-1-\lambda)(4-\lambda) + 12 - 6\right] = (3-\lambda)(\lambda^2 - 3\lambda + 2)$

$= (3-\lambda)(\lambda-1)(\lambda-2)$

SOLVE: $(3-\lambda)(\lambda-1)(\lambda-2) = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$

- Get $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

2) GET P (if possible)

FOR $\lambda_1 = 1$, SOLVE $(A - \lambda_1 I_3)x = 0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$$\left(\begin{array}{ccc|c} -2 & 4 & -2 & 0 \\ -3 & 3 & 0 & 0 \\ -3 & 1 & 2 & 0 \end{array}\right) \xrightarrow[R_3 - R_2]{-\frac{1}{2}R_1} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -3 & 3 & 0 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right) \xrightarrow[-\frac{1}{2}R_3]{\frac{1}{3}R_2}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \xrightarrow{R_2 + R_1} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \xrightarrow{R_3 + R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- $$\xrightarrow{(-1)R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
$x_1 \quad x_2 \quad x_3$

$x_3 = t$, x_3 is free

$x_2 = t$; $x_1 = 2x_2 - x_3 = 2t - t = t$

$X = \begin{pmatrix} t \\ t \\ t \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; t scalar; choose $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

For $\lambda_2=2$ $(A-\lambda_2 I_3)x=0$; $(A-2I_3)x=0$.

$$\left(\begin{array}{ccc|c} -3 & 4 & -2 & 0 \\ -3 & 2 & 0 & 0 \\ -3 & 1 & 1 & 0 \end{array}\right) \xrightarrow[R_3-R_1]{R_2-R_1} \left(\begin{array}{ccc|c} -3 & 4 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right)$$

$$\xrightarrow[\frac{1}{-3}R_3]{\frac{1}{-2}R_2} \left(\begin{array}{ccc|c} -3 & 4 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \xrightarrow{R_3-R_2} \left(\begin{array}{ccc|c} -3 & 4 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_3=t$, free
$x_2=x_3=t$

$-3x_1=-4x_2+2x_3=-4t+2t=-2t$; $x_1=\frac{2}{3}t \Rightarrow X=\begin{pmatrix} 2/3\,t \\ t \\ t \end{pmatrix}$

$=t\begin{pmatrix} 2/3 \\ 1 \\ 1 \end{pmatrix}$; t scalar ; $t=3 \Rightarrow v_2=\begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$.

For $\lambda_3=3$ SOLVE: $(A-\lambda_3 I_3)x=0=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$;

$$\left(\begin{array}{ccc|c} -4 & 4 & -2 & 0 \\ -3 & 1 & 0 & 0 \\ -3 & 1 & 0 & 0 \end{array}\right) \xrightarrow[\frac{1}{-2}R_1]{R_3-R_2} \left(\begin{array}{ccc|c} +2 & -2 & 1 & 0 \\ -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1+R_2}$$

$$\left(\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2-3R_1} \left(\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & 4 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{4}R_2} \left(\begin{array}{ccc|c} -1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_3=t$, t scalar ; $x_2=\frac{3}{4}t$; $-x_1=x_2-x_3=\frac{3}{4}t-t=-\frac{1}{4}t$

So $X=\begin{pmatrix} 1/4\,t \\ 3/4\,t \\ t \end{pmatrix}=t\begin{pmatrix} 1/4 \\ 3/4 \\ 1 \end{pmatrix}$; t scalar. Choose $t=4 \Rightarrow$

$v_3=\begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$.

Now construct $P=\begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{pmatrix}$.

Check yourself that $A=PDP^{-1}$

(THM) if an $n \times n$ mx has $\underline{n}$ DISTINCT eigenvalues, then it is diagonalizable.

- (see the above exc!) WHAT if the eigenvalues are <u>not</u> distinct??

(THM) Let A be an $n \times n$ mx. whose DISTINCT eigenvalues are $\lambda_1, \lambda_2, ---, \lambda_p$.
The mx. A <u>is</u> diagonalizable <u>IF AND ONLY IF</u> the sum of the dimensions of the <u>DISTINCT</u> eigenspaces equals n

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$$A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix} ; \quad A - \lambda I_3 = \begin{pmatrix} 4-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{pmatrix}$$

- $$\det \begin{pmatrix} 4-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{pmatrix} = (4-\lambda) \cdot \det \begin{pmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix} + 2(-1) \cdot \det \begin{pmatrix} 2 & 2 \\ 2 & 4-\lambda \end{pmatrix}$$

$$+2 \cdot \det \begin{pmatrix} 2 & 4-\lambda \\ 2 & 2 \end{pmatrix} = (4-\lambda)\left[(4-\lambda)^2 - 2^2\right] - 2\left[8 - 2\lambda - 4\right]$$

$$+2\left[4 - 8 + 2\lambda\right] = (4-\lambda)\left[(4-\lambda-2)(4-\lambda+2)\right] - 2\left[4 - 2\lambda\right] + 2 \cdot$$

$$\cdot (2\lambda - 4) = (4-\lambda)(2-\lambda)(6-\lambda) - 4(2-\lambda) + 4(\lambda - 2)$$

$$= (2-\lambda)\left[24 - 4\lambda - 6\lambda + \lambda^2 - 4 - 4\right] =$$

$$= (2-\lambda)\left[\lambda^2 - 10\lambda + 16\right] = (2-\lambda)(\lambda-2)(\lambda-8)$$

<u>SOLVE:</u> $(2-\lambda)(\lambda-2)(\lambda-8) = 0 \Rightarrow \lambda_1 = \lambda_2 = 2 \; ; \; \lambda_3 = 8$

- Construct $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}$.

TRY to get P : For $\lambda_1 = 2$; solve $(A - \lambda_1 I_3)x = 0 = \begin{pmatrix}0\\0\\0\end{pmatrix}$;

$$\left(\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$$

$x_2 = t$, $x_3 = s$; t, s scalars
$x_1 = -t - s$.

So $X = \begin{pmatrix}-t-s\\t\\s\end{pmatrix} = t\begin{pmatrix}-1\\1\\0\end{pmatrix} + s\begin{pmatrix}-1\\0\\1\end{pmatrix}$. BASIS for the eigenspace: $\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}; \begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$; dimension is: 2

For $\lambda_2 = 8$: solve $(A - \lambda_2 I_3)x = 0 = \begin{pmatrix}0\\0\\0\end{pmatrix}$;

$$\left(\begin{array}{ccc|c} -4 & 2 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right) \xrightarrow[\frac{1}{-2}R_1]{R_3 - R_2} \left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 2 & -4 & 2 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right) \xrightarrow{R_2 - R_1} \left(\begin{array}{ccc|c} 2 & -1 & -1 & \\ 0 & -3 & 3 & \\ 0 & 6 & -6 & \end{array}\right)$$

$$\to\to \left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$$

$x_3 = t$; t scalar; $x_2 = t$;
$2x_1 = x_2 + x_3 = 2t \Rightarrow x_1 = t$.

So $X = t\begin{pmatrix}1\\1\\1\end{pmatrix}$; BASIS for eigenspace: $\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$; dimension is 1.

Since $2 + 1 = 3 \Rightarrow$ Diagonolizoble ; $P = \begin{pmatrix}-1 & -1 & 1\\ 1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}$.

PLEASE Attend the

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