Lecture -1- TUE - JAN-2009

- INTRODUCE yourself
- TALK ABOUT OUTLINE, DISTRIBUTE IT

1.1 SYSTEMS OF LINEAR EQUATIONS

Q: What is the equation of a line?

A: $ax+by=c$, where a,b,c #s

We generealize:

Def: A linear equation (in variables $x_1, x_2, \dots, x_n$) is an equation of the form:

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b, \text{ where:}$$

$\begin{cases} a_1, a_2, \dots, a_n & \text{are \#s called the coefficients} \\ b & \text{is a \#, called the free term.} \end{cases}$

Here n is a positive integer!

EXP
- $3x_1 - 2x_2 + 7 = x_3$ is an eq. because:

$3x_1 - 2x_2 - x_3 = -7$

- $2x_1 - \frac{1}{2}x_2 + \sqrt{7}\,x_3 - x_4 = 0$ IT IS an eq.
- $\sqrt{x_1} - 2x_2 + \frac{1}{3} = 0$ IT IS not a lin. equation

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Def: A system of linear equations is a collection (bunch) of linear equations involving the same variables!

EXP: $\begin{cases} 3x_1 - \frac{1}{2}x_2 + x_3 = -3 \\ 2x_1 \qquad\quad + x_3 = 7 \end{cases}$

Def: A solution of a linear system is a list $(s_1, s_2, \ldots, s_n)$ of #s that makes each equation a true statement: when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$ respectively.

Def: The set of ALL possible solutions is called: THE SOLUTION SET.

SURPRISE: $0, \quad 1, \quad \infty$

OUR GOAL: How many? What are them?

DEF: TWO linear systems are called equivalent if They have the same solution set.

IDEA: Replace a system by an equivalent one, but easier to solve!!

EXP: SOLVE:
$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

SOL: We use x_1 (from the Eq. 1) to eliminate x_1 from the other equations:

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_2 - 8x_3 = 8 \\ -3x_2 + 13x_3 = -9 \end{cases}$$

STEP (1): Add 4 times Eq. 1 to Eq. 3

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ -3x_2 + 13x_3 = -9 \end{cases}$$

STEP ②

Divide Eq. 2 by 2

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

STEP ③

Add 3 times Eq. 2 to Eq. 3

SOLVE (BACK SUBSTITUTION)

$x_3 = 3$, $x_2 = 4 + 4x_3 = 16$; $x_1 = 2x_2 - x_3 = 32 - 3 = 29$

So: $(29, 16, 3)$ is the solution!!

TOO STUFFY!! We introduce the mx. notation

DEF: A matrix is a rectangular array. The numbers in the mx. are called entries

(EXP) $\begin{bmatrix} 3 & 1 & -2 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 7 & -1 & 0 \\ 1 & 2 & 2 & 1 \end{bmatrix}$ OR $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\longrightarrow$ ROWS, $\downarrow$ columns

So: Create the AUGMENTED MATRIX:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & +5 & 9 & -9 \end{array}\right]$$

LET us perform the same "operations": on the mx:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & +9 & -9 \end{array}\right] \xrightarrow{R_3 + 4R_1} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \xrightarrow{\frac{1}{2}R_2}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right] \xrightarrow{R_3+3R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$x_1 \quad x_2 \quad x_3$

STAIRS pattern (Echelon)

USE BACK SUBSTITUTION:

$x_3 = 3$; $x_2 = 4 + 4x_3 = 16$, $x_1 = 2x_2 - x_3 = 29$ $\Rightarrow (29, 16, 3)$

SAME answer!!

ROW OPERATIONS

1. Interchange: Interchange 2 Rows
2. Scaling: Multiply all entries in a Row by a non-zero constant.
3. Replacement: Replace one Row by the sum of itself and a multiple of another Row.

NOTE! DOING (performing) these 3 operations on an augmented mx (coming from a system) WILL NOT CHANGE the SOLUTION SET.

DEF: 2 matrices are called ROW equivalent if there is a sequence of Row Operations that transforms one mx into the other.

SOLVE: 12, 14 / pg 11

IF TIME:
- A system is called consistent if it has either one solution or infinitely many solutions
- A system is called inconsistent if it has NO solutions.

IF TIME: 16 / 11

NOTE: Row Operations are Reversible:

$A \xrightarrow{op} B \Rightarrow B \xrightarrow{\tilde{op}} A$

§1.1 LECTURE 2 – FRI – JAN 9TH 2009

ROW OPERATIONS

(1) INTERCHANGE : Interchange 2 Rows.

(2) SCALING : Multiply all entries in a Row by a non-zero constant

(3) Replacement: Replace one Row by the sum of itself and a multiple of another Row

NOTE: DOING (performing) these 3 operations on an augmented mx (coming from a SYSTEM) WILL NOT change the SOLUTION SET.

Def: 2 matrices are called ROW Equivalent if there is a sequence of Row Operations that transforms one mx. into the other.

SOLVE: $\boxed{12, 14 / 11}$

DEF:
- A SYSTEM is called CONSISTENT if it has either one solution or infinitely many solutions.
- A SYSTEM is called INCONSISTENT if it has NO SOLUTIONS.

NOTE: Row Operations Are Reversible: $A \xrightarrow{op} B \Rightarrow B \xrightarrow{\tilde{op}} A$

§ $\boxed{1.2}$ ROW REDUCTION and ECHELON FORMS

Def: A non-zero Row in a mx means a Row that contains at least one non-zero entry.

DEF: A LEADING ENTRY of a ROW is the leftmost nonzero entry (!! non zero Row)

DEF: (REF) A mx. is in Row Echelon Form if it has the following 3 properties:

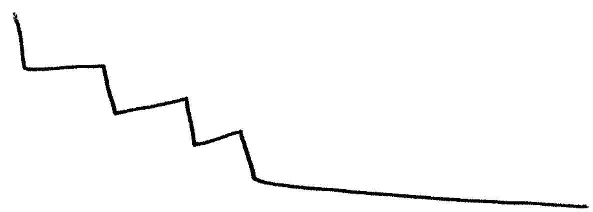
1. ALL nonzero Rows are above any Rows of all zeros.
2. Each leading entry of a Row is in a Column to the Right of the leading entry of the Row above it.
3. ALL entries in a Column below a leading entry are zeros

DEF: (RREF) If a mx. in REF satisfies:

4. The leading entry in each non-zero Row is 1
5. Each leading 1 is the only nonzero entry in its column

then the mx. is in Reduced Row Echelon Form

Recall: the stairpattern:



EXP:

$$\begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ RREF}$$

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \text{ REF NOT RREF}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ NOT REF !!}$$

NOTE: We know that each mx. can be transformed into REF mx. (using Row Op.), BUT the Result (= mx.) is NOT UNIQUE.

THM/15 Each mx. is Row Equivalent to one and only one REDUCED ROW ECHELON FORM mx.

Because of this uniqueness $\Rightarrow$ the leading entries are ALWAYS in the same positions in ANY ROW ECHELON FORM. SO:

Def: a) A PIVOT position in a mx. A is a location in A that corresponds to a leading 1 in the Reduced Row Echelon Form of A.

b) A PIVOT COLUMN is a Column of A that contains a pivot position.

DO 4/25.

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§ SOLUTIONS OF LINEAR SYSTEMS

EXP: Suppose that a system was given; we constructed the augmented mx; we Row Reduced, and get:

$$\begin{array}{c} \left(\begin{array}{ccc|c} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \\ \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \end{array}$$

- SAY the variables Are x_1, x_2, x_3
- Circle the pivot positions
- The variables x_1, x_2 correspond to pivot columns, they are called BASIC (leading / independent) variables
- The rest of variables, x_3, is called FREE VARIABLES.

WE DO: $\boxed{x_3 = \text{free},\ x_3 = t,\ t \text{ scalar}}$ $\boxed{x_2 = 4 - x_3 = 4 - t}$

$\boxed{x_1 = 1 + 5x_3 = 1 + 5t}$ $\boxed{\text{SO } \infty\text{-ly many solutions}}$

We are LED to:

DEF: The variables that correspond to pivot columns are called BASIC VARIABLES; the rest of variables are called FREE.

DO 14/25

§ HOW many SOLUTIONS?

STORY: SITUATION: SYSTEM ⇝ AUG. MX ⇝ $\boxed{REF}$

- IF THERE IS A ROW of the form: $[0\ 0\ 0 \dots 0 | b]$, $b \neq 0$, there are NO SOLUTIONS
- EITHER: ALL VAR. ARE BASIC → 1 (unique) SOL
  OR: AT LEAST one free var → ∞-ly many SOL

THM A linear SYSTEM is consistent if and only if an echelon form of the augmented mx. has NO row of the form $[0\ 0 \dots 0 | b]$, $b \neq 0$.

IF THE SYSTEM is consistent, then the solution set contains either 1 SOL or ∞-ly many SOL

ALGORITHM

1°) CONSTRUCT the augmented MX.
2°) Reduce to REF OR RREF
3°) CIRCLE the pivot positions
4°) Decide what var. ARE FREE, and what var. ARE BASIC
5°) SOLVE: FIND BASIC VAR. in terms of free variables/constants = #s

DO 18, 20/26

Recall: (THM) A linear system is consistent if and only if an echelon form of the augmented mx. has NO ROW of the form $[0\ 0 \ldots 0 \mid b]$, $b \neq 0$.

IF THE SYSTEM is consistent, then the solution set contains either 1 solution OR ∞-ly many solutions.

ALGORITHM (1°) CONSTRUCT the AUGMENTED mx.

(2°) Reduce to REF or RREF.

(3°) Circle the pivot positions.

(4°) Decide what variables ARE FREE, and what variables ARE BASIC.

(5°) SOLVE: FIND BASIC VARIABLES in terms of FREE VARIABLES/constants.

DO: 18, 20/26

18/26
$$\left[\begin{array}{cc|c} 1 & -3 & -2 \\ 5 & h & -7 \end{array}\right] \xrightarrow{R_2 - 5R_1} \left[\begin{array}{cc|c} 1 & -3 & -2 \\ 0 & h+15 & 3 \end{array}\right]$$

We Want Consistent: if $h+15=0 \Rightarrow [0\ \ 0 \mid 3] \Rightarrow$ NO SOL. $\Rightarrow$ INCONSISTENT. HENCE $h+15 \neq 0$; i.e. $h \neq -15$

20/26
$$\begin{cases} x_1 + 3x_2 = 2 \\ 3x_1 + hx_2 = k \end{cases}$$

SOL:
$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & h & k \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & h-9 & k-6 \end{array}\right]$$

a) NO SOL: I am looking for $[0 \ 0 \mid b]$; $b \neq 0$.

ONLY Row #2 CAN GIVE SUCH A ROW:

$$[\underbrace{0}_{h-9} \quad 0 \mid \underbrace{b}_{k-6}] \quad b \neq 0 \qquad \text{So: } \begin{cases} h-9=0 \\ k-6 \neq 0 \end{cases} \Rightarrow \begin{cases} h=9 \\ k \neq 6 \end{cases}$$

b) CIRCLE The PIVOTS !! We need to have a PIVOT POSITION where is $h-9$ IN ORDER TO HAVE A UNIQUE SOL: $\boxed{h-9 \neq 0}$ or $\boxed{h \neq 9}$

c) ∞-ly many SOLUTIONS: ONLY 1 pivot position in our case; so $h-9=0$ and $k-6=0$. In other WORDS: $\boxed{h=9, \quad k=6}$

A different TYPE of PROBLEM:

$\boxed{14/25}$
$$\left[\begin{array}{ccccc|c} 1 & 2 & -5 & -6 & 0 & -5 \\ 0 & 1 & -6 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

SOL: IT is already in REF. Circle the PIVOT positions

$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5$

$x_1, x_2, x_5 \rightarrow$ BASIC AND $x_3, x_4 \rightarrow$ FREE

Set $x_3 = t$, t scalar, $x_4 = s$, scalar

The third Row gives us: $\boxed{x_5 = 0}$; The 2^{nd} Row gives us: $x_2 - 6x_3 - 3x_4 = 2 \Rightarrow \boxed{x_2 = 6t + 3s.}$

The 1^{st} Row tells us: $x_1 = -2x_2 + 5x_3 + 6x_4 + (-5) =$

$= -2(6t+3s) + 5t + 6s - 5 = \boxed{-7t - 5}$.

DO: 24/26

$$\left(\begin{array}{cccc|c} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \to \boxed{\text{INC}}$$

$$\left(\begin{array}{cccc|c} * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$ same!

$\boxed{\text{DO:}}$ ~~1~~6b/26 15 a,b / 25

$\boxed{1.3}$ $\boxed{\text{VECTOR EQUATIONS}}$

§ VECTORS in $\mathbb{R}^n$

if n is a positive integer, $\mathbb{R}^n$ denotes the collection of all lists of n real #s, written as Column matrices:

$$\mathbb{R}^n = \left\{ u \;\middle|\; u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix};\ u_1, u_2, \dots, u_n \text{ are \#s} \right\}$$

u is called VECTOR.

EXP: $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is in $\mathbb{R}^2$; $\begin{pmatrix} 1/2 \\ -1/3 \\ 0 \end{pmatrix}$ is in $\mathbb{R}^3$; $\begin{pmatrix} 1 \\ -1 \\ 2 \\ 7 \end{pmatrix}$ is in $\mathbb{R}^4$!

DEF: ① Two vectors in $\mathbb{R}^n$ are equal if and only if their corresponding entries are equal.

EXP: $\begin{pmatrix} 2 \\ 3 \\ 1 \\ 4 \end{pmatrix} \neq \begin{pmatrix} 2 \\ 3 \\ 0 \\ 4 \end{pmatrix}$

② The ZERO VECTOR is the vector whose entries are ALL zero. It is denoted by 0.

EXP: $0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ in $\mathbb{R}^4$; $\quad 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ in $\mathbb{R}^2$.

(3) $\boxed{\text{SUM}}$ Given 2 vectors in $\mathbb{R}^n$, their sum is the vector $u+v$ obtained by adding the corresponding entries:

$$u+v = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{pmatrix}$$

EXP: $\begin{pmatrix} 3 \\ -7 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 4 \\ -1 \\ 1 \end{pmatrix}$

(4) Scalar Multiplication

if c is a #, u in $\mathbb{R}^n$, $u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$, then the scalar multiple of u by c is the VECTOR obtained by multiplying each entry of u by c:

$$cu = c\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{pmatrix}$$

$\boxed{\text{EXP}}$ $(-2)\begin{pmatrix} 1/4 \\ 2 \\ 3.1 \\ 7 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -4 \\ -6.2 \\ -14 \\ 0 \end{pmatrix}$.

NOTATION:

$-v = (-1)v$

$\boxed{\text{THM}}$ (Algebraic Properties of $\mathbb{R}^n$) FOR ALL u, v, w in $\mathbb{R}^n$, and all c, d #s: (i) $u+v = v+u$; (ii) $(u+v)+(w) = u+(v+w)$; (iii) $u+0 = 0+u = u$; (iv) $u+(-u) = 0$

(v) $c(u+v) = cu + cv$; (vi) $(c+d)u = cu + du$;

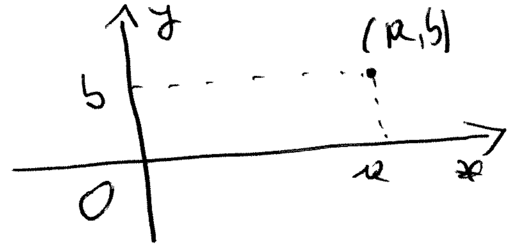
(vii) $(cd)u = c(du)$; (viii) $1 \cdot u = u$

PROVE one of them!!

We close:

GEOMETRIC DESCRIPTION of $\mathbb{R}^2$

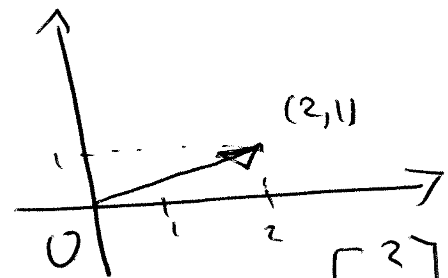
- Consider the plane : $\begin{cases} \text{ORIGIN} \\ \text{2 perpendicular axes} \\ \text{scale} \end{cases}$



Each point in the plane determines an ordered pair of #s.

So we identify a point (a,b) with the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbb{R}^2$.

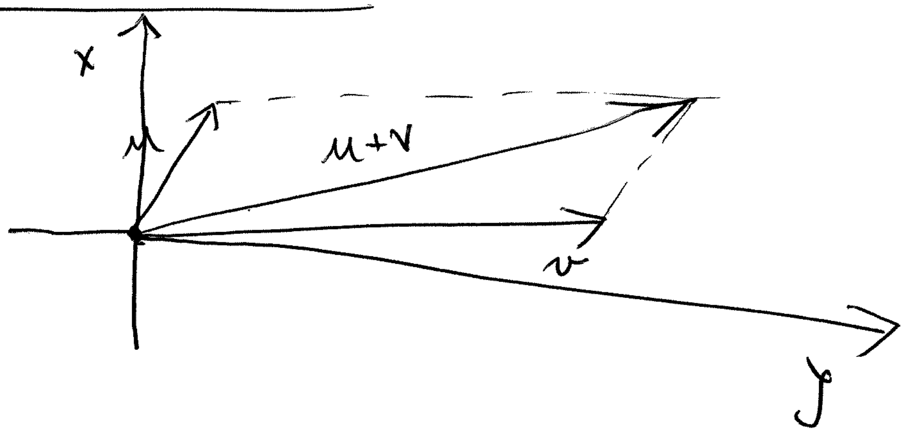
So $\mathbb{R}^2$ "is" the set of all points in the plane.



$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\boxed{\text{TO VISUALIZE: use an ARROW from the ORIGIN to our point}}$

- PARALLELOGRAM LAW:



— Lecture 4 — ~~THU~~ — FRI — 16th 2009

§ 1.3 Vector Equations

DEF: Given vectors $v_1, v_2, \ldots, v_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$, the vector $c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$ is called a linear combination of $v_1, v_2, \ldots, v_p$ with weights $c_1, c_2, \ldots, c_p$.

EXP: Given v_1, v_2, v_3, we may form: $3v_1 - \sqrt{3}v_2 + \frac{1}{2}v_3$;

$-v_1 + v_3$; $\quad v_2 + v_3$; $\quad 0 = 0v_1 + 0 \cdot v_2$;

EXC: Let $a_1 = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$, $a_2 = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}$; $b = \begin{pmatrix} -7 \\ -4 \\ 3 \end{pmatrix}$. Determine if b can be written as a linear combo of a_1 and a_2.

SOL: → Can we find x_1, x_2 #s such that:

$$x_1 a_1 + x_2 a_2 = b \, ?$$

→ $$x_1 \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} -7 \\ -4 \\ 3 \end{pmatrix}$$

→ $$\begin{pmatrix} -x_1 \\ 2x_1 \\ 5x_1 \end{pmatrix} + \begin{pmatrix} 2x_2 \\ 5x_2 \\ 6x_2 \end{pmatrix} = \begin{pmatrix} -7 \\ -4 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} -x_1 + 2x_2 \\ 2x_1 + 5x_2 \\ 5x_1 + 6x_2 \end{pmatrix} = \begin{pmatrix} -7 \\ -4 \\ 3 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -x_1 + 2x_2 = -7 \\ 2x_1 + 5x_2 = -4 \\ 5x_1 + 6x_2 = 3 \end{cases} \Rightarrow \left[\begin{array}{cc|c} -1 & 2 & -7 \\ 2 & 5 & -4 \\ 5 & 6 & 3 \end{array}\right] \xrightarrow[R_3 + 5R_1]{R_2 + 2R_1}$$

$$\left[\begin{array}{cc|c} -1 & 2 & -7 \\ 0 & 9 & -18 \\ 0 & 16 & -32 \end{array}\right] \xrightarrow{\frac{1}{16}R_3;\ \frac{1}{9}R_2} \left[\begin{array}{cc|c} -1 & 2 & -7 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{array}\right] \xrightarrow{R_3 - R_2}$$

$$\left(\begin{array}{cc|c} -1 & 2 & -7 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right)$$

consistent; moreover:

$$\begin{cases} x_2 = -2 \\ -x_1 = -7 - 2x_2 = -7 + 4 = -3; \quad x_1 = 3 \end{cases}$$

weights

What DID we learn? The answer was found by setting: $[a_1 \; a_2 \,|\, b]$; then Row Reduction $\boxed{\text{etc}}$

FACT A vector equation: $x_1 a_1 + x_2 a_2 + \dots + x_n a_n = b$

HAS THE SAME SOLUTION SET AS the linear system whose augmented mx is: $[a_1 \; a_2 \dots a_n \,|\, b]$.

In particular: b can be written (generated) by a linear combo of $a_1, a_2, \dots, a_n$ $\iff$ there is a solution of $[a_1 \; a_2 \dots a_n \,|\, b]$.

TO fix the ideas:

$\boxed{\text{DO } 14/38}$

$$\left[\begin{array}{ccc|c} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|c} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & 11 & -2 \end{array}\right]$$

3 PIVOT POS $\Rightarrow$ Unique sol $\Rightarrow$ Answer: $\boxed{\text{YES}}$

IF we want the weights: $11x_3 = -2 \Rightarrow x_3 = -2/11$; $3x_2 = -5 - 7x_2$

$\Rightarrow x_2 = -\frac{5}{3} - \frac{7}{3} \cdot \left(-\frac{2}{11}\right) = \frac{-55 + 14}{33} = -\frac{41}{33}$; $x_1 = 2x_2 + 6x_3 + 11$ etc

DO 18/38

$$\left[\begin{array}{cc|c} 1 & -3 & h \\ 0 & 1 & -5 \\ -2 & 8 & -3 \end{array}\right] \xrightarrow{R_3 + 2R_1} \left[\begin{array}{cc|c} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 2 & -3+2h \end{array}\right] \xrightarrow{R_3 - 2R_2}$$

$$\left[\begin{array}{cc|c} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 0 & 7+2h \end{array}\right]$$

We WANT: consistent. Hence: $7 + 2h = 0$

$\Downarrow$

$h = -7/2$

IMPT:
Def: if $v_1, v_2, \ldots, v_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $v_1, v_2, \ldots, v_p$ is denoted by Span $\{v_1, v_2, \ldots, v_p\}$ and is called: the subset of $\mathbb{R}^n$ spanned (or generated) by $v_1, v_2, \ldots, v_p$:

$$\text{Span}\{v_1, v_2, \ldots, v_p\} = \left\{ c_1 v_1 + c_2 v_2 + \cdots + c_p v_p \mid \underset{\#s}{c_1, \ldots, c_p} \right\}$$

FACT: Asking if a vector b is in Span $\{v_1, v_2, \ldots, v_p\}$

amounts to

Asking if the vector equation $x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b$ has a SOLUTION

and this is equivalent to

Asking if the linear system with augmented mx. $[v_1 \; v_2 \; \cdots \; v_p \mid b]$ has a solution

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DO 26/38 a)

$$\left(\begin{array}{ccc|c} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 3 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ -1 & 8 & 5 & 3 \\ 2 & 0 & 6 & 10 \end{array}\right)$$

$$\xrightarrow[R_3 - 2R_1]{R_2 + R_1} \left(\begin{array}{ccc|c} \cancel{1} & \cancel{-2} & \cancel{1} & 3 \\ \cancel{-1} & \cancel{8} & \cancel{5} & 3 \\ & & & \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 6 & 6 & 6 \\ 0 & 4 & 4 & 4 \end{array}\right) \cdots \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

YES

$x_3 = t$, t scalar, x_3 = free

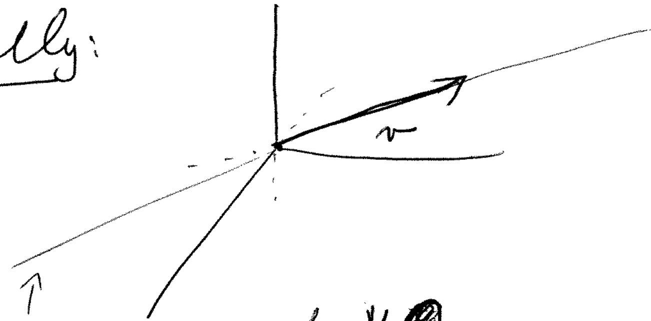
$x_2 = 1 - x_3 = 1 - t$

$x_1 = 2x_2 - x_3 + 3 = 2 - 2t - t + 3 = 5 - 3t$. Choose $t = 100$ to get some weights

A geometric description of Span$\{v\}$, Span$\{u,v\}$

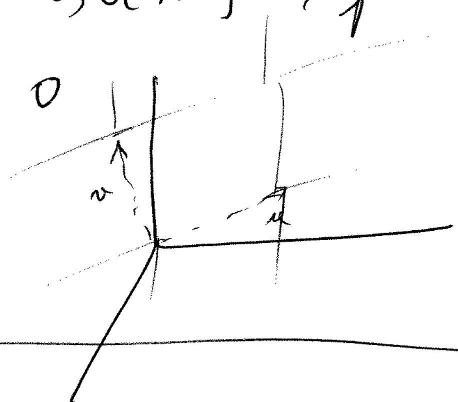
(1°) Span$\{v\} \stackrel{\Delta}{=} \{cv \mid c \in \mathbb{R}\}$ → all scalar multiples of v

Geometrically:



line through $v, 0$.

(2°) Span$\{u,v\} \stackrel{\Delta}{=} \{cu + dv \mid c, d \in \mathbb{R}\}$ → plane of $\mathbb{R}^3$ that contains $u, v, 0$



IF TIME 22/38 $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

6/37 ; 10/37

LONGER 11/38 ← IF EXTRA TIME

Lecture -5- TUE -

§ 1.4 THE MATRIX EQUATION $Ax=b$

Def A matrix is called a square mx. if it has the same # of Rows as the # of Columns

EXP: $\begin{pmatrix} -1 & 2 & 0 \\ 0 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}$

DEF: The size of a mx is: #Rows x #of Columns

EXP: $\begin{pmatrix} 3 & -1 & 2 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{pmatrix}$ 3x3 $\quad \begin{bmatrix} 2 & 1 & 3 & 2 \end{bmatrix}$ 1x4 $\quad \begin{bmatrix} 2 \\ 1 \\ 3 \\ 5 \\ 0 \end{bmatrix}$ 5x1

DEF: (MATRIX times VECTOR) If A is an $m \times n$ matrix, with columns $a_1, a_2, \dots, a_n$ and if x is in $\mathbb{R}^n$ then the product of A and x, denoted Ax, is the linear combination of Columns of A using the entries of x as weights:

$$Ax = [a_1 \; a_2 \; \dots \; a_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = a_1x_1 + a_2x_2 + \dots + a_nx_n$$
$$(\text{OR } x_1a_1 + x_2a_2 + \dots + x_na_n).$$

DO EXC 4,8/47 $\quad$ 8 → TRICK

DEF: A mx. equation is an equation of the form: $Ax=b$, where A is a mx, b is a vector, x – the unknown, a vector.

NOTE: ANY LINEAR SYSTEM can be written as a mx equation

Q: What is A? what is b? what is x?

$$\begin{cases} 8x_1 - x_2 = 4 \\ 5x_1 + 4x_2 = 1 \\ x_1 - 3x_2 = 2 \end{cases} \Rightarrow \begin{pmatrix} 8x_1 - x_2 = 4 \\ 5x_1 + 4x_2 = 1 \\ x_1 - 3x_2 = 2 \end{pmatrix} \Rightarrow x_1\begin{pmatrix} 8 \\ 5 \\ 1 \end{pmatrix} + x_2\begin{pmatrix} -1 \\ 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} 8 & -1 \\ 5 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

Answer: • A is the mx of coefficients of the SYSTEM.

• $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$; $b = \begin{pmatrix} \text{free} \\ \text{terms} \end{pmatrix}$

BASED on the def. above, on 1.3 we get:

THM/42) If A is an $m \times n$ mx, with columns $a_1, a_2, \dots, a_n$ and if b is in $\boxed{\mathbb{R}^m}$, the mx. equation: $Ax=b$ HAS the same solution set as the vector equation:

$$x_1 a_1 + x_2 a_2 + \dots + x_n a_n = b,$$

which, in turn, has the same solution set as the linear system whose augmented mx is $[a_1 \; a_2 \dots a_n \mid b]$.

SO: The equation $Ax=b$ has a solution if and only if b is a linear combination of the Columns of A

DO 14/48

$$\left(\begin{array}{ccc|c} 5 & 8 & 7 & 2 \\ 0 & 1 & -1 & -3 \\ 1 & 3 & 0 & 2 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 5 & 8 & 7 & 2 \end{array}\right) \xrightarrow{R_3-5R_1}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & -7 & 7 & -8 \end{array}\right) \xrightarrow{R_3+7R_2} \left(\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & -29 \end{array}\right)$$

NO SOL.
SO The answer is NOT in that SUBSET

THM/43 Let A be an $m \times n$ mx. Then the following are equivalent.

(1) FOR EACH b in $\mathbb{R}^m$, the eq. $Ax=b$ has a solution.

(2) EACH b in $\mathbb{R}^m$ is a linear combination of the columns of A.

(3) The columns of A span $\mathbb{R}^m$

!!! (4) A HAS A PIVOT position in every ROW

DO: 15/48 $\left(\begin{array}{cc|c} 2 & -1 & b_1 \\ -6 & 3 & b_2 \end{array}\right) \xrightarrow{R_2+3R_1} \left(\begin{array}{cc|c} 2 & -1 & b_1 \\ 0 & 0 & b_2+3b_1 \end{array}\right)$

NOTE: if $b_2+3b_1 \neq 0 \Rightarrow$ NO SOL; EXP $b_2=1, b_1=0$.

The set of vectors b for which one gets a solution:

$\left\{ b=\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \;\middle|\; b_2+3b_1=0 \right\}$.

DO: 21/48 USE THM/43

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3+R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_4+R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_4-R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

NO PIVOT POSITION in Row 4, so the answer IS: NO!

22/48 $\begin{pmatrix} 0 & 0 & 4 \\ 0 & -3 & -1 \\ -2 & 8 & -5 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} -2 & 8 & -5 \\ 0 & -3 & -1 \\ 0 & 0 & 4 \end{pmatrix} \xrightarrow[\frac{1}{-3}R_2]{\frac{1}{-2}R_1} \frac{1}{4}R_3$

$\begin{pmatrix} 1 & -4 & 5/4 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{pmatrix}$ Since we have a PIVOT position in every Row $\Rightarrow$ the answer is: YES

30/49 $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

29/49 $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

STOP

PROPERTIES: (of the product)

TH/45 If A is an $m \times n$ mx., u, v in $\mathbb{R}^n$, c scalar; then:

a) $A(u+v) = Au + Av$; b) $A(cu) = cAu$

MEMO: $\underset{[m \times n]\cdot[n \times 1]}{Ax} = b \leftarrow [m \times 1]$

24/48 T/F a) T b) T; c) T; d) T; e) F; f) T

– Lecture 6 – FRI –

PROPERTIES (of the PRODUCT)

THM/45 if A is an $m \times n$ mx., u, v in $\mathbb{R}^n$, c scalar,

then: a) $A(u+v) = Au + Av$; b) $A(cu) = c\,Au$.

MEMO: $A\,x = b \rightarrow [m \times 1]$

A: $[m \times n]$, x: $[n \times 1]$

24/48 T/F a) T, b) T, c) T, d) T, e) F; f) T.

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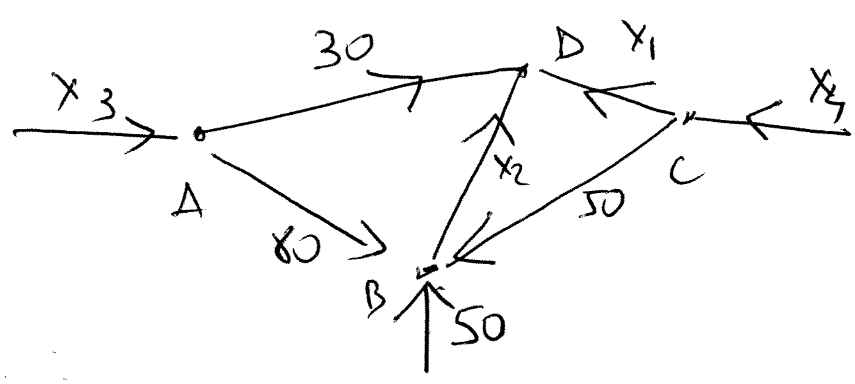
§ 1.6 Applications of LINEAR SYSTEMS

§ NETWORK FLOW

- Def. A network consists of a set of points called junctions (or NODES), with lines (called branches) connecting some of the junctions.
  - The Direction is indicated, AND the flow amount is shown or is denoted by a variable.



BASIC ASSUMPTIONS of network flow:

- TOTAL FLOW INTO the network = total flow out of network
- TOTAL flow into a junction = total flow out of the junction.

IN OUT

SO:

$$x_3 + x_4 + 50 = 0$$

A: $x_3 = 30 + 80$

B: $50 + 50 + 80 = x_2$

C: $x_4 = 50 + x_A$

D: $x_2 + 30 + x_4 = 0$

$\Rightarrow$ AUG MX. SOLVE.

DO: 11/64 ; 12/64

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§ Balancing Chemical Equations

Chemical equations describe chemical reactions:

$$x_1 C_3H_8 + x_2 O_2 \longrightarrow x_3 CO_2 + x_4 H_2O$$

(propane) oxy carbondioxide water

To "balance" we must find integers $x_1, \ldots, x_4$ s.t.

atoms C_L = # C_R ; # O_L = # O_R , # H_L = # H_R

(NO ATOMS are created or destroyed)

C: $3x_1 + 0 = x_3 + 0$

H: $8x_1 + 0 = 0 + 2x_4$

O: $0 + 2x_2 = 2x_3 + x_4$

$$\begin{cases} 3x_1 + 0 - x_3 = 0 \\ 8x_1 \qquad\qquad - 2x_4 = 0 \\ 0 + 2x_2 - 2x_3 - x_4 \end{cases}$$

$\boxed{\text{ROW}}$

$$\left[\begin{array}{cccc|c} 3 & 0 & -1 & 0 & 0 \\ 8 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right] \xrightarrow[\frac{1}{2}R_2]{} \left[\begin{array}{cccc|c} 3 & 0 & -1 & 0 & 0 \\ 4 & 0 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cccc|c} 3 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - 3R_2} \left[\begin{array}{cccc|c} 0 & 0 & -4 & 3 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & -4 & 3 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 0 & 0 & -4 & 3 & 0 \end{array}\right] \xrightarrow[-\frac{1}{4}R_3]{\frac{1}{2}R_2} \begin{array}{c} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{3}{4} & 0 \end{array}\right] \\ \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \end{array}$$

$x_4 = t$, t scalar free

$x_3 = \frac{3}{4}t$; $\quad x_2 = x_3 + \frac{1}{2}x_4 = \frac{3}{4}t + \frac{1}{2}t = \frac{5}{4}t$

$x_1 = -x_3 + x_4 = -\frac{3}{4}t + t = \frac{1}{4}t$

We need integers: $t = 4 \Rightarrow (1, 5, 3, 4)$

DO 5/63 $\qquad$ DO 7/63

- LECTURE - TUE - 27 -

1.5 SOLUTION SETS OF LINEAR SYSTEM

§ HOMOGENEOUS SYSTEMS

DEF: A system is called homogeneous if it can be written in the form: $Ax = 0$; A is an $m \times n$ $m x$; 0 - the zero vector in $\mathbb{R}^m$.

NOTE: → At least one solution: $X = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0$ in $\mathbb{R}^n$.

→ It is called: The Trivial solution.

→ so: consistent.

FACT: The homogeneous system $Ax = 0$ has a nontrivial solution if and only if there is at least one free variable

DO: 1/55 $\left(\begin{array}{ccc|c} 2 & -5 & 8 & 0 \\ -2 & -7 & 1 & 0 \\ 4 & 2 & 7 & 0 \end{array}\right) \xrightarrow{R_2 + R_1} \left(\begin{array}{ccc|c} 2 & -5 & 8 & 8 \\ 0 & -12 & 9 & 0 \\ 0 & +12 & -9 & 0 \end{array}\right)$

$\xrightarrow{R_3 + R_2} \left(\begin{array}{ccc|c} 2 & -5 & 8 & 8 \\ 0 & -12 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$. Since At least one free variable ⟹ we get a nontrivial solution (in fact so many)

§ PARAMETRIC VECTOR FORM (of solutions)

EXP: Solve: $\begin{cases} 10x_1 - 3x_2 + 2x_3 = 0 \\ 20x_1 - 6x_2 + 4x_3 = 0. \end{cases}$

SOL: $\left[\begin{array}{ccc|c} 10 & -3 & +2 & 0 \\ 20 & -6 & +4 & 0 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{ccc|c} 10 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{10}R_1} \left[\begin{array}{ccc|c} 1 & -\frac{3}{10} & \frac{2}{10} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_2 =$ free, $x_2 = s$, s scalar; $x_3 =$ free, $x_3 = t$, t scalar

$x_1 = \frac{3}{10}s + \frac{2}{10}t \Rightarrow X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{3}{10}s + \frac{2}{10}t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} 3/10 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2/10 \\ 0 \\ 1 \end{pmatrix}$

This is called: PARAMETRIC VECTOR FORM.

ONE MORE: 6/55 $\left(\begin{array}{ccc|c} 1 & 3 & -5 & 0 \\ 1 & 4 & -8 & 0 \\ -3 & -7 & 9 & 0 \end{array}\right) \xrightarrow[R_3+3R_1]{R_2-R_1} \left(\begin{array}{ccc|c} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & -6 & 0 \end{array}\right) \xrightarrow{R_3-2R_2}$

$\left(\begin{array}{ccc|c} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $x_3 =$ free; $x_3 = t$; t #; $x_2 = 3x_3 = 3t$.

$x_1 = -3x_2 + 5x_3 = -9t + 5t = -4t$ So:

$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -4t \\ 3t \\ t \end{pmatrix} = t \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}$; t scalar.

---

§ SOLUTIONS to NONHOMOGENEOUS SYSTEMS

(THM) Suppose the eq. $Ax=b$ has a solution (≥ 1), and let p be a solution. Then the ~~solution~~ set of $Ax=b$ is the set of all vectors of the form: $w = p + v_h$, where v_h is any solution of the homogeneous system $Ax=0$

IDEA: $Aw = A(p+v_h) = Ap + Av_h = b + 0 = b$

EXP. 18/55 $Ax=b$, $Ax=0$, where $A = [1 \ -3 \ 5]$ size 1×3;

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; 3×1; $b = [4]$; 1×1.

SOL: • A particular solution of $Ax=b$ is $\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$ = p:

$4\cdot 1 + 0\cdot(-3) + 5\cdot 0 = 4 = b$

• solve $Ax=0$: $[1 \ -3 \ 5 \,|\, 0]$ $x_2 = t$, $x_3 = s$; t, s scalar
$x_1 = 3t - 5s$.

$X = \begin{pmatrix} 3t-5s \\ t \\ s \end{pmatrix} = t\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$; t, s scalars.

• So: Solution set (of $Ax=b$): $\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$; t, s scalars

---

MID-TERM equations

1°) Find all solutions of the eq. $X + 5Y = 16$, where X, Y are integers greater than or equal than 0.

20) $4x-3y=24$, where $x \geq 0$ integer; $y \leq 0$ integer

22/55 SOL: $\Rightarrow$ $ax+by=c$ $\Rightarrow$ $\begin{cases} -6a+3b=c \\ -4b=c \end{cases}$ $\Rightarrow$ $\begin{cases} a=-\frac{1}{6}[c-3b \\ b=\frac{c}{-4} \end{cases}$

$\Rightarrow \begin{cases} a=\frac{-7}{24}c \\ b=-\frac{c}{4} \end{cases} \Rightarrow -\frac{7}{24}cx-\frac{1}{4}c\cdot y=c \Rightarrow y=\left[-\frac{7}{24}x-1\right]4$

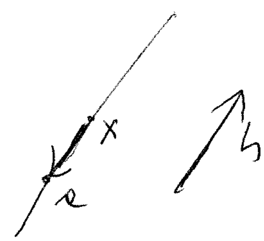
$\Rightarrow y=-4-\frac{7}{6}x \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -4-\frac{7}{6}x \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \end{bmatrix} + x\begin{bmatrix} 1 \\ -7/6 \end{bmatrix}$

x #.

20/55 Recall: the line through a and parallel to b is:

$X = a+tb$, t scalar.

Why?
A: $X-a$ is a multiple of b.



So: $X=\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} + t\begin{pmatrix} -7 \\ 8 \end{pmatrix}$ where t scalar

IF TIME: 36/56 $A=\begin{bmatrix} +1 & 1 & 1 \\ +1 & 1 & 1 \\ +1 & 1 & 1 \end{bmatrix}$ one column is enough to get the whole m.

31/56 a) No free var $\Rightarrow$ no non trivial sol
b) NO (2 piv in 3 Rows)

32/56 a) Free var $\rightarrow$ non triv sol
b) YES 2 piv pos in 2 Rows

34/56 if time

§1.7 Linear Independence

Def The set of vectors $\{v_1, v_2, ---, v_p\}$ in $\mathbb{R}^n$ is said to be: LINEARLY INDEPENDENT if (the vector eq.):
$x_1 v_1 + x_2 v_2 + --- + x_p v_p = 0$ has only the trivial solution.

The set of vectors $\{v_1, v_2, ---, v_p\}$ in $\mathbb{R}^n$ is said to be: LINEARLY DEPENDENT if there exist weights $c_1, c_2, --, c_p$ NOT ALL zero, such that $c_1 v_1 + c_2 v_2 + --- + c_p v_p = 0$

DO 1/1.7 We need to solve $x_1 \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 7 \\ 2 \\ -6 \end{pmatrix} + x_3 \begin{pmatrix} 9 \\ 4 \\ -8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

We get: $\begin{pmatrix} 5x_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 7x_2 \\ 2x_2 \\ -6x_2 \end{pmatrix} + \begin{pmatrix} 9x_3 \\ 4x_3 \\ -8x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 5x_1 + 7x_2 + 9x_3 \\ 2x_2 + 4x_3 \\ -6x_2 - 8x_3 \end{pmatrix}$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 5x_1 + 7x_2 + 9x_3 = 0 \\ 2x_2 + 4x_3 = 0 \\ -6x_2 - 8x_3 = 0 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 5 & 7 & 9 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & -6 & -8 & 0 \end{array}\right] \rightarrow$$

$$\xrightarrow{R_3 + 3R_2} \left[\begin{array}{ccc|c} 5 & 7 & 9 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 7/5 & 9/5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow$$

$x_3 = x_2 = x_1 = 0 \Rightarrow$ L.I.

NOTE: We used the columns = vectors IDEA!!

FACT: The columns of a mx. A are linearly IND. if and only if the eq. $AX = 0$ HAS ONLY the trivial solution.

$\longrightarrow$ EXC

DO 7/71 (Use the fact)

$$\left[\begin{array}{cccc|c} 1 & 4 & -3 & 0 & 0 \\ -2 & -7 & 5 & 1 & 0 \\ -4 & -5 & 7 & 5 & 0 \end{array}\right] \xrightarrow[R_3+4R_1]{R_2+2R_1} \left[\begin{array}{cccc|c} 1 & 4 & -3 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 11 & -5 & 5 & 0 \end{array}\right] \xrightarrow{R_3-11R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 4 & -3 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 6 & -6 & 0 \end{array}\right] \xrightarrow{\frac{1}{6}R_3} \left(\begin{array}{cccc|c} 1 & 4 & -3 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right)$$

1 free variable is OBTAINED! SO: LIN. DEP.

Let us find some weights that will create a lin. combo (of the columns) that gives the ZERO VECTOR:

$x_4 = t$, t #; $x_3 = t$, $x_2 = x_3 - x_4 = 0$; $x_1 = -4x_2 + 3x_3 = 3t$

So $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3t \\ 0 \\ t \\ t \end{pmatrix} = t \cdot \begin{pmatrix} 3 \\ 0 \\ 1 \\ 1 \end{pmatrix}$. Choose $t=1$; so $\begin{pmatrix} 3 \\ 0 \\ 1 \\ 1 \end{pmatrix}$. Hence

$$3 \cdot Col_1 + 0 \cdot Col_2 + 1 \cdot Col_3 + 1 \cdot Col_4 = 0$$

THM: Any set of vectors $\{v_1, v_2, \ldots, v_p\}$ in $\mathbb{R}^n$ is lin. Dependent if $p > n$.

$\boxed{\text{Another THM}}$ If a set of vectors $\{v_1, v_2, \ldots, v_p\}$ in $\mathbb{R}^n$ contains the ZERO VECTOR, the the set is lin. dependent.

Pf: SAY $v_1 = 0$; then $1 \cdot v_1 + 0 \cdot v_2 + \ldots + 0 \cdot v_p = 0$, and NOT all weights are zero; so LD.

DO: 10/71 b) $\left(\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ -5 & 10 & -3 & 0 \\ -3 & 6 & h & 0 \end{array}\right) \xrightarrow[R_3+3R_1]{R_2+5R_1} \left(\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & h+6 & 0 \end{array}\right)$

ANYWAY: AT least one free VAR: x_2. So h can be anything

DO: 12/71 $\left(\begin{array}{ccc|c} 2 & -6 & 8 & 0 \\ -4 & 7 & h & 0 \\ 1 & -3 & 4 & 0 \end{array}\right) \xrightarrow{R_1 \Leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ -4 & 7 & h & 0 \\ 2 & -6 & 8 & 0 \end{array}\right)$

$\xrightarrow{R_3-2R_1} \left(\begin{array}{ccc|c} 1 & -3 & 4 & 0 \\ -4 & 7 & h & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$ h can be anything (since already a free var.)

PARTICULAR CASE:

- A set of 2 vectors $\{v_1, v_2\}$ is lin. Dependent if AT least one of the vectors is a multiple of the other.
- The set of vectors $\{v_1, v_2\}$ is linearly IND. if neither of the vectors is a multiple of the other.

SOL: SAY $v_1 = 3v_2$; then $1 \cdot v_1 + (-3)v_2 = 0 \Rightarrow$ L[illegible]

In general: A set $S = \{v_1, v_2, \ldots, v_p\}$ of 2 or more vectors is linearly dependent if and only if:

At least one of the vectors is a linear combo of the others

NOTE: We DO NOT SAY: Every, we DO SAY: At least one.

$\boxed{\text{DO: } 28/71 \Rightarrow 5}$ $\boxed{27/71 \Rightarrow 5}$

~~35/72 ⇒ T~~ ~~37/72 ⇒ T~~

---

if TIME DO:

35/56 $A = \begin{pmatrix} | & | & | \\ | & | & | \\ | & | & | \end{pmatrix}$

31/56 a) No free var → no nontriv. sol
b) NO (2 piv. in 3 Rows)

32/56 a) Free var → non triv. sol
b) YES: 2 piv. in 2 Rows

34/56

TUE - FEB - 3 - 2009

CHAPTER 2 MATRIX ALGEBRA

§2.1 Matrix Operations

Def Let A be an $m \times n$ mx. The scalar entry in the ith Row and jth Column of A is denoted by a_{ij}; and is called the (i,j)-entry of A.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & - & - \\ - & - & - & - \\ a_{mn} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

NOTATION $A = (a_{ij})_{ij}$

Def: The Diagonal entries in an $m \times n$ mx $A = (a_{ij})$ are $a_{11}, a_{22}, \ldots, a_{..}, \ldots$ They form the main diagonal of A

EXP: $\underset{n \times n}{I_n} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ - & - & - & - & - \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$ - called the identity mx

$\underset{m \times n}{0} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$ - called the zero mx.

Def A mx. is called a square mx. if # Rows = # of Columns.

A diagonal mx. is a square mx. whose nondiagonal entries are zero: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 7 \end{pmatrix}$.

OPERATIONS

- **SUM** If A and B are $m \times n$ matrices, then the sum $A+B$ is the $m \times n$ mx whose each entry is the sum of the corresponding entries of A and B.

EXP: $\begin{pmatrix} 3 & 0 & 1 \\ -1 & 2 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 2 & 0 \\ 1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 5 & 2 \end{pmatrix}$

• SCALAR MULTIPLICATION If r is a scalar, A is an $m \times n$ mx., then the scalar multiple rA is the $m \times n$ mx obtained by multiplying each entry of A by r.

EXP: $(-2)\begin{pmatrix} 3 & 0 \\ 1 & 2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} -6 & 0 \\ -2 & -4 \\ 2 & -6 \end{pmatrix}$

NOTATION: $-A = (-1)A$

$\boxed{1/116 \quad B-2A}$

$\boxed{2/116 \quad 3C-E \text{ IS NOT Defined}}$

(THM) (Properties) Let A, B, C be matrices of the same size; r, s #s.

Then: a) $A+B = B+A$; b) $(A+B)+C = A+(B+C)$; c) $A+0=A$;

d) $r(A+B) = rA + rB$; e) $(r+s)A = rA + sA$; f) $(rs)A = r(sA)$

Pf: Do one: f) $LS = (rs)a_{ij} = rsa_{ij}$ ♡ $RS = r(s\,a_{ij}) = rsa_{ij}$

---

§§ MATRIX MULTIPLICATION

Def: Let A be an $m \times n$ mx; let B be an $n \times p$ mx, with columns $B = [b_1 \; b_2 \; \dots \; b_p]$. Then The product of $AB = A[b_1 \; b_2 \; \dots \; b_p] \overset{\Delta}{=} [Ab_1 \; Ab_2 \; \dots \; Ab_p]$. The size of AB is $m \times p$.

NOTE: EACH Column of AB is a linear Combo of the columns of A USING weights from the corresponding column of B (just read the ABOVE Def)

EXP: $\overset{A}{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 0 & 1 \end{bmatrix}} \cdot \overset{B}{\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$

$2 \times 4 \quad 4 \times 2$

$C_1 = A\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$; $C_2 = A\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

ROW times Column RULE:

Rule: The (i,j)-entry in the mx AB is the SUM of the products of corresponding entries from Row i of A and Column j of B:

Do: EXC 6/116 $AB = ?$

PROPERTIES of MULTIPLICATION: if A, B, C are matrices such that the indicated sums, products are defined, then:

1) $A(BC) = (AB)C$; 2) $A(B+C) = AB + AC$; 3) $(B+C)A = BA + CA$;
4) $r(AB) = (rA)\cdot B = A(rB)$; 5) $I_m \cdot A = A = A I_n$ if A has size $m \times n$.

WARNING: ① $AB \neq BA$ $\quad A = \begin{pmatrix} 1 & 9 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

② NO cancellation: $\boxed{AB = AC \not\Rightarrow B = C}$ ⇔ DO 10/116

③ $AB = O \not\Rightarrow A = O$ OR $B = O$: DO 12/116

$A = \begin{pmatrix} 3 & -6 \\ -1 & 2 \end{pmatrix}$; $\begin{pmatrix} 22 \\ 11 \end{pmatrix} = B$ ~~Done~~

§ POWERS of a square mx

Def If A is a square mx, say $n \times n$, $k > 0$ - a positive integer

Then: $A^k = \underbrace{A \cdot A \cdot A \cdots A}_{k \text{ times}}$

Convention: $A^0 = I_n$.

§ TRANSPOSE of a mx. Given a mx. A of size $m \times n$, the transpose of A is just the $n \times m$ mx, denoted A^T, whose columns are formed from the corresponding rows of A.

EXP: 1) if $A = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$, find A^3

2) if $A = \begin{pmatrix} 2 & 3 & -1 \\ \frac{1}{2} & 5 & \sqrt{2} \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ find A^T, B^T.

PROPERTIES (1) $(A^T)^T = A$, (2) $(A+B)^T = A^T + B^T$; (3) $(rA)^T = r(A^T)$; (4) $(AB)^T = B^T A^T$ (reverse order)

$\boxed{23/117}$ a) Suppose x is a solution; then $Ax = 0$. But then $CAx = I_n x \Rightarrow C(Ax) = x \Rightarrow C\,0 = x \Rightarrow \boxed{0 = x}$.

b) if more columns than rows $\Rightarrow$ ∞-ly many sol (because of free var.) ↯.

24/117 a) Since $AD = I_m \Rightarrow ADb = I_m b \Rightarrow A(Db) = b$.
So for any b in $\mathbb{R}^m$, the eq. $Ax = b$ has at least one solution: $\boxed{Db}$.

b) by Thm 4/43 (Sect 1.4): A has a pivot position in each ROW; So A can NOT have more Rows than Columns.

25/117 b) by 23, 24 $\Rightarrow$ # Rows = # of Columns.

a) $CAD = C(AD) = C I_m = C$
$\quad \|$
$(CA)D = I_n D = D$

IF time, DO mx. multiplication

TUE : 10 FEB 2009

2.1 §§ POWERS of a square mx.

Def: if A is a square mx, say $n \times n$, $k > 0$ --- a positive integer, then $A^k = \underbrace{A \cdot A \cdot \ldots \cdot A}_{k \text{ times}}$.

CONVENTION: $A^0 = I_n$

§§ TRANSPOSE of a mx.

Def: Given a mx. A of size $m \times n$, the transpose of A is the $n \times m$ mx., denoted A^T, whose columns are formed from the corresponding rows of A.

EXP: 1) if $A = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$, find A^3.

2) if $A = \begin{pmatrix} 2 & 3 & -1 \\ \frac{1}{2} & 5 & \sqrt{2} \end{pmatrix}$; $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ find A^T, B^T.

PROPERTIES: ① $(A^T)^T = A$; ② $(A \pm B)^T = A^T \pm B^T$;

③ $(rA)^T = r A^T$; ④ $(AB)^T = B^T A^T$. (reverse order)

ONE more: $\boxed{27/117}$ $u \cdot v^T = ?$

2.2 THE INVERSE OF A MATRIX

Recall: $5 \cdot 5^{-1} = 1 = 5^{-1} \cdot 5$;

Def: An $n \times n$ mx. A is said to be invertible if there is an $n \times n$ mx. C s.t. $CA = I_n = AC$.

Def: In this case, C is called an inverse of A.

Since: C is uniquely determined, this unique inverse is denoted by: A^{-1}; $A \cdot A^{-1} = I_n = A^{-1} \cdot A$.

Terminology: $\begin{cases} \text{NOT invertible} \leftrightarrow \text{singular} \\ \text{invertible} \leftrightarrow \text{nonsingular} \end{cases}$

THM (- a lot of examples ...) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. if $ad - bc \neq 0$, then A is invertible and $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. if $ad-bc$, then A is NOT invertible.

Def The quantity $ad-bc$ is called the determinant of A;
We write: $\det A = ad - bc$

EX: 2/12 6 $A = \begin{pmatrix} 3 & 2 \\ 7 & 4 \end{pmatrix}$ $\det A \neq 0 \to A^{-1}$ exists

EXC: $A = \begin{pmatrix} 1 & 2 \\ 7 & 14 \end{pmatrix}$; $\det A = 0 \Rightarrow$ A is NOT inv.

THM: (link: inverses and systems)
if A is an invertible mx. of size $n \times n$, then for each b in $\mathbb{R}^n$, the eq. $Ax = b$ has the unique solution: $x = A^{-1}b$

Imagine: $Ax = b \Rightarrow A^{-1}Ax = A^{-1}b \Rightarrow \boxed{x = A^{-1}b}$

DO 5/12 6 We need to solve: $\begin{cases} 8x_1 + 6x_2 = 2 \\ 5x_1 + 4x_2 = -1 \end{cases}$

SOL: The system is equivalent to $Ax = b$, where $A = \begin{pmatrix} 8 & 6 \\ 5 & 4 \end{pmatrix}$; $b = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$; $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. NOTE: $\det A = 8 \cdot 4 - 5 \cdot 6 = -6 \neq 0$. So A is invertible. Hence $x = A^{-1}b = \frac{1}{-6}\begin{pmatrix} 4 & -6 \\ -5 & 8 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} =$

$= -\frac{1}{6}\begin{pmatrix} 8 + 6 \\ -10 - 8 \end{pmatrix} = \begin{pmatrix} -7/3 \\ 3 \end{pmatrix}$. Hence $x_1 = -7/3$; $x_2 = 3$.

PROPERTIES of invertible matrices are recorded in the following:

THM (a) if A is invertible, then A^{-1} is invertible and $(A^{-1})^{-1} = A$.

(b) if A, B are $n \times n$ matrices, then AB is invertible, AND $(AB)^{-1} = B^{-1}A^{-1}$.

(c) if A is invertible, then A^T is invertible, and $(A^T)^{-1} = (A^{-1})^T$.

(b') if $A_1, A_2, \dots, A_m$ are invertible: $(A_1 A_2 \cdots A_m)^{-1} =$
$= A_m^{-1} A_{m-1}^{-1} \cdots A_2^{-1} \cdot A_1^{-1}$

§ ELEMENTARY MATRICES:

Def: An elementary matrix is one that is obtained by performing a SINGLE Row operation on an identity mx.

EXP: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{pmatrix}$: $R_3 - 5R_1$

$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$: $R_1 \Leftrightarrow R_3$; $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$: $(-5)R_3$.

NOTE: if a Row operation is performed on a mx. A (size $m \times n$), the resulting mx. can be written as: EA, where the $m \times m$ mx. E is created by performing the same Row op. on I_m.

EXP: Suppose $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ • Perform: $R_2 - 2R_1$. We get the mx. $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 7 & 8 & 9 \end{pmatrix}$; • NOW Create E: $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 - 2R_1}$

$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E$. DO: $E \cdot A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 7 & 8 & 9 \end{pmatrix}$

– match – !!

NOTE: 1) Each Elementary mx. is invertible.

2) ITS INVERSE is an elementary mx.

Pf: Know $I \xrightarrow{\text{Row Op}} E$; So $E \xrightarrow[\text{R.O}]{\text{Another}} I$. NOW apply the above note! Get F, elementary mx, st $F \cdot E = I$

(THM!) An $n \times n$ mx. A is invertible IF AND ONLY IF A is Row equivalent to I_n

From its proof one has an algorithm for finding A^{-1}!

- construct $[A|I]$
- ROW reduce, and

if $[A|I]$ is Row equivalent to $[I|M]$; then $M = A^{-1}$, and A is inv

if $[A|I]$ is NOT Row eq. to $[I|\cdots]$ then A is NOT invertible

32/127 $\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3+2R_1]{R_2-4R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{array}\right]$

$\xrightarrow{R_3-2R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & x & x & x \end{array}\right] \Rightarrow$ NOT inv.

31/127 $\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2+3R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_3+3R_2}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{array}\right] \xrightarrow[R_1+2R_3]{R_2+2R_3}$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{array}\right]$, with the left block I_3. Hence A is inv., $A^{-1} = \begin{pmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{pmatrix}$

Do: 21/126 $A = \text{inv} \Rightarrow$ Columns are L.I.

SOL: $x_1 \, Col_1 + x_2 \, Col_2 + \dots + x_n \, Col_n = 0 \Rightarrow Ax = 0$; $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$,

$\Rightarrow A^{-1}Ax = A^{-1}0 \Rightarrow x = 0 \Rightarrow x_1, \dots, x_n = 0 \Rightarrow$ L.I.

14 | 126 $(B-C)D=0 \Rightarrow (B-C)\underbrace{DD^{-1}}_{I_n}=0D^{-1} \Rightarrow B-C=0 \Rightarrow \underline{B=C}$

19 | 126 $C^{-1}(A+X)B^{-1}=I_n \Rightarrow \underbrace{CC^{-1}}_{I_n}(A+X)B^{-1}=\underbrace{CI_n}_{C}$

$(A+X)B^{-1}=C \Rightarrow (A+X)\underbrace{B^{-1}B}_{I}=CB \Rightarrow A+X=CB \Rightarrow X=CB-A$

18 / 126 $A=PBP^{-1} \Rightarrow P^{-1}A=\underbrace{P^{-1}P}_{I}BP^{-1} \Rightarrow P^{-1}A=BP^{-1} \Rightarrow$

$P^{-1}AP=B\underbrace{P^{-1}P}_{I} \Rightarrow \boxed{P^{-1}AP=B}$

Recall the PLAN

FRI - 13TH OF FEB - 1302 -

DO: 21/126 A = inv. $\Rightarrow$ Columns of A are L.I.

SOL: $x_1 \cdot C_1 + x_2 C_2 + \cdots + x_n \cdot C_n = 0$ (where $A = [C_1\ C_2 \cdots C_n]$

Then $Ax = 0$, where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. It follows: $A^{-1}Ax = A^{-1}0 \Rightarrow$

$I_n x = 0 \Rightarrow x = 0 \Rightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow$ $\boxed{x_1 = 0, \ldots, x_n = 0}$; so L.I.

DO: 14/126:

$(B-C)D = 0 \Rightarrow (B-C)DD^{-1} = 0D^{-1} \Rightarrow (B-C)I = 0$

$\Rightarrow B - C = 0 \Rightarrow \boxed{B = C}$

DO 19/126: $C^{-1}(A+X)B^{-1} = I_n \Rightarrow CC^{-1}(A+X)B^{-1} = CI_n$

$\Rightarrow I_n(A+X)B^{-1} = C \Rightarrow I_n(A+X)B^{-1}B = CB \Rightarrow$

$(A+X)I_n = CB \Rightarrow A + X = CB \Rightarrow X = CB - A$

DO 18/126: $A = PBP^{-1} \Rightarrow P^{-1}A = P^{-1}PBP^{-1} \Rightarrow P^{-1}A = IBP^{-1}$

$\Rightarrow P^{-1}A = BP^{-1} \Rightarrow P^{-1}AP = BP^{-1}P \Rightarrow P^{-1}AP = BI \Rightarrow$

$\boxed{P^{-1}AP = B}$

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§ 2.3 Characterizations OF INVERTIBLE MATRICES

THM /129 (The Invertible Matrix Theorem)

Let A be an $n \times n$ mx. The following are equivalent

1. A is invertible
2. A is ROW Equivalent to I_n
3. A has n pivot positions
4. The eq. $Ax = 0$ has only the trivial solution.
5. The columns of A are Lin. independent
6. The eq. $Ax = b$ has at least one solution for each b in $\mathbb{R}^n$

(7) The Columns of A span $\mathbb{R}^n$.

(8) There is an $n\times n$ mx. C such that $CA=I_n$

(9) There is an $n\times n$ mx. D such that $AD=I_n$

(10) A^T is invertible

$\boxed{\text{FACT}}$: Let A, B be 2 square matrices. If $AB=I_n$, then A and B are invertible and $A^{-1}=B$ $\boxed{\text{AND}}$ $A=B^{-1}$

Do: 8/132 Q: HOW MANY PIV. POSITIONS?

A: 4; so INV; (Thm (a) $\longleftrightarrow$ (c))

7/132

$$A \xrightarrow[R_3-2R_1]{R_2+3R_1} \begin{bmatrix} -1 & 3 & 0 & 1 \\ 0 & -4 & 8 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix} \xrightarrow[\frac{1}{3}R_3]{R_2 \Leftrightarrow R_4} \begin{bmatrix} -1 & 3 & 0 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -4 & 8 & 0 \end{bmatrix}$$

$$\xrightarrow[(-1)R_1]{R_4-4R_2} \begin{bmatrix} 1 & -3 & 0 & -1 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \Rightarrow \quad 4 \text{ PIV. POS} \Rightarrow \underline{\text{INV}}$$

17/133 Since A is inv. $\Rightarrow$ A^{-1} is inv $\Rightarrow$ its Columns are L.I. (Thm.)

19/133 The eq. $Ax=b$ has AT least one sol.

24/133 $\boxed{\text{NOT}}$ $\rightarrow$ we need "ONLY the trivial sol."

$\rightarrow$ Think about $L=O$ $(n\times n)$

28/133 Since AB is inv. $\Rightarrow \exists\, C$ s.t. $C(AB)=I_n \Rightarrow (CA)B=I_n$

FACT $\Rightarrow$ B is inv. (You may use Thm.)

$\boxed{21/133}$ Suppose $x_1 \neq x_2$ are 2 sol: $Gx_1 = y$, $Gx_2 = y$

So $Gx_1 - Gx_2 = y - y \Rightarrow G(x_1 - x_2) = 0$.

So the eq. $Gx = 0$ has a nontrivial sol. $\boxed{x_1 - x_2} \neq 0$

So by Thm $\rightarrow$ $\boxed{NO}$

---

if time $\begin{cases} 20/133 \\ 18/133 \end{cases}$ 26/133

2.6 The Leontief INPUT-OUTPUT MODEL

- Suppose that a nation's economy is divided in n sectors that produce goods or services. Let x be a production vector in $\mathbb{R}^n$ that lists the output of each sector for one year.
- Suppose that another part of the economy (the open sector) DOES NOT produce goods or services, but only consumes them, let d be a FINAL DEMAND VECTOR that lists the value of goods and services demanded from the sectors by the non-productive part of economy.
- As the sectors produce goods/services to meet consumer demand, the producers create additional intermediate DEMAND for goods they need as inputs for their own production.

$\boxed{\text{W. Leontief}}$ asked: is there a production level x such that the amounts produced will EXACTLY match (balance) the total demand?

$$\underbrace{\left\{\begin{array}{c}\text{AMOUNT}\\ \text{Produced}\end{array}\right\}}_{x} = \left\{\begin{array}{c}\text{intermediate}\\ \text{demand}\end{array}\right\} + \underbrace{\left\{\begin{array}{c}\text{FINAL}\\ \text{Demand}\end{array}\right\}}_{d}$$

We need an Assumption: FOR EACH SECTOR there is a UNIT CONSUMPTION VECTOR in $\mathbb{R}^n$ that lists the inputs needed per unit of output of the sector.

EXP: $\Rightarrow$ SAY one economy is made up of 3 sectors:
M $\to$ manufacturing, A $\to$ agriculture, S $\to$ services.
$\Rightarrow$ SAY M, A, S have the following vectors as unit consumption vectors:

$$c_1 = \begin{pmatrix} 0.5 \\ 0.2 \\ 0.1 \end{pmatrix} ; \quad c_2 = \begin{pmatrix} 0.4 \\ 0.3 \\ 0.1 \end{pmatrix} ; \quad c_3 = \begin{pmatrix} 0.2 \\ 0.1 \\ 0.3 \end{pmatrix}$$

NOTE:

$$\begin{array}{c c} & \begin{array}{ccc} M & A & S \end{array} \\ \begin{array}{c} M \\ A \\ S \end{array} & \begin{pmatrix} 0.5 & 0.4 & 0.2 \\ 0.2 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.3 \end{pmatrix} \end{array}$$

Meaning: To get 1 unit of OUTPUT, A needs
0.4 units from M
0.3 units from A
0.1 units from S.
etc

So, to get 100 units, M needs

$$100 \cdot c_1 = 100 \begin{pmatrix} 0.5 \\ 0.2 \\ 0.1 \end{pmatrix} = \begin{pmatrix} 50 \\ 20 \\ 10 \end{pmatrix}, \text{ i.e. } \begin{cases} 50 \text{ units from M} \\ 20 \text{ units from A} \\ 10 \text{ units from S.} \end{cases}$$

$\Rightarrow$ if M decides to produce x_1 units of OUTPUT, then $x_1 \cdot c_1$ represents the intermediate demand of M, because the amounts $x_1 \cdot c_1$ will be consumed in the process of creating x_1 units of output.

LIKEWISE: if A plans to produce x_2 outputs, S plans to produce x_3 outputs, then $x_2 c_2$, $x_3 c_3$ list their corresponding intermediate demands.

Hence: INTERMEDIATE DEMAND $= x_1 c_1 + x_2 c_2 + x_3 c_3 \overset{!!}{=}$ Cx, where C is the "consumption mx",

$$C = [c_1 \quad c_2 \quad c_3] = \begin{pmatrix} 0.5 & 0.4 & 0.2 \\ 0.2 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.3 \end{pmatrix}.$$

Def The Leontief input-output model is:

amount produced $\rightarrow$ $x = \underbrace{Cx}_{\text{Int. Demand}} + d \leftarrow$ final Demand

So: $x - Cx = d$, $Ix - Cx = d$, $(I-C)x = d$

Q: when can we find x?

A: Thm 1/154 if C, d have non-negative entries, and if each column sums of C is less than 1, then $(I-C)^{-1}$ exists, and $x = (I-C)^{-1}d$.

EXC: 1/156

$$\begin{array}{c|ccc} & M & A & S \\ \hline M & 0.1 & 0.6 & 0.6 \\ A & 0.3 & 0.2 & 0 \\ S & 0.3 & 0.1 & 0.1 \end{array}$$

ANS: $100 \cdot c_2 = 100 \begin{pmatrix} 0.6 \\ 0.2 \\ 0.1 \end{pmatrix} = \begin{pmatrix} 60 \\ 20 \\ 10 \end{pmatrix}$

2/156 $d = \begin{pmatrix} 0 \\ 18 \\ 0 \end{pmatrix} \begin{matrix} M \\ A \\ S \end{matrix}$

Sol: $I_3 - C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.1 & 0.6 & 0.6 \\ 0.3 & 0.2 & 0 \\ 0.3 & 0.1 & 0.1 \end{pmatrix} \Rightarrow$

$$\Rightarrow \left(\begin{array}{ccc|c} 0.9 & -0.6 & -0.6 & 0 \\ -0.3 & 0.8 & 0 & 18 \\ -0.3 & -0.1 & 0.9 & 0 \end{array}\right) \xrightarrow[R_1+3R_2]{R_3-R_2}$$

$$\left(\begin{array}{ccc|c} 0 & 1.8 & -0.6 & 48 \\ -0.3 & 0.8 & 0 & 18 \\ 0 & -0.9 & 0.9 & -18 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2}$$

$$\left(\begin{array}{ccc|c} -0.3 & 0.8 & 0 & 18 \\ 0 & 1.8 & -0.6 & 48 \\ 0 & -0.9 & 0.9 & -18 \end{array}\right) \xrightarrow{R_2+2R_3}$$

$$\left(\begin{array}{ccc|c} -0.3 & 0.8 & 0 & 18 \\ 0 & 0 & 1.2 & 12 \\ 0 & -0.9 & 0.9 & -18 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left(\begin{array}{ccc|c} -0.3 & 0.8 & 0 & 18 \\ 0 & -0.9 & 0.9 & -18 \\ 0 & 0 & 1.2 & 12 \end{array}\right) \Rightarrow$$

$x_3 = \frac{12}{1.2} = 10$; $x_2 = 30$

$x_1 = 20$

Different EXC:

5/156 $\quad x = Cx + d \Rightarrow (I-C)x = d$

$$I - C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0.5 \\ 0.6 & 0.2 \end{pmatrix} = \begin{pmatrix} 1 & -0.5 \\ -0.6 & 0.8 \end{pmatrix} \Rightarrow$$

$$(I-C)^{-1} = \frac{1}{1\cdot(0.8) - (0.5)\cdot(0.6)} \begin{pmatrix} 0.8 & 0.5 \\ 0.6 & 1 \end{pmatrix}$$

$$= \frac{1}{0.8 - 0.3} \begin{pmatrix} 0.8 & 0.5 \\ 0.6 & 1 \end{pmatrix} = \frac{1}{0.5} \begin{pmatrix} 0.8 & 0.5 \\ 0.6 & 1 \end{pmatrix}$$

$$\Rightarrow x = \frac{1}{0.5} \begin{pmatrix} 0.8 & 0.5 \\ 0.6 & 1 \end{pmatrix} \begin{pmatrix} 50 \\ 30 \end{pmatrix}$$

$$= \frac{1}{\frac{1}{2}} \begin{pmatrix} 40 + 15 \\ 30 + 30 \end{pmatrix} = 2 \begin{pmatrix} 55 \\ 60 \end{pmatrix} = \begin{pmatrix} 110 \\ 120 \end{pmatrix}$$

Done

TRY: 6/156 ; 7 a, or b /157

4, 3/156

6, 7 a,b /157

8/173

25, 26/174

18/174

If time 13, 17/174

- FRI - 2009 - LEC

§2.8 SUBSPACES OF $\mathbb{R}^n$

A subspace of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has 3 properties:

1. The zero vector is in H;
2. if u, v are in H, then $u+v$ is in H;
3. if u is in H, if c is a scalar, then cu is in H.

i.e., CLOSED Under $+, \cdot$

EXP: $\mathbb{R}^n$ itself ; $\{0\}$.

EXP: if $v_1, v_2, ---, v_p$ are in $\mathbb{R}^n$, then the set span$\{v_1, v_2, ---, v_p\}$ is a subspace.

Proof: • $0 = 0 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_p$ is in span$\{v_1, ---, v_p\}$.

• Suppose $u = \alpha_1 v_1 + \cdots + \alpha_p v_p$ and $v = \beta_1 v_1 + \cdots + \beta_p v_p$. Then $u + v = (\alpha_1 + \beta_1) v_1 + \cdots + (\alpha_p + \beta_p) v_p$ is in span$\{v_1, ---, v_p\}$

• If c is a #, then $cu = c(\alpha_1 v_1 + \cdots + \alpha_p v_p) = (c\alpha_1) v_1 + \cdots + (c\alpha_p) v_p$ is in span$\{v_1, ---, v_p\}$.

Def: The column space of a $m \times n$ A is the set col A of all linear Combinations of the columns of A.

So, if A has size $m \times n$, then col A is a SUBSPACE of $\mathbb{R}^m$.

DO: 7/173 a) 3 ; b) ∞-ly many ; c) $\begin{pmatrix} 2 & -3 & -4 & | & 6 \\ -8 & 8 & 6 & | & -10 \\ 6 & -7 & -7 & | & 11 \end{pmatrix} \xrightarrow[R_3 - 3R_1]{R_2 + 4R_1}$

$$\begin{pmatrix} 2 & -3 & -4 & | & 6 \\ 0 & -4 & -10 & | & 14 \\ 0 & 2 & 5 & | & -7 \end{pmatrix} \xrightarrow{R_3 + \frac{1}{2} R_2} \begin{pmatrix} 2 & -3 & -4 & | & 6 \\ 0 & -4 & -10 & | & 14 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

So:

∞-ly many solutions (X_3 is free!) Hence (since At least one solution) p is in Col A!!!

DEF The null space of a mx. A is the set Nul A of ALL solutions of the eq. $Ax=0$:

$$\text{Nul}(A) = \{ X \mid AX = 0 \}.$$

NOTE: if A is an $m \times n$ $\Rightarrow$ $\text{nul}\, A \subseteq \mathbb{R}^n$.

THM: The null space of an $m \times n$ mx. is a subspace of $\mathbb{R}^n$.

Pf: • $A \cdot 0 = 0$; •• $\underbrace{Ax + Ay}_{A(x+y)} = 0 + 0 = 0$

• $Acx = cAx = c0 = 0$

BASIS FOR A SUBSPACE

DEF: A basis for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H.

EXP: $H = \mathbb{R}^n$; $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$. Then $\{e_1, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$.

FINDING A BASIS FOR THE NULL SPACE

- SOLVE The Equation $Ax = 0$.
- WRITE the solution in parametric vector form.
- GET the basis

23/174

$$\begin{array}{c} \left(\begin{array}{cccc|c} 1 & 2 & 6 & -5 & 0 \\ 0 & 1 & 5 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \\ \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \end{array}$$

$x_3 = t$, x_3 is free

$x_4 = s$, x_4 is free

$\boxed{t, s \text{ are scalars}}$

$x_2 = -5t + 6s$; $x_1 = -2(-5t+6s) - 6t + 5s = 4t - 7s \Rightarrow$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4t-7s \\ -5t+6s \\ t \\ s \end{pmatrix} = t\begin{pmatrix} 4 \\ -5 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -7 \\ 6 \\ 0 \\ 1 \end{pmatrix}; \quad t, s \text{ #s.}$$

GET the basis: $\left\{ \begin{pmatrix} 4 \\ -5 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} -7 \\ 6 \\ 0 \\ 1 \end{pmatrix} \right\}$.

FINDING A BASIS FOR THE COL space

- ROW REDUCE A
- CIRCLE the pivot positions
- GO BACK to A (= ORIGINAL mx.) and pick-up the pivot columns of A. They form a basis of Col A.

$\boxed{\text{DO: } 23/174:}$ $A \longrightarrow \begin{pmatrix} 1 & 2 & 6 & -5 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. So Column 1, Column 2 form a basis of Col A

$\left\{ \begin{pmatrix} 4 \\ 6 \\ 3 \end{pmatrix}; \begin{pmatrix} 5 \\ 5 \\ 4 \end{pmatrix} \right\}$.

DO: 28/174 $\quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; $\quad b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$;

SOL: Just imagine: $[A|b] = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$ has No SOL.

DO: 29/174:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

§2.9 Dimension and Rank

it can be shown that if a subspace H has a basis of p vectors, then every basis of H MUST HAVE exactly p vectors. So it makes sense:

DEF: The dimension of a non-zero subspace H, denoted dim H, is the # of vectors in any basis of H.

Convention: The dimension of the zero-space $\{0\}$ is 0.

EXP: $\mathbb{R}^n$ has dimension n. (Recall 2.8)

DEF: The RANK of A, denoted rank (A), is ~~the~~ the dimension of the Column space of A.

So: (1) Since the pivot columns of A form a basis for Col A, the rank A is just the # of pivot columns of A

---

(2) Recall HOW DO we get a basis for Nul(A):

TO EACH VECTOR in the basis it corresponds a free variable.

HENCE:
THM (Rank thm) if a m×n A has n columns, then

$$\text{rank } A + \dim(\text{Nul } A) = n$$

DO now 11/181 : Check the thm!

Lecture TUE 3rd of MARCH (POST ASSIG#

§2.9 DIMENSION AND RANK

it can be shown that if a subspace H has a basis of p vectors, then every basis of H MUST have exactly p vectors!! So it makes sense:

DEF: The dimension of a nonzero subspace H, denoted dim H, is the # of vectors in any basis of H.

Convention: $\mathbb{R}^n$ has dimension n (Recall 2.8).

DEF: The rank of A, denoted rank (A), is the dimension of the Column space of A.

So: (1) Since the pivot columns of A form a basis for Col A, the rank A is just the # of Pivot Columns of A.

(2) Recall: How Do we get a basis for Null A?

TO EACH VECTOR in the basis it corresponds a free variable.

Hence:

THM (RANK Thm) if a m x. A has n columns, then

rank A + dim (null A) = n.

Do NOW: 11/181 (i.e., check the theorem...)

Then DO: 14/181 → Col A → basis ----

THMS (BASIS Thm) Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. • Also, any set of p elements of H that spans H is automatically a basis for H.

DO 1/243 → Check: LI, use Thm → BASIS

Do: 5/243; NOT: $\mathbb{R}^3 \to 3$, $\to 4$, $3 \neq 4$. You may Do: 6!!

THM (The Invertible Mx. Theorem) Let A be an $\boxed{n \times n}$ mx. TFAE:

(1) A is invertible

(2) The Columns of A form a basis of $\mathbb{R}^n$

(3) $\dim \operatorname{Col} A = n$

(4) $\operatorname{Col} A = \mathbb{R}^n$

(5) $\operatorname{rank} A = n$

(6) $\operatorname{nul} A = \{0\}$

(7) $\dim (\operatorname{Nul} A) = 0$.

DO 15/181; 16/181

20/182 rank $= 5 - 3 = 2$

21/182 $\dim \operatorname{nul} A = 6 - 4 = 2$

DO 13/234 + 11/234 + 17/235

§ (2.9) COORDINATES

DEF: Let $B = \{b_1, b_2, \ldots, b_p\}$ be a basis of the subspace H. For each x in H, the coordinates of x relative to the basis B are the weights $c_1, c_2, \ldots, c_p$ such that $x = c_1 b_1 + \cdots + c_p b_p$. The vector in $\mathbb{R}^p$:

$$[x]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

is called the COORDINATE VECTOR of x (relative to B).

2/180 $X = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$

6/180 $\left(\begin{array}{cc|c} -3 & 7 & 11 \\ 1 & 5 & 0 \\ -4 & -6 & 7 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 5 & 0 \\ -3 & 7 & 11 \\ -4 & -6 & 7 \end{array}\right)$

$\xrightarrow[R_3+4R_1]{R_2+3R_1} \left(\begin{array}{cc|c} 1 & 5 & 0 \\ 0 & 22 & 11 \\ 0 & 14 & 7 \end{array}\right) \xrightarrow[\frac{1}{7}R_3]{\frac{1}{11}R_2} \left(\begin{array}{cc|c} 1 & 5 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{array}\right) \xrightarrow{R_3-R} $

$\left(\begin{array}{cc|c} 1 & 5 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{array}\right)$ $b_2 = \frac{1}{2}$; $b_1 = -5b_2 = -5/2$

---

iF time $\left\langle \begin{array}{l} 11/243 \\ 4/180 \end{array} \right.$

24/182

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

T/F

17/181

c, d, e

18/182 c, d,