COHOMOLOGY OPERATIONS FOR LIE ALGEBRAS

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Abstract. If L is a Lie algebra over $\mathbb{R}$ and Z its centre, the natural inclusion $Z \hookrightarrow (L^*)^*$ extends to a representation $i^*\colon \Lambda Z \to \mathrm{End}\, H^*(L, \mathbb{R})$ of the exterior algebra of Z in the cohomology of L. We begin a study of this representation, by examining its Poincaré duality properties, its associated higher cohomology operations and its relevance to the toral rank conjecture. In particular, by using harmonic forms we show that the higher operations of [9] form a subalgebra of End $H^*(L, \mathbb{R})$, and that they can be assembled to yield an explicit Hirsch-Brown model for the Borel construction associated to $0 \to Z \to L \to L/Z \to 0$.

§1. Introduction

Let L be a finite dimensional real Lie algebra and let Z be its centre. For given $z \in \mathbb{Z}$, the homomorphism $x^* \mapsto x^*(z)$ from $L^* = \Lambda^1 L^*$ to $\mathbb{R}$ extends to a derivation i_z of degree -1 of the algebra ΛL^*. For $a = z_1 \cdots z_k$ set $i_a = i_{z_1} i_{z_2} \cdots i_{z_k}$ and extend by linearity to obtain an algebra homomorphism $i\colon \Lambda Z \to \mathrm{End}\, \Lambda L^*$. Recall that the differential d on ΛL^* is the unique derivation of degree 1 extending the transpose of the Lie bracket $[\,,]\colon \Lambda^2 L \to L$. The cohomology of L (with trivial coefficients) is defined as $H^*L := H^*(\Lambda L^*, d)$. It is important to note that if $x \in L$,then $i_x d + d\, i_x = 0$ if and only if $x \in Z$. This shows that i induces a homomorphism of algebras $i^*\colon \Lambda Z \to \mathrm{End}\, H^*L$ which we will call the *central representation*, making H^*L a ΛZ-module. We conjecture that the central representation of every nilpotent Lie algebra is nontrivial; see Conjecture 5.9.

Every finitely generated ΛZ-module is a direct sum of indecomposables [6], but, ΛZ being nilpotent, it is not completely reducible. Indeed, all irreducibles have dimension one. The question of duality seems to be subtle; Poincaré duality may not exist in a strong sense (see Example 2.10 and Conjecture 2.11). Nevertheless, one does have a weak form of duality. If we consider the adjoint action of ΛZ on H^*L determined by an inner product on L, then this together with the central representation makes H^*L a bi-ΛZ-module. If $*\colon H^*L \to H^*L$ denotes the Hodge star involution on L, one has:

Theorem 1.1. *If L is a unimodular Lie algebra, then H^*L can be written in a $*$-invariant manner as a direct sum of irreducible bimodules for the central representation; that is $H^*L = \bigoplus_i M_i \;\oplus\; \bigoplus_j (N_j \oplus *N_j)$, where the M_i are self-dual; i.e., $*M_i = M_i$ for each i.*

The situation regarding the higher cohomology operations is by comparison more satisfying. Recall that higher operations are usually only defined on subspaces of

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the cohomology, and take values in quotients. In our case, if $z \in Z$, then the secondary operation (corresponding to the primary operation i_z) is defined on the kernel of i_z^* and takes its image in the cokernel of i_z^*; explicitly, if $i_z\alpha = d\beta$, then the corresponding secondary operation, evaluated on $[\alpha]$, is $[i_z\beta]$, which a priori, is only defined modulo the image of i_z^*. As a consequence, there are ambiguities when one attempts to compose higher operations. We avoid this by choosing an inner product on L and identifying H^*L with the space $\mathcal{H}$ of harmonic forms on L, as has been done for example in a different context in [10] and [11]. This enables one to define the higher operations naturally as linear maps in End $\mathcal{H}$, and has the added advantage that the operations are given by explicit formulas involving the Green's function of the associated Laplacian. In this way, for each element s of the polynomial ring $S = \mathbb{R}[\xi_1, \dots, \xi_{\dim Z}]$, we define an associated higher cohomology operation $\delta_s \in$ End $\mathcal{H}$, and the map $s \mapsto \delta_s$ is linear. The algebra generated by the maps δ_s enjoys a form of Poincaré duality similar to that of Theorem 1.1 (see Corollary 3.3 and the subsequent remark). Moreover, the operators δ_s can be assembled into a differential δ on $S \otimes \mathcal{H}$. The following theorem is the main result in the paper (see Section 3 for more details):

Theorem 1.2. *$(S \otimes \mathcal{H}, \delta)$ is a Hirsch-Brown model for the Borel construction associated to the short exact sequence $0 \to Z \to L \to L/Z \to 0$.*

One motivation for this study is Steve Halperin's *Toral rank conjecture* (TRC) [Ha]. The toral rank r of a manifold is the largest rank of a torus acting freely on it, and the TRC states that $\dim H^*(M;\mathbb{R}) \geq 2^r$. By analogy with nilmanifolds and their corresponding nilpotent Lie algebras over $\mathbb{Q}$, the TRC for a Lie algebra L, is: $\dim H^*L \geq 2^{\dim Z}$ [4]. As far as we are aware, this conjecture is open for Lie algebras over an arbitrary field k, though we will assume $k = \mathbb{R}$ throughout this paper. The TRC for Lie algebras is known to hold for a large class of nilpotent Lie algebras; it is true for algebras L possessing a grading $L = \oplus_{i=0}^k L_i$, with $[L_i, L_j] \subseteq L_{i+j}$, such that $Z(L) = L_k$ [7] (see also [18]). In particular, it is true for nilpotent Lie algebras of dimension at most 14 [4], for free n-step nilpotent Lie algebras, and for all 2-step nilpotent Lie algebras (see [4] and [17] for other proofs of this fact).

The central representation enables an approach to the TRC in which the centre plays a natural role. If the central representation is faithful, then H^*L contains a copy of the exterior algebra ΛZ and so the TRC holds for L (see Lemma 4.1). Computer computations using Mathematica show that, at least in low dimensions, most nilpotent Lie algebras have faithful central representations; in dimension ≤ 7, there are only a handful of exceptions. As well, there are a number of general families of algebras having faithful central representations; we give examples in §4. As we show in some examples in §5, when the central representation is not faithful, one can often use higher operations to complete a "cube" in H^*L and in this way establish the TRC for the given algebra. In fact, we have not found an algebra where the TRC couldn't be obtained in this manner.

Throughout this paper, we assume that $(L, [\,,])$ is a unimodular Lie algebra of dimension n with centre Z.

§2 Duality for Primary Operations

Choose an isomorphism $\chi\colon L \to L^*$ and extend it to an isomorphism $\Lambda L \to \Lambda L^*$ as follows: set $\chi(1) = 1$ and for $a = x_1 \cdots x_k$, set $\chi(a) = \chi(x_k) \cdots \chi(x_2)\chi(x_1)$, and extend by linearity. While this isn't an algebra homomorphism, one has $\chi(ab) =$

$(-1)^{pq}\chi(a)\chi(b)$ for all $a \in \Lambda^p L, b \in \Lambda^q L$. Note that χ determines an inner product on ΛL^* for which the spaces $\Lambda^p L^*$ are mutually orthogonal, and $\langle \alpha, \beta \rangle = \alpha(\chi^{-1}\beta)$ for all $\alpha, \beta \in \Lambda^p L^*$. Choose an ordered orthonormal basis $y_1, \ldots, y_n$ for L^*, set $\mathfrak{t} = y_1 \cdots y_n$, and consider the map $* : \Lambda L^* \to \Lambda L^*$ defined by $*\alpha = i_{\chi^{-1}\alpha}\mathfrak{t}$. This is the standard Hodge star map on ΛL^*; one has $*\colon \Lambda^p L^* \to \Lambda^{n-p} L^*$, for all p and if α is a monomial in the variables $y_1, \ldots, y_n$, then $*\alpha$ is the unique monomial such that $\alpha * \alpha = \mathfrak{t}$. It is well known that $*$ verifies the following [8]:

Lemma 2.1.
(a) $** = (-1)^{p(n-p)}$ *on* $\Lambda^p L^*$,
(b) $\langle \alpha, \beta \rangle = *(\alpha * \beta)$ *for all* $\alpha, \beta \in \Lambda^p L^*$.
(c) $\langle *\alpha, *\beta \rangle = \langle \alpha, \beta \rangle$, *for all* $\alpha, \beta \in \Lambda L^*$. *(That is,* $*$ *is an isometry.)*

If φ is a map on ΛL^*, let $\bar{\varphi}$ denote the adjoint of φ relative to $\langle\, , \rangle$. The following lemma is a straightforward consequence of Lemma 2.1:

Lemma 2.2. *Let* φ *be a linear map on* ΛL^* *of degree* k.
(a) If φ *is a derivation (i.e.,* $\varphi(\alpha\beta) = (\varphi\alpha)\beta + (-1)^{k \deg(\alpha)}\alpha\varphi\beta$, *for all* α, β*) and* $\bar{\varphi}(\mathfrak{t}) = 0$, *then* $\bar{\varphi} = (-1)^{(n-p)(p+k)+1} * \varphi *$ *on* $\Lambda^p L^*$.
(b) If $\bar{\varphi}$ *is multiplication on the left by an element of* ΛL^* *(i.e.,* $\bar{\varphi}(\alpha) = (\bar{\varphi}1)\alpha$ *for all* $\alpha \in \Lambda L^*$*), then* $\bar{\varphi} = (-1)^{(n-p)(p+k)+k} * \varphi *$ *on* $\Lambda^p L^*$.

Now recall the representation $i : \Lambda Z \to \mathrm{End}\, \Lambda L^*$. For each $a \in \Lambda^k Z$, i_a is a linear map of degree $-k$. For each $a \in \Lambda Z$, let μ_a denote the adjoint of i_a. Using Lemma 2.2, one easily obtains:

Lemma 2.3. $\mu_a \alpha = (\chi a)\alpha = (-1)^{(n-p)(p+k)+k} * i_a * \alpha$, *for all* $a \in \Lambda^k Z, \alpha \in \Lambda L^*$.

The following result is a variation on the classical *Hodge Decomposition Theorem* [19, Chapter 6]. In our context, its proof is a straightforward exercise in linear algebra. We state it in some generality as we will use it both on ΛL^* and $H^* L$:

Proposition 2.4. *If* $(\Omega, \langle\, , \rangle)$ *is a finite dimensional inner product space and* φ *is a differential on* Ω *(i.e.,* $\varphi^2 = 0$*), then for the Laplacian* $\Delta_\varphi := \varphi\bar{\varphi} + \bar{\varphi}\varphi$, *one has:*
(a) $\ker \Delta_\varphi = \ker \varphi \cap \ker \bar{\varphi}$, *and one has the following orthogonal decompositions:*
(i) $\Omega = \ker \Delta_\varphi \oplus \mathrm{im}\ \Delta_\varphi$, *(ii)* $\mathrm{im}\ \Delta_\varphi = \mathrm{im}\ \varphi \oplus \mathrm{im}\ \bar{\varphi}$, *(iii)* $\ker \varphi = \ker \Delta_\varphi \oplus \mathrm{im}\ \varphi$,
(b) Δ_φ *restricts to an isomorphism on both* $\mathrm{im}\ \varphi$ *and* $\mathrm{im}\ \bar{\varphi}$,
(c) if moreover, Ω *is a graded algebra possessing a map* $*$ *with the properties of Lemma 2.1, and if* φ *satisfies either of the hypotheses (a) or (b) of Lemma 2.2, then* Δ_φ *commutes with* $*$, *and thus* $*$ *restricts to an isomorphism on both* $\ker \Delta_\varphi$ *and* $\mathrm{im}\ \Delta_\varphi$.

The differential d on ΛL^* is a derivation of degree 1. As is customary, we denote the adjoint of d by ∂. The hypothesis that L is unimodular is equivalent to the condition that $d\colon \Lambda^{n-1} L^* \to \Lambda^n L$ is zero, where $n = \dim L$, and this is equivalent to the condition $\partial\mathfrak{t} = 0$. Thus d verifies the hypothesis of Lemma 2.2(a) and so:

$$\partial = (-1)^{n(p+1)+1} * d* \tag{2.1}$$

on $\Lambda^p L^*$. We denote the Laplacian Δ_d simply by Δ and we denote the kernel of the restriction of Δ to $\Lambda^p L^*$ by $\mathcal{H}^p$. (The Laplacian of free 2-step nilpotent Lie algebras is examined in [16]). The space $\mathcal{H} = \oplus \mathcal{H}^p$ is the space of *harmonic* forms, and from Proposition 2.4(a)(iii), $\mathcal{H} \cong \ker d / \mathrm{im}\ d = H^* L$. In particular, by Proposition 2.4(c), $*$ induces an isomorphism $*\colon \mathcal{H}^p \to \mathcal{H}^{n-p}$, which yields the Poincaré duality for H^*L.

Let $\pi\colon \Lambda L^* \to \mathcal{H}$ denote the orthogonal projection. By Proposition 2.4(c), $*$ preserves $\mathcal{H}$ and im Δ and so $*$ commutes with π. Notice that $\mathcal{H}$ is not always a subalgebra of ΛL^*, but it can be made into an associative graded-commutative algebra by composing the product in ΛL^* with the projection π. We also denote the restriction of $*$ to $\mathcal{H}$ by $*$.

The space $\mathcal{H}$ is a natural setting to study the central representation. When $\mathcal{H}$ is given the above algebra structure, the vector space isomorphism $\mathcal{H} \to H^*L$ is an algebra isomorphism. Under this isomorphism, the map i_a^* induced by i_a on H^*L corresponds to the map πi_a on $\mathcal{H}$; thus, the central representation $i^*\colon \Lambda Z \to \operatorname{End} H^*L$ is isomorphic to the representation $\pi i\colon \Lambda Z \to \operatorname{End} \mathcal{H}$. Notice that for $a \in \Lambda Z$, the adjoint μ_a of i_a in ΛL^* does not naturally induce a map in H^*L. However, relative to the inner product inherited by $\mathcal{H}$ from ΛL^*, the adjoint of πi_a on $\mathcal{H}$ is $\pi\mu_a$; indeed, for all $a \in \Lambda Z, \alpha, \beta \in \mathcal{H}$, one has $\langle \pi i_a \alpha, \beta\rangle = \langle i_a\alpha, \beta\rangle = \langle \alpha, \mu_a\beta\rangle = \langle \alpha, \pi\mu_a\beta\rangle$. Notice that $\pi\mu$ defines a "right action" $\pi\mu\colon \Lambda Z \to \operatorname{End} \mathcal{H}$, in that $\pi\mu_a\pi\mu_b = \pi\mu_{ba}$. Also, since π commutes with $*$, Lemma 2.3 gives $\pi\mu_a\alpha = (-1)^{(n-p)(p+k)+k} * \pi i_a * \alpha$, for all $a \in \Lambda^k Z, \alpha \in \mathcal{H}^p$. In particular, for homogeneous $a \in \Lambda Z$, the maps πi_a are $*$-*symmetric* in the following sense:

Definition 2.5. We say that a graded linear map φ on H^*L is $*$-*symmetric* if $\bar{\varphi}\alpha = \pm * \varphi * \alpha$, for all homogeneous $\alpha \in H^*L$.

We now take a slightly more abstract approach, so that we may use the results of this section again in §3 for the higher operations. Suppose that we have an associative algebra A and a representation $i\colon A \to \operatorname{End} H^*L$. Let $\mu\colon A \to \operatorname{End} H^*L$ denote the corresponding right action, defined by $\mu(a) = \overline{i(a)}$.

Definition 2.6. We say that a graded subspace M of H^*L is
(a) an *i-module* (resp. a *μ-module*) if $i(A).M \subseteq M$, (resp. $\mu(A).M \subseteq M$).
(b) a *bimodule* if it is both an i-module and a μ-module,
(c) a $*$-*bimodule* if it is a $*$-invariant bimodule.

Recall that an i-module (resp. bimodule, resp. $*$-bimodule) M is said to be *irreducible* if M has no non-zero proper sub-i-modules (resp. sub-bimodules, sub-$*$-bimodules), and it is *indecomposable* if it cannot be written as the direct sum of two non-zero i-modules (resp. bimodules, $*$-bimodules). In fact, one has:

Lemma 2.7. *A bimodule* (*resp.* $*$-*bimodule*) $M \subseteq H^*L$ *is irreducible iff it is indecomposable.*

Proof. One direction is obvious. For the other, if $N \subseteq M$ is a bimodule, consider its orthogonal complement $N^\perp$ in M, relative to the inner product $\langle\, ,\rangle$. Clearly $M = N \oplus N^\perp$ and $N^\perp$ is a bimodule. Moreover, if N is a $*$-bimodule, it follows from Lemma 2.1 that $N^\perp$ is also a $*$-bimodule. □

The next lemma follows easily from Lemma 2.1(a):

Lemma 2.8. *Suppose that A is generated as an algebra by elements a for which $i(a)$ is $*$-symmetric. If $M \subseteq H^*L$ is a bimodule, then $*M$ is a bimodule.*

We can now give a result which has Theorem 1.1 as an immediate corollary, and which we will apply again in §3 for the higher operations.

Theorem 2.9. *Suppose i is a representation of an algebra A in H^*L and that A is generated as an algebra by elements a for which $i(a)$ is $*$-symmetric. Then, H^*L can be written in a $*$-invariant manner as a direct sum of irreducible bimodules; that is, $H^*L = \bigoplus_i M_i \oplus \bigoplus_j (N_j \oplus *N_j)$, where the M_i are self-dual; i.e., $*M_i = M_i$ for each i.*

Proof. From the proof of Lemma 2.7, H^*L is an orthogonal direct sum of irreducible $*$-bimodules. Let $M \subseteq H^*L$ be an irreducible $*$-bimodule. It suffices to show that either M is irreducible as a bimodule, or there is an irreducible bimodule $N \subseteq M$ such that $M = N \oplus *N$, as $*$-bimodules.

Suppose that $N \subseteq M$ is a proper sub-bimodule. From Lemma 2.8, $*N$ is also a bimodule, and hence the vector space sum $N + *N$ is a $*$-bimodule and so $N + *N = M$, as M is irreducible as a $*$-bimodule. Moreover, the intersection $N \cap *N$ is also an irreducible $*$-bimodule, and so either $N \cap *N = M$ or $N \cap *N = 0$. But the former case is impossible since $N \neq M$. Hence $M = N \oplus *N$ and it remains to show that N is irreducible as a bimodule. Suppose that N contains some non-zero bimodule Q. Arguing as above, $Q \oplus *Q$ is a non-zero $*$-bimodule and thus $Q \oplus *Q = M$, as M is irreducible as a $*$-bimodule. Therefore $Q = N$, and we are done. $\square$

The disappointing aspect of the above result is that irreducible bimodules may be decomposable as i-modules. We give an abstract example which shows the sort of phenomenon that can occur:

Example 2.10. Consider the 6 dimensional associative graded-commutative algebra Ω with generators $\alpha_1 = 1, \alpha_2, \ldots, \alpha_6$, of degree $0, 1, 1, 1, 1, 2$ respectively, and defining relations: $\alpha_2\alpha_3 = \alpha_4\alpha_5 = \alpha_6$. Choose the inner product for which the basis $\{\alpha_1, \ldots, \alpha_6\}$ is orthonormal. Consider the map $*$ defined by $\alpha_2 \mapsto \alpha_3 \mapsto -\alpha_2$, $\alpha_4 \mapsto \alpha_5 \mapsto -\alpha_4$ and $*\colon \alpha_1 \leftrightarrow \alpha_6$. It is easy to see that $*$ verifies conditions analogous to those of Lemma 2.1 (by replacing $\Lambda^p L^*$ by Ω^p). Consider the algebra A generated by the maps a and b whose non-zero images on the basis elements are as follows: $a\alpha_2 = \alpha_1$, $a\alpha_6 = \alpha_3$, $b\alpha_3 = -\alpha_1$, $b\alpha_4 = \alpha_1$, $b\alpha_6 = \alpha_2 + \alpha_5$. We have $A \cong \Lambda\mathbb{R}^2$. Note that a and b are derivations of degree -1, and they are $*$-symmetric. As an i-module, Ω decomposes as a direct sum of three irreducible i-modules: $\Omega = \operatorname{span}\{\alpha_1, \alpha_3, \alpha_2 + \alpha_5, \alpha_6\} \oplus \operatorname{span}\{\alpha_5\} \oplus \operatorname{span}\{\alpha_3 + \alpha 4\}$. However, it is not difficult to show that Ω is irreducible as a bimodule.

As the above example indicates, it is not possible in general to decompose H^*L as a direct sum of i-modules in a $*$-invariant manner. There is however, an analogous (conjectured) decomposition for i-modules. Call an i-module $M \subseteq H^*L$ *n.d.-indecomposable* if the multiplication is non-degenerate on M but it cannot be written as a direct sum of two i-submodules, on each of which the multiplication is still non-degenerate.

Conjecture 2.11. *If L is a unimodular Lie algebra, then H^*L can be written as a direct sum of indecomposable i-modules in the form $H^*L = \bigoplus_i M_i \oplus \bigoplus_j (N_j \oplus \overline{N}_j)$, where each M_i and $N_j \oplus \overline{N}_j$ is n.d.-indecomposable, and the multiplication is trivial on N_j and $\overline{N}_j$.*

§3 Higher Operations

Recall from Proposition 2.4(b) that the Laplacian Δ on ΛL^* restricts to an isomorphism on im Δ. The *Green's function* $G\colon \Lambda L^* \to \operatorname{im} \Delta$ is the linear function which is zero on $\mathcal{H}$ and equals $(\Delta\,|_{\mathrm{im}\;\Delta})^{-1}$ on im Δ. So on ΛL^* one has:

$$\mathrm{id} = \pi + \Delta\, G. \tag{3.1}$$

It is easy to see that since π and Δ are self-adjoint, so too is G. Moreover, since Δ commutes with $*, d, \partial, \pi$ and itself, so too does G.

For $z \in Z$, let $P_z = \pi i_z$ and $Q_z = \partial G i_z$. For each element $z = (z_1, \ldots, z_k)$ of the k-fold Cartesian product $\prod_k Z$, consider the symmetric multilinear function $z \mapsto \frac{1}{k!}\sum_{\sigma\in\Sigma_k} P_{z_{\sigma 1}} Q_{z_{\sigma 2}} Q_{z_{\sigma 3}} \ldots Q_{z_{\sigma k}}$, where Σ_k denotes the symmetric group. If $S = \sum_{k\geq 0} S^k$ denotes the symmetric tensor algebra on Z, this defines a linear function $S \to \mathrm{End}\,\mathcal{H}, s \mapsto \delta_s$, where we take the map $s \mapsto \delta_s$ to be 0 on S^0. (Note that the map $s \mapsto \delta_s$ is not a homomorphism of algebras.) Adopting notation similar to that in [9] and [1], let $\tilde{Z}^*$ be a copy of Z^* regarded as being in degree two. Choose a basis $\{x_1, \ldots, x_m\}$ for Z, let $\{\xi_1, \ldots, \xi_m\}$ be the corresponding dual basis of $\tilde{Z}^*$, and define an isomorphism $\tilde{\chi}\colon Z \to \tilde{Z}^*$ by $\tilde{\chi}(x_i) = \xi_i$ for all i. We identify S with the polynomial ring in the variables $\xi_1, \ldots, \xi_m$. So

$$\delta_{\xi_1^{i_1}\ldots\xi_m^{i_m}} = \frac{1}{k!}\sum_{\sigma\in\Sigma_k} P_{y_{\sigma 1}} Q_{y_{\sigma 2}} Q_{y_{\sigma 3}} \ldots Q_{y_{\sigma k}},$$

where $k = \sum_{j=1}^m i_j$ and $\{y_1, \ldots, y_k\} = \{\overbrace{x_1, \ldots, x_1}^{i_1 \text{ times}}, \ldots, \overbrace{x_m, \ldots, x_m}^{i_m \text{ times}}\}$.

Definition 3.1. For $s \in S$, we call δ_s the *higher operation* associated to s.

If s is a homogeneous polynomial of degree $k > 0$, then δ_s has degree $1 - 2k$. The following lemma follows easily from Lemma 2.3, equation (2.1) and the fact that G commutes with $*$.

Lemma 3.2. *If s is a homogeneous polynomial of degree $k > 0$, then the adjoint of δ_s is $\bar{\delta}_s = (-1)^{n(p+1)+1} * \delta_s *$.*

Now consider the *algebra of higher operations*: $A = (\delta_s \mid s \in S) \subseteq \mathrm{End}\,\mathcal{H}$. If s is homogeneous, δ_s is $*$-symmetric by the previous lemma. Hence Theorem 2.9 gives:

Corollary 3.3. *If L is a unimodular Lie algebra, then for the action of the algebra A of higher operations, the space of harmonic forms $\mathcal{H}$ can be written in a $*$-invariant manner as a direct sum of indecomposable bimodules.*

A remarkable feature of the operations δ_s is that they can be assembled into a differential, which yields a Hirsch-Brown model for $(S \otimes \Lambda L^*, D)$, where $D = d + \sum_{j=1}^{\dim Z} \xi_j i_{z_j}$. Define the S-linear function δ on $S \otimes_{\mathbb{R}} \mathcal{H}$ by setting

$$\delta(\alpha) = \sum_{s\in\mathcal{M}(S)} (-1)^{\deg(s)+1} c(s)\, s\delta_s(\alpha)$$

on $\mathcal{H}$, where the sum is taken over the set $\mathcal{M}(S)$ of monomials (of degree ≥ 1) in the variables $\xi_1, \ldots, \xi_m$, and $c(s)$ is the multinomial coefficient:

$$c(\xi_1^{j_1}\ldots\xi_m^{j_m}) = \binom{\sum_i j_i}{j_1, \ldots, j_m}.$$

The sum is finite on any α because $|\delta_s| = 1 - 2s < 0$, and $\mathcal{H}^{<0} = 0$. One has:

Theorem 3.5. $\delta^2 = 0$.

Proof. We are required to show that for each monomial s,

$$\sum_{\substack{s_1, s_2 \in \mathcal{M}(S) \\ s = s_1 s_2}} c(s_1)c(s_2)\delta_{s_1}\delta_{s_2} = 0. \tag{3.2}$$

Note that these relations are analogous to those in Remark 1.3 of [1]. We first prove that (3.2) holds for all s of the form $s = \xi^k$, with $\xi \in \tilde{Z}^*$ and $k \geq 1$. We will use:

Lemma 3.6. *For all $z \in Z$, the following hold on ΛL^*:*
(a) $P_z = i_z + Q_z d - dQ_z$,
(b) $Q_z^i P Q_z^j = Q_z^i i_z Q_z^j + Q_z^{i+1} dQ_z^j - Q_z^i dQ_z^{j+1}$, *for all* $i, j \geq 0$,
(c) $\sum_{j=1}^{k-1} Q_z^{j-1} P Q_z^{k-j-1} = i_z Q_z^{k-2} + Q_z^{k-1} d - dQ_z^{k-1}$, *for all* $k \geq 2$, *(where* $Q_z^0 := \mathrm{id}$*).*
(d) $\sum_{j=1}^{k-1} P_z Q_z^{j-1} P_z Q_z^{k-j-1} = P_z Q_z^{k-1} d$, *for all* $k \geq 2$.

Proof of Lemma 3.6. By equation (3.1), $P_z = (\mathrm{id} - \Delta\, G)i_z$ and so

$$P_z = i_z - d\partial G i_z - \partial d G i_z = i_z - dQ_z + \partial G i_z d = i_z - dQ_z + Q_z d,$$

which gives (a). Thus,

$$Q_z^i P_z Q_z^j = Q_z^i(i_z + Q_z d - dQ_z)Q_z^j = Q_z^i i_z Q_z^j + Q_z^{i+1} dQ_z^j - Q_z^i dQ_z^{j+1},$$

which gives (b). By summing, (b) gives

$$\sum_{j=1}^{k-1} Q_z^{j-1} P Q_z^{k-j-1} = \sum_{j=1}^{k-1} Q_z^{j-1} i_z Q_z^{k-j-1} + Q_z^{k-1} d - dQ_z^{k-1}.$$

Thus, as $Q_z i_z = \partial G i_z i_z = 0$, we obtain (c). Multiplying (c) on the left by P and using the facts that $P_z i_z = \pi i_z i_z = 0$ and $P_z d = \pi i_z d = -\pi d i_z = 0$, one obtains (d). $\square$

Returning now to the proof of the theorem, let $z \in Z$ and set $\xi = \tilde{\chi}(z)$. When restricted to $\mathcal{H}$, equation (d) of the above lemma gives $\sum_{j=1}^{k-1} P_z Q_z^{j-1} P_z Q_z^{k-j-1} = 0$. This yields:

$$\sum_{j=1}^{k-1} \delta_{\xi^j}\delta_{\xi^{k-j}} = 0, \tag{3.3}$$

which establishes (3.2) for $s = \xi^k$. We now polarize (3.3); let $z = \sum_{i=1}^m t_i x_i$ and $\xi = \sum_{i=1}^m t_i \xi_i$, where $\{x_1, \dots, x_m\}$ is the basis for Z chosen above, and $t_1, \dots, t_m$ are real variables. From (3.3):

$$\begin{aligned} 0 = \sum_{j=1}^{k-1} \delta_{\xi^j}\delta_{\xi^{k-j}} &= \sum_{j=1}^{k-1} \delta_{(\sum_{i=1}^m t_i \xi_i)^j} \delta_{(\sum_{i=1}^m t_i \xi_i)^{k-j}} \\ &= \sum_{j=1}^{k-1} \sum_{\substack{i_1 + \cdots + i_m = j \\ j_1 + \cdots + j_m = k-j}} c(\xi_1^{i_1} \dots \xi_m^{i_m}) c(\xi_1^{j_1} \dots \xi_m^{j_m})(t_1^{i_1+j_1} \dots t_m^{i_m+j_m}) \delta_{\xi_1^{i_1} \dots \xi_m^{i_m}} \delta_{\xi_1^{j_1} \dots \xi_m^{j_m}}. \end{aligned}$$

Regarding this as a polynomial in $t_1, \dots, t_m$, the coefficient of $s = \xi_1^{l_1} \dots \xi_m^{l_m}$ is:

$$0 = \sum_{\substack{0 \le i_1 \le l_1 \\ \vdots \\ 0 \le i_m \le l_m}} c(\xi_1^{i_1} \dots \xi_m^{i_m}) c(\xi_1^{l_1 - i_1} \dots \xi_m^{l_m - i_m}) \delta_{\xi_1^{i_1} \dots \xi_m^{i_m}} \delta_{\xi_1^{l_1-i_1} \dots \xi_m^{l_m - i_m}}$$

$$= \sum_{\substack{s_1, s_2 \in \mathcal{M}(S) \\ s = s_1 s_2}} c(s_1) c(s_2) \delta_{s_1} \delta_{s_2},$$

as required. This completes the proof of the theorem. $\square$

We can now prove Theorem 1.2. Consider the differential complex $(S \otimes \Lambda L^*, D)$, where D is S-linear and is defined on ΛL^* by $D = d + \sum_{j=1}^{\dim Z} \xi_j i_{z_j}$. We have:

Theorem 3.7. *There is a map of differential complexes $(S \otimes \mathcal{H}, \delta) \to (S \otimes \Lambda L^*, D)$ which induces an isomorphism in cohomology. In particular, $H(S \otimes \mathcal{H}, \delta) \cong H(L/Z)$ and so $H(S \otimes \mathcal{H}, \delta)$ is independent of the basis and the inner product used in the definition of δ.*

Proof. Consider the linear function $\zeta \colon S \to \operatorname{End} \mathcal{H}$, defined on monomials $s = \xi_1^{i_1} \dots \xi_m^{i_m}$ by:

$$\zeta \colon s \mapsto \zeta_s = \frac{1}{k!} \sum_{\sigma \in S_k} Q_{z_{\sigma 1}} Q_{z_{\sigma 2}} Q_{z_{\sigma 3}} \dots Q_{z_{\sigma k}}$$

where $k = \sum_{j=1}^m i_j$, $\{z_1, \dots, z_k\} = \{\overbrace{x_1, \dots, x_1}^{i_1 \text{ times}}, \dots, \overbrace{x_m, \dots, x_m}^{i_m \text{ times}}\}$, and $\zeta_1 := \mathrm{id}$. Now define an S-linear map $\phi \colon S \otimes \mathcal{H} \to S \otimes \Lambda L^*$ by $\phi = \mathrm{id}|_{\mathcal{H}} + \sum_{s \in \mathcal{M}(S)} (-1)^{\deg(s)} c(s)\, s \zeta_s$, where $\mathcal{M}(S)$ and $c(s)$ are as before. One has:

$$\phi\delta = \left(\mathrm{id}|_{\mathcal{H}} + \sum_{s_1 \in \mathcal{M}(S)} (-1)^{\deg(s_1)} c(s_1)\, s_1 \zeta_{s_1} \right) \sum_{s_2 \in \mathcal{M}(S)} (-1)^{\deg(s_2)+1} c(s_2)\, s_2 \delta_{s_2}$$

and

$$D\phi = \left(d + \sum_{j=1}^m \xi_j i_{z_j} \right) \left(\mathrm{id}|_{\mathcal{H}} + \sum_{s \in \mathcal{M}(S)} (-1)^{\deg(s)} c(s)\, s \zeta_s \right),$$

both of which we regard as a polynomial in $\xi_1, \dots, \xi_m$ with values in $\operatorname{End} \mathcal{H}$. The constant term of $\phi\delta - D\phi$ is $-d\,\mathrm{id}|_{\mathcal{H}} = 0$. From Lemma 3.6(a), $P_{z_i} = i_{z_i} + Q_{z_i} d - dQ_{z_i}$ and hence $\delta_{\xi_i} = i_{z_1} + \zeta_{\xi_i} d - d\zeta_{\xi_i}$ Hence the first order term of $\phi\delta - D\phi$ is:

$$\sum_{i=1}^m \delta_{\xi_i} - \left(-d \sum_{i=1}^m \zeta_{\xi_i} + \sum_{j=1}^m \xi_j i_{z_j} \right) = \sum_{i=1}^m \left(\delta_{\xi_i} + d\zeta_{\xi_i} - i_{z_j} \right) = \sum_{i=1}^m \zeta_{\xi_i} d,$$

which is zero in $\operatorname{End} \mathcal{H}$. In order to complete the proof that $\phi\delta = D\phi$, it remains to show that

$$c(s)\delta_s + \sum_{\substack{s_1, s_2 \in \mathcal{M}(S) \\ s = s_1 s_2}} c(s_1) c(s_2) \zeta_{s_1} \delta_{s_2} = -c(s) d\zeta_s + \sum_{\substack{j=1 \\ \xi_j | s}}^m c(s/\xi_j) i_{z_j} \zeta_{s/\xi_j} \tag{3.4}$$

for all $s \in \mathcal{M}(S)$ for degree ≥ 2. From Lemma 3.6(c), for all $z \in Z$ and $k \geq 2$, one has $\sum_{j=1}^{k-1} Q_z^{j-1} P Q_z^{k-j-1} = i_z Q_z^{k-2} + Q_z^{k-1} d - dQ_z^{k-1}$; that is, for $\xi = \zeta^*(z)$,

$$0 = \sum_{j=1}^{k-1} \zeta_{\xi^{j-1}} \delta_{\xi^{k-j}} - i_z \zeta_{\xi^{k-2}} - \zeta_{\xi^{k-1}} d + d\zeta_{\xi^{k-1}}. \tag{3.5}$$

Polarizing (3.5), in the same way that we polarized (3.3) in the proof of Theorem 3.5, one obtains:

$$c(s)(\delta_s - \zeta_s d + d\zeta_s) + \sum_{\substack{s_1, s_2 \in \mathcal{M}(S) \\ s = s_1 s_2}} c(s_1)c(s_2)\zeta_{s_1}\delta_{s_2} - \sum_{\substack{j=1 \\ \xi_j | s}}^{m} c(s/\xi_j) i_{z_j} \zeta_{s/\xi_j} = 0. \tag{3.6}$$

When restricted to End $\mathcal{H}$, one has $c(s)\zeta_s d = 0$ and so (3.6) gives (3.4); so ϕ is a differential map.

We now show that ϕ induces an isomorphism in cohomology. To see this, note that the complexes $(S \otimes \mathcal{H}, \delta)$ and $(S \otimes \Lambda L^*, D)$ both have bounded decreasing filtrations determined by the polynomial degree on S and moreover, ϕ is a morphism of filtered differential graded modules. In the spectral sequences of the complexes, one obviously has $E_1(S \otimes \mathcal{H}, \delta) = E_1(S \otimes \Lambda L^*, D) = S \otimes \mathcal{H}$, and the map induced by ϕ between the two E_1 terms is the identity. Thus, by the comparison theorem (see [13, Theorem 3.2]), ϕ induces an isomorphism $H(S \otimes \mathcal{H}, \delta) \to H(S \otimes \Lambda L^*, D)$.

Lastly, consider the inclusion of complexes $(\Lambda U, d) \to (S \otimes \Lambda L^*, D)$, where we have written $\Lambda L^* = \Lambda U \otimes \Lambda Z^*$ with $d(U \oplus Z^*) \subseteq \Lambda U$. Filtering by word length in ΛU, it is straightforward to see that this induces an isomorphism between the E_1 terms of the associated spectral sequences (both are ΛU) and so $H(\Lambda U, d) \cong H(S \otimes \Lambda L^*, D) \cong H(L/Z)$ and thus $H(S \otimes \mathcal{H}, \delta) \cong H(L/Z)$, which is independent of the basis and the inner product. $\square$

Thus $H(S \otimes \mathcal{H}, \delta)$ is finite dimensional, and the following proposition shows that while all the primary operations may be trivial (see Examples 5.6 and 5.7), sufficiently many higher operations must be non-zero in order to render the cohomology finite dimensional:

Proposition 3.8. *For all $s \in S$ of degree 1, there exists $l > 0$ such that δ_{s^l} is non-zero.*

Proof. Consider ξ_j and let $i > 0$ be an integer such that ξ_j^i is zero in the cohomology $H(S \otimes \mathcal{H}, \delta)$; that is, $\xi_j^i = \delta\alpha$ for some $\alpha \in S \otimes \mathcal{H}$, say $\alpha = \sum_{s \in \mathcal{M}(S)} s\alpha_s$ with $\alpha_s \in \mathcal{H}$ for all $s \in \mathcal{M}(S)$. Then clearly there exists $k < i$ such that $\delta_{\xi_j^{i-k}} \alpha_{\xi_j} \neq 0$. In particular, $\delta_{\xi_j^{i-k}} \neq 0$. Thus, for each $j \in \{1, \ldots, \dim Z\}$, there exists $l > 0$ such that δ_{ξ_j} is non-zero. Since this holds for any basis $\{\xi_1, \ldots, \xi_m\}$ of $\tilde{Z}^*$, and since the definition of δ_s is independent of the choice of basis, we conclude that for all $s \in S$ of degree 1, there exists $l > 0$ such that δ_{s^l} is non-zero. $\square$

§4. Examples with Faithful Central Representations

As usual, by an *orientation* for Z, we mean a choice of a non-zero element $\zeta \in \Lambda^{\dim Z} Z$. The connection with the TRC is given by the straightforward but key

Lemma 4.1. *The central representation i^* is faithful if and only if there exists $\alpha \in H^*L$ such that $i^*_\zeta\alpha \neq 0$ for any orientation ζ. Moreover, if i^* is faithful, L satisfies the TRC.*

Proof. The "only if" part of the first statement is clear, so suppose there is such an α, and choose an orientation ζ. By the Poincaré duality of ΛZ, for any $0 \neq \beta \in \Lambda Z$, there exists $\delta \in \Lambda Z$ with $\beta\delta = \zeta$, so that $0 \neq i^*_\zeta\alpha = i^*_\beta i^*_\delta\alpha$. In particular, $i^*_\beta \neq 0$, so i^* is injective. By the first part, choose $\alpha \in H^*L$, an orientation ζ for Z with $i^*_\zeta\alpha \neq 0$ and consider the linear map $e_\alpha : \Lambda Z \to H^*L, e_\alpha : \beta \mapsto i^*_\beta\alpha$. Exactly as above, the Poincaré duality of ΛZ shows that e_α is injective. Indeed this shows that $i^*(\Lambda Z)\alpha$ is an indecomposable ΛZ-submodule of H^*L which is isomorphic to ΛZ^*, establishing the result. □

If $\alpha \in H^*L$ is such that $i^*_\zeta\alpha \neq 0$ for any orientation ζ, we say that the central representation is *faithful on* α. In this section we give a number of examples of families of nilpotent Lie algebras whose central representations are faithful. These examples all use the following idea: let $p\colon \Lambda(L/Z)^* \to \Lambda L^*$ be the injection induced by the projection $L \to L/Z$, and let ζ and ζ^* be orientations for Z and Z^* respectively. If $[\beta] \in H^*(L/Z)$ satisfies $[p(\beta)] \neq 0$, and if $d\alpha = 0$ in $(\Lambda L^*, d)$, where $\alpha = \zeta^* p(\beta)$, then the central representation is faithful on $[\alpha]$, since $i^*_\zeta[\alpha] = [p(\beta)] \neq 0$.

In the following examples, the bases will be chosen to be orthonormal, and their dual bases will as usual be denoted using $*'s$.

Example 4.2. Lie algebras possessing a codimension 1 Abelian ideal. These algebras are all of the form $\mathrm{span}\{x, y_{i,j} \mid [x, y_{i,j}] = y_{i,j+1}, i = 1, \dots, k,\ j = 1, \dots, n_i\}$; see [2] for a description of their cohomology. Here the centre is $\mathrm{span}\{y^*_{i,n_i} \mid i = 1, \dots, k\}$ and has dimension k. Their central representations are faithful on the harmonic $(k+1)$-form $\alpha = x^* y^*_{1,n_1} \cdots y^*_{k,n_k}$.

Example 4.3 Algebras of truncated upper triangular matrices. Choose $k, n \in \mathbb{N}$ with $k < n$ and consider the algebra $\mathfrak{t}_{n,k}$ of $n \times n$ (strictly) upper triangular matrices with k "off-diagonals". Let $x_{i,j}$ denote the $n \times n$ matrix with 1 in the i^{th}-row and $(i+j)^{\text{th}}$-column, and 0's elsewhere. So $\mathfrak{t}_{n,k}$ has basis $\{x_{1,j}, \dots x_{n-j,j} \mid 1 \leq j \leq k\}$ and relations $[x_{i,j}, x_{l,m}] = \delta_{i+j,l} x_{i,j+m}$, for all $i < l, 1 \leq j, m \leq k$, where δ is the Kronecker delta. The centre of $\mathfrak{t}_{n,k}$ is $\mathrm{span}\{x_{i,k} \mid 1 \leq i \leq n-k\}$ and has dimension $n-k$. Let α be the product of the 1-forms $x^*_{i,j}$ for all $1 \leq j \leq k$ and $[i] \geq k-j$, where $[i] \in \{0, \dots, k-1\}$ denotes the remainder of i on division by k. A computation shows that α is closed. Consider the orientation $\zeta^* = x^*_{1,k} \cdots x^*_{n-k,k}$ of Z^*, and let $\beta = i_{\zeta^*}\alpha$; note that β is the product of the 1-forms $x^*_{i,j}$ where $j < k$ and $[i] \geq k-j$. It is not difficult to verify that β is not exact, and thus the central representation is faithful on $[\alpha]$. We remark that $\mathfrak{t}_{n,k}$ has an obvious grading (for which $x^*_{i,j}$ is in level j), and the TRC for $\mathfrak{t}_{n,k}$ can also be deduced from [7].

Example 4.4 Algebras of the form $K/Z(K)$ with one dimensional centres. As shown in [5], a Lie algebra L is of the form $K/Z(K)$ for some algebra K if and only if the restriction $i^*\colon Z(L) \to Hom(H^2L, H^1L)$ of the central representation to H^2L is injective. In this case, the central representation is non-trivial (provided $Z(L) \neq 0$), and it is faithful if $Z(L)$ has dimension 1. Well known examples of nilpotent Lie algebras of this kind include:

1. The standard filiform algebra $\mathfrak{f}_m = \mathrm{span}\{x, y_1, \dots, y_m \mid [x, y_i] = y_{i+1}\}$, for which $Z(\mathfrak{f}_m) = \mathrm{span}\{y_m\}$ and $\mathfrak{f}_m = \mathfrak{f}_{m+1}/Z(\mathfrak{f}_{m+1})$.

2. The truncated nilpotent algebra of polynomial differential operators on the real line: $\mathfrak{s}_n = \operatorname{span}\{x^i \frac{\mathrm{d}}{\mathrm{d}x} \mid 2 \le i \le n+1\}$, for which $Z(\mathfrak{s}_n) = \operatorname{span}\{x^{n+1}\frac{\mathrm{d}}{\mathrm{d}x}\}$ and $\mathfrak{s}_n = \mathfrak{s}_{n+1}/Z(\mathfrak{s}_{n+1})$.

When $Z(L)$ has dimension 1, the map $i\colon Z(L) \to Hom(H^2L, H^1L)$ is injective precisely when there exists $\Omega \in H^2L$ such that the evaluation map $\Omega\colon Z(L)\otimes L \to \mathbb{R}$ is non-zero. Such a class Ω is called an *affine cohomology class* in [3]. In summary, one has: *if a Lie algebra has a centre of dimension 1 and it possesses an affine cohomology class, then its central representation is faithful.*

The algebras $\mathfrak{f}_n$ and $\mathfrak{s}_n$ are all filiform; that is, the possess an element of maximum possible nilpotency class. Filiform algebras all have centres of dimension 1, but while many possess an affine cohomology class, some do not [3].

The Heisenberg algebra $\mathfrak{h}_m = \operatorname{span}\{x_i, y_i, z \mid i = 1, \dots, m,\ [x_i, y_i] = z\}$ has a centre of dimension 1 but for $m > 1$ it is not of the form $K/Z(K)$. Nevertheless, its central representation is faithful; i_z^* is zero except from $H^{m+1}\mathfrak{h}_m \to H^m\mathfrak{h}_m$, where it is an isomorphism (this is clear from the explicit computation of the cohomology of $\mathfrak{h}_m$ given in [14], and is also proven in [12]).

§5 Examples where the Central Representation is not Faithful

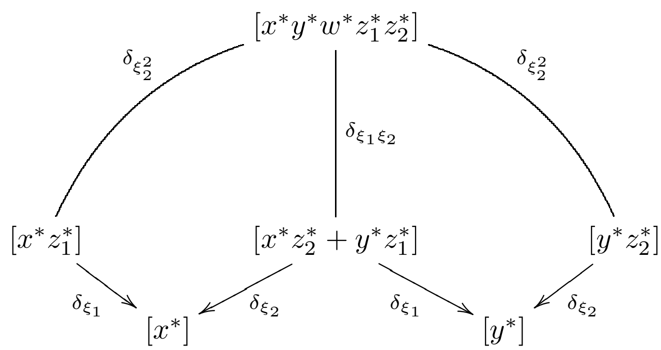
Recall that a central extension $0 \to C \to L \to K \to 0$ of a Lie algebra K is determined up to isomorphism by a linear map $C^* \xrightarrow{t} H^2K$. In several of the examples below, we will consider the canonical central extension determined by $H^2K \xrightarrow{t=\mathrm{id}} H^2K$; in this case, we will call L the *full central extension* of K, and denote it $\mathcal{F}(K)$.

Example 5.1. Consider the 5-dimensional algebra $\mathcal{F}(\mathfrak{h}_1) = \operatorname{span}\{x, y, w, z_1, z_2 \mid [x, y] = w, [x, w] = z_1, [y, w] = z_2\}$; this is the free 3-step nilpotent Lie algebra on 2 generators (also denoted $F_2(3)$). It is a 2-dimensional central extension of $\mathfrak{h}_1$, with centre $Z = \operatorname{span}\{z_1, z_2\}$, and its central representation is not faithful. Its $\Lambda(z_1, z_2)$-module structure is depicted in the diagram below (in this and subsequent diagrams in this paper, arrows marked $[x^*z_1^*] \xrightarrow{i_{z_1}^*} [x^*]$, for example, are to be interpreted to mean $0 \neq i_{z_1}^*[x^*{z_1}^*] \in \operatorname{span}\{[x^*]\}$):

$$\begin{tikzcd}[column sep=small]
 & & {[x^*y^*w^*z_1^*{z_2}^*]} & & \\
 & {[x^*w^*z_1^*{z_2}^*]} \arrow[dl, "i_{z_2}^*"'] \arrow[dr, "i_{z_1}^*"] & & {[y^*w^*z_1^*{z_2}^*]} \arrow[dl, "i_{z_2}^*"'] \arrow[dr, "i_{z_1}^*"] & \\
{[x^*w^*z_1^*]} & & {[x^*w^*{z_2}^*] = [y^*w^*z_1^*]} & & {[y^*w^*{z_2}^*]} \\
{[x^*z_1^*]} \arrow[dr, "i_{z_1}^*"'] & & {[x^*{z_2}^* + y^*z_1^*]} \arrow[dl, "i_{z_2}^*"] \arrow[dr, "i_{z_1}^*"'] & & {[y^*{z_2}^*]} \arrow[dl, "i_{z_2}^*"] \\
 & {[x^*]} & & {[y^*]} &
\end{tikzcd}$$

1

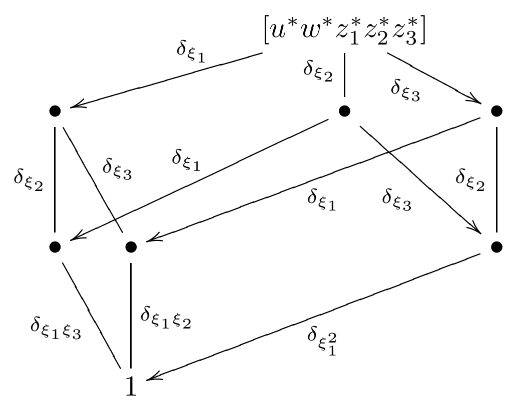
Under the algebra A of higher operations, $H^*\mathcal{F}(\mathfrak{h}_1)$ decomposes as the direct sum of two (dual) irreducible bimodules; here is one of the them:



Example 5.2. The algebra $3, 5, 7_B$ from Seeley's classification ([15]) of nilpotent Lie algebras of dimension 7 is

$$\mathrm{span}\{u, v, w, x, z_1, z_2, z_3 \mid [u, v] = w, [u, w] = z_1, [v, w] = z_2, [u, x] = z_3\},$$

which has centre $\mathrm{span}\{z_1, z_2, z_3\}$. The central representation is not faithful, but the cohomology contains the following "cube" (which is an i-module, but not a bimodule), where the lower three edges are secondary operations:



Example 5.4. The Heisenberg algebras $\mathfrak{h}_m$ are examples of algebras with higher operations of arbitrary high degree; here $Z(\mathfrak{h}_m) = \mathrm{span}\{z\}$, and in $H^*\mathfrak{h}_m$ the operations δ_{ξ^i} are non-zero for $1 \leq i \leq m$. Details are given in [12].

Example 5.5. Consider the 6-dimensional nilpotent Lie algebras:

$$\begin{aligned} L_1 = \mathrm{span}\{u, v, w, x, y, z \mid &[u, v] = w, [u, w] = x, [u, x] = y, \\ &[v, y] = z, [w, x] = -z\}, \\ L_2 = \mathrm{span}\{u, v, w, x, y, z \mid &[u, v] = w, [u, w] = x, [u, x] = y, \\ &[v, w] = y, [v, y] = z, [w, x] = -z\}. \end{aligned}$$

L_1 and L_2 are not isomorphic since their quotients $L_1/Z(L_1) \cong \mathfrak{f}_4$ and $L_2/Z(L_2) \cong \mathfrak{s}_5$ are not isomorphic. However, it is easy to see that the cohomology spaces H^*L_1 and H^*L_2 are isomorphic as A-modules, where $A = (\delta_s \mid s \in S)$ is the algebra of all higher operations. It follows from [1] that the complexes ΛL_1^* and ΛL_2^* are homotopic as differential ΛZ-modules, and in this case one can construct such a homotopy equivalence by first defining it from $\mathcal{H}(L_1)/A^+\mathcal{H}(L_1) \to \Lambda L_2^*$, extending as an A-module map, and then pre-composing with π_{L_1}. Furthermore, H^*L_1 and H^*L_2 inherit an algebra structure from ΛL_1^* and ΛL_2^* respectively, and it is not difficult to show that there is an A-module isomorphism $H^*L_1 \to H^*L_2$ which is also an algebra isomorphism.

Example 5.6. Consider the 7-dimensional nilpotent Lie algebras:

$$\begin{aligned} L_1 = \operatorname{span}\{y_1, \ldots, y_7 \,|[y_1, y_i] = y_{i+1}, \text{ for } i = 1, \ldots, 6,\\ [y_2, y_5] = y_7, [y_3, y_4] = -y_7\},\\ L_2 = \operatorname{span}\{y_1, \ldots, y_7 \,|[y_1, y_i] = y_{i+1}, \text{ for } i = 1, \ldots, 6,\\ [y_2, y_3] = y_6, [y_2, y_4] = y_7, [y_2, y_5] = y_7, [y_3, y_4] = -y_7\}. \end{aligned}$$

(These are respectively algebras $1, 2, 3, 4, 5, 7_C$ and $1, 2, 3, 4, 5, 7_F$ of [15].) For both algebras, $Z = \operatorname{span}\{y_7\}$. One finds that H^*L_1 and H^*L_2 are isomorphic as ΛZ-modules, but not as A-modules, where $A = (\delta_s \mid s \in S)$ is the algebra of all higher operations. Indeed, $\delta_{\xi^3} \equiv 0$ for L_1, while $\delta_{\xi^3} \not\equiv 0$ for L_2.

Example 5.7. Consider the following semi-direct product of $sl_2(\mathbb{R})$ and $\mathfrak{h}_1$:

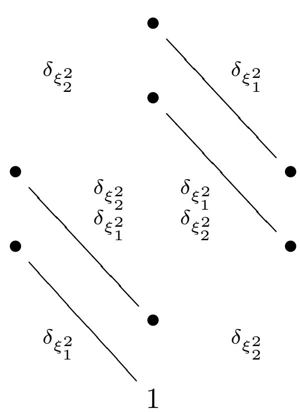
$$\begin{aligned} L = \operatorname{span}\{u, v, w, x, y, z \,|[u, v] = w, [u, w] = -2u, [v, w] = 2v,\\ [u, y] = x, [v, x] = y, [w, x] = u, [w, v] = -y, [x, y] = z\}. \end{aligned}$$

This algebra is unimodular and non-solvable; its centre is $\operatorname{span}\{z\}$. The Betti numbers of L are $1, 0, 0, 2, 0, 0, 1$, and so the central representation is completely trivial, as are the higher operations δ_s for polynomials s of (polynomial) degree ≥ 3. There are only two non-zero secondary operations: $\delta_\xi\colon [u^*v^*w^*x^*y^*z^*] \mapsto -\frac{1}{2}[u^*v^*w^*]$ and $\delta_\xi\colon [x^*y^*z^*] \mapsto -\frac{1}{2}$, which give 2 dual irreducible bimodules of dimension 2.

Example 5.8. The 7-dimensional solvable Lie algebra

$$L = \operatorname{span}\{w, x_i, y_i, z_i \,|[x_i, y_i] = z_i, [w, x_i] = i\, x_i, [w, y_i] = -i\, y_i, \text{ for } i = 1, 2\}$$

has centre $Z = \operatorname{span}\{z_1, z_2\}$. One finds that H^*L has dimension 8, the central representation is trivial, as are the higher operations δ_s for polynomials s of degree ≥ 3. The non-zero secondary operations give 2 dual irreducible bimodules of dimension 4:



As we saw in Proposition 3.8, every Lie algebra has non-trivial cohomology operations, of some sufficiently high degree. In the previous two examples, which are non-nilpotent, the central representation (i.e., the primary operations) is completely trivial. We conclude with:

Conjecture 5.9. Every nilpotent Lie algebra has a non-trivial central representation.

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