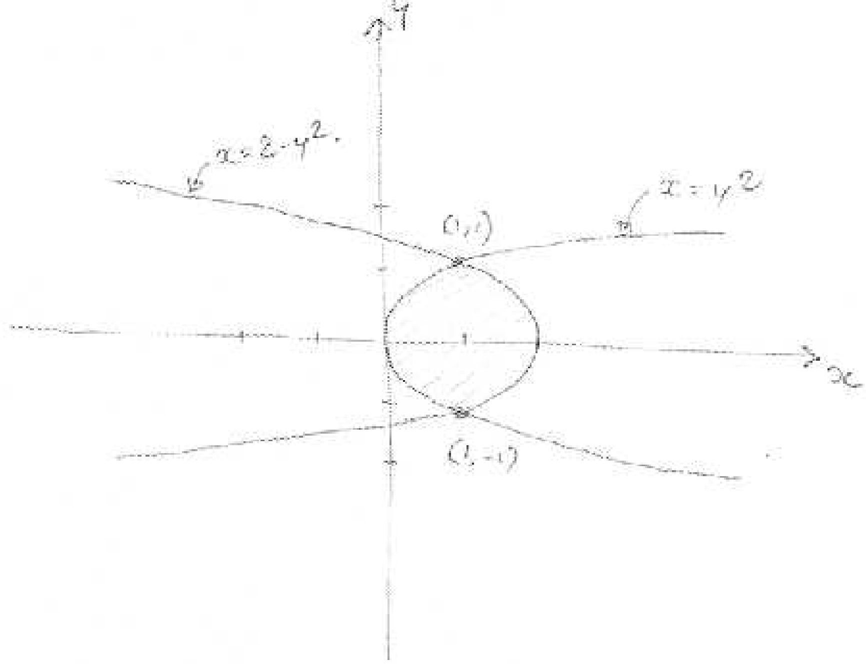
MAT 1322, Summer 2007.

HMW # 2

SOLUTIONS

① (i) area between $\begin{cases} y^2 = x \\ x = 2 - y^2 \end{cases}$

→ the curves are functions of y



Intersection points : $y^2 = 2 - y^2$

$\Leftrightarrow 2y^2 = 2$

$\Leftrightarrow y^2 = 1$

$\Leftrightarrow y = 1$ or $y = -1$

So the curves intersect at $(1,1)$ and $(1,-1)$

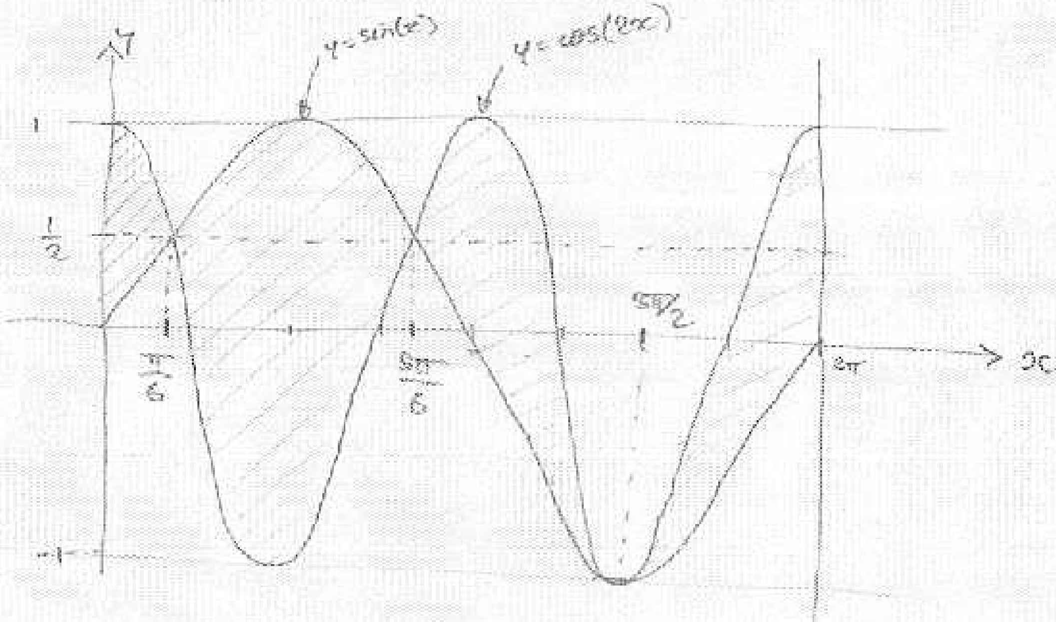
When $y \in [-1,1]$, $2 - y^2 \geq y^2$

So the area between the 2 curves is

$$\int_{-1}^{1} \left((2-y^2) - y^2\right) dy = \int_{-1}^{1} (2 - 2y^2)\, dy$$

$$= \left[2y - \frac{2y^3}{3} \right]_{-1}^{1} = \left(2 - \frac{2}{3}\right) - \left(-2 + \frac{2}{3}\right) = 4 - \frac{4}{3} = \boxed{\frac{8}{3}}$$

(ii) $\begin{cases} y = \cos(2x) \\ y = \sin(x) \end{cases}$ $\quad x \in [0, 2\pi]$



The intersection points are when $\cos(2x) = \sin x$ $\quad (*)$

$\cos(2x) = 1 - 2\sin^2 x$, thus $(*)$ becomes

$$1 - 2\sin^2 x = \sin x$$

$$2\sin^2 x + \sin x - 1 = 0$$

Since $2X^2 + X - 1 = (X+1)(2X-1)$, $(*)$ becomes

$$(\sin x + 1)(2\sin x - 1) = 0$$

ie $\sin x = -1$ or $\sin x = \frac{1}{2}$.

So the intersection points are $x = \frac{3\pi}{2}$ (when $\sin x = -1$)

and $x = \frac{\pi}{6}, \frac{5\pi}{6}$ (when $\sin x = \frac{1}{2}$)

Now, between $x=0$ and $x=\frac{\pi}{6}$, $\cos 2x \geq \sin x$

$x=\frac{\pi}{6}$ and $x=\frac{5\pi}{6}$, $\sin x \; x \geq \cos 2x$

$x=\frac{5\pi}{6}$ and $x=\frac{3\pi}{2}$, $\cos 2x \geq \sin x$

$x=\frac{3\pi}{2}$ and $x = 2\pi$, $\cos 2x \geq \sin x$.

Thus the area is

$$\int_0^{\pi/6} (\cos 2x - \sin x)\,dx + \int_{\frac{\pi}{6}}^{5\pi/6} (\sin x - \cos 2x)\,dx + \int_{\frac{5\pi}{6}}^{2\pi} (\cos 2x - \sin x)\,dx$$

$$= \left[\tfrac{1}{2}\sin 2x + \cos x\right]_0^{\pi/6} + \left[-\cos x - \tfrac{1}{2}\sin 2x\right]_{\frac{\pi}{6}}^{5\pi/6} + \left[\tfrac{1}{2}\sin 2x + \cos x\right]_{\frac{5\pi}{6}}^{2\pi}$$

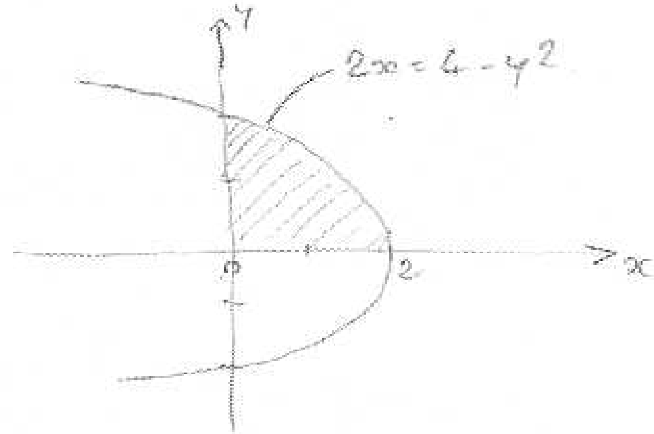
$$= \left(\tfrac{3}{4}\sqrt{3} - 1\right) + \left(\tfrac{3}{2}\sqrt{3}\right) + \left(\tfrac{3}{4}\sqrt{3} + 1\right)$$

$$= \boxed{3\sqrt{3}}$$

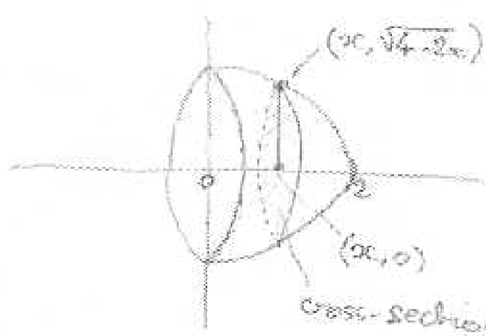
(2) Solid of revolution generated by

$$\begin{cases} 2x = 4 - y^2 \\ x = 0,\ y = 0 \end{cases}$$

(1) rotated around the x-axis



rotated around the x-axis: we take vertical cross-sections. The crossections form discs, and the one at x looks like this



Thus the disc has radius $\sqrt{4-2x}$, and it has area $\pi\left(\sqrt{4-2x}\right)^2 = 2\pi(2-x)$.
If it has thickness dx, it has volume
$2\pi(2-x)\,dx$.

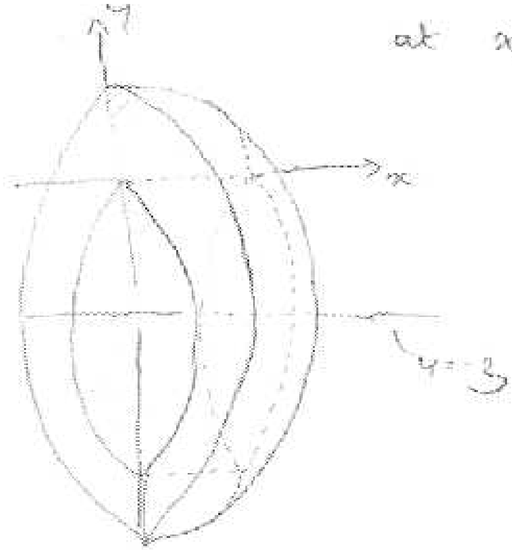
Thus the total volume is

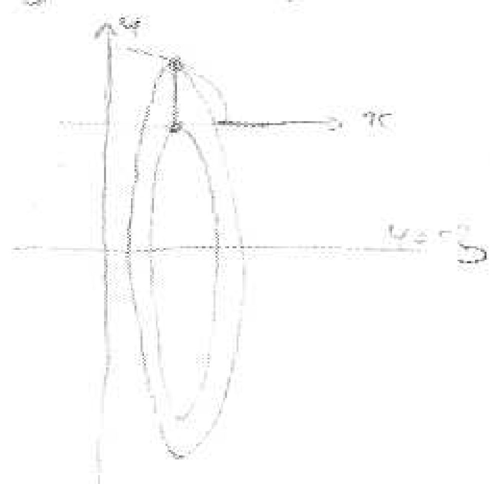
$$\int_0^2 2\pi(2-x)\,dx = 2\pi\left[2x - \frac{x^2}{2}\right]_0^2$$

$$= 2\pi\left(4 - \frac{4}{2}\right) = \boxed{4\pi}$$

(2) rotated around $y=-3$:

This time, a cross-section at x is giving an annulus.





the internal radius is $\cancel{(x+3)}$ 3

the external one is $\left(\sqrt{4-2x} - (-3)\right)$

Thus the area of the annulus is

$$\pi\left(\left(\sqrt{4-2x}+3\right)^2 - \overset{9}{\cancel{(x+3)^2}}\right)$$

$$= \pi\left(4 - 2x + 9 + 6\sqrt{4-2x} - \cancel{x^2 - 6x} - 9\right)$$

$$= \pi\left(4 - \underset{2x}{\cancel{8x}}\ \cancel{-x^2} + 6\sqrt{4-2x}\right)$$

$$= \pi\left(4 - 2x + 6\sqrt{4-2x}\right)$$

The volume of the solid of revolution is

$$\pi \int_0^2 (4 - 2x + 6\sqrt{4-2x})\, dx$$

$$= \pi \left[4x - x^2 - 2(4-2x)^{3/2} \right]_0^2$$

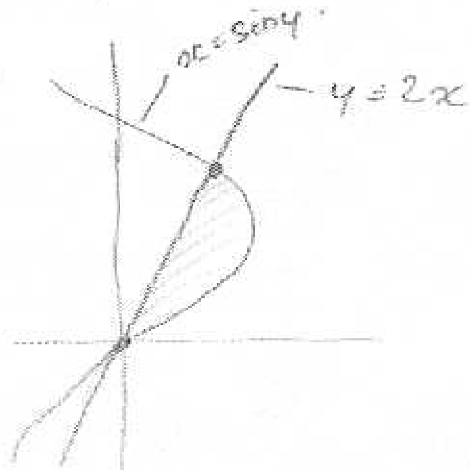
$$= \pi (8-4) - \pi(-2\sqrt{4}^{\,3})$$

$$= \pi (4) + \pi (16) = 20\pi$$

(3) Compute the volume of the solid generated by equilateral triangles standing on the edge of the region:

$$\begin{cases} x = \sin y \\ y = 2x \\ y = 0 \end{cases}$$

(and perpendicular to the y-axis)

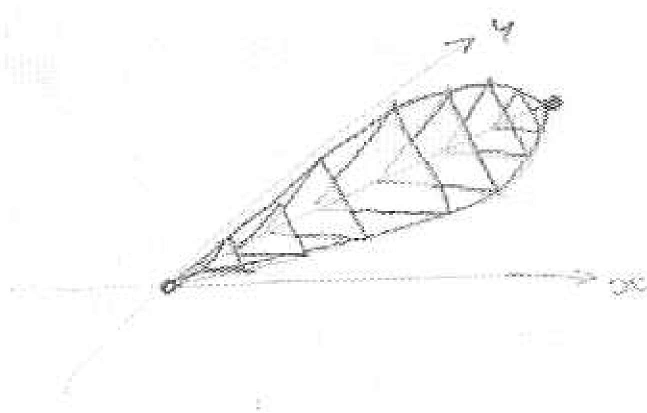


the intersection points are $\sin y = \frac{y}{2}$

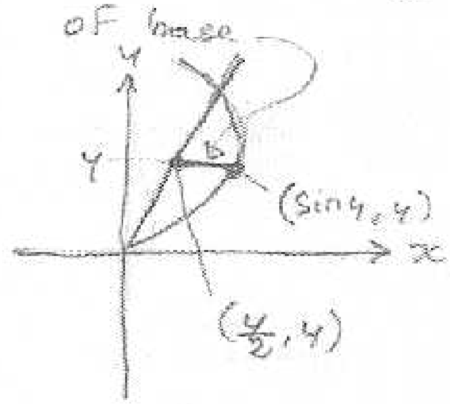
which is $y = 0$

$y \simeq 1{,}8954$

The solid looks like



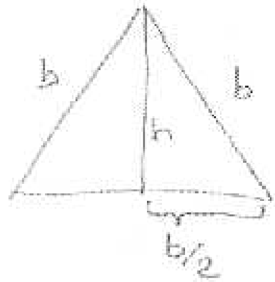
The crossection at y is an equilateral triangle of base



the base has length

$b = \left(\sin y - \frac{y}{2}\right)$.

the triangle looks like



Thus $b^2 = h^2 + \frac{b^2}{4}$

and $h = \frac{\sqrt{3}}{2} b = \frac{\sqrt{3}}{2}\left(\sin y - \frac{y}{2}\right)$.

The area of the triangle is $\frac{bh}{2} = \frac{\sqrt{3}}{4}\left(\sin y - \frac{y}{2}\right)^2$

Thus the volume of the total solid is

$$\frac{\sqrt{3}}{4}\int_0^{1,8954}\left(\sin y - \frac{y}{2}\right)^2 dy$$

$$= \frac{\sqrt{3}}{4}\int_0^{1,8954}\left(\sin^2 y - y\sin y + \frac{y^2}{4}\right) dy$$

* $\int \sin^2 y \, dy = \frac{1}{2}\int (1 - \cos(2y))\, dy$

  $= \frac{y}{2} - \frac{1}{4}\sin(2y)$

* $\int y \sin y \, dy = -y\cos y + \int \cos y \, dy = \sin y - y\cos y$

  $u = y \quad u' = 1$

  $v' = \sin y \quad v = -\cos y$

* $\int \frac{y^2}{4} dy = \frac{y^3}{12}$

Thus $\frac{\sqrt{3}}{4}\int_0^{1,8954}\left(\sin y - \frac{y}{2}\right)^2 dy = \frac{\sqrt{3}}{4}\left[\frac{y}{2} - \frac{\sin(2y)}{4} - \sin y + y\cos y + \frac{y}{1}\right.$

$\left.\simeq 0,04936\right]$

(4) Compute the length of the curve

$$y = \frac{x^3}{6} + \frac{1}{2x}, \quad x \in [1,2]$$

The formula says it is

$$\int_1^2 \sqrt{1 + f'(x)^2}\, dx$$

where $f(x) = \frac{x^3}{6} + \frac{1}{2x}$.

thus $f'(x) = \frac{x^2}{2} - \frac{1}{2x^2}$

Thus $1 + f'(x)^2 = 1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}$

$$= \frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4} = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2$$

Hence

$$\int_1^2 \sqrt{1 + f'(x)^2}\, dx = \int_1^2 \left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx$$

$$= \left[\frac{x^3}{6} - \frac{1}{2x}\right]_1^2 = \left(\frac{8}{6} - \frac{1}{4}\right) - \left(\frac{1}{6} - \frac{1}{2}\right)$$

$$= 17/12$$