Explicit formulae for the rational L-S category of some Homogeneous spaces

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Abstract

Explicit formulae for rational L-S category (cat_0) are rare, but some are available for a class of spaces which includes homogeneous spaces G/H when H is a product of at most 3 rank 1 groups, and rank $G-$ rank $H \leq 1$. We extend the applicability of these formulae to the case when rank $G = 5$ and H is a $4-$torus or $(SU_2)^4$. With a Sullivan minimal model as data, implementing the formula requires the selection of a regular subsequence of length 4 from a sequence $f_1, \ldots, f_5$ of homogeneous polynomials in 4 variables satisfying

$$\dim \mathbf{Q}[x_1, \ldots, x_4]/(f_1, \ldots, f_5) < \infty.$$

Such subsequences are readily obtainable, and the ease of computation is in contrast to most available methods for determining rational L-S category, which usually involve both upper and lower bounds and a good measure of luck.

The proof of the formula is a pretty application of ideal class groups in algebraic topology. We also present some examples to illustrate our result.

1 Introduction

The Lusternik-Schnirelmann [LS] category $\mathrm{cat}T$ of a topological space T is the least number of contractible (in T) open sets need to cover T, less one. It is a subtle homotopy invariant which is usually difficult to compute. The difficulties are attenuated somewhat by localizing at the rationals, where Felix and Halperin [FH] used Sullivan models [S] to provide a tractable algebraic characterization of $\mathrm{cat}_0 T := \mathrm{cat}\, T_{\mathbf{Q}}$, the L-S category of the rationalization $T_{\mathbf{Q}}$ of a simply connected CW complex.

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Moreover, Félix, Halperin and Lemaire [FHL] recently established a long-standing conjecture that the rational L-S category of an *elliptic* space (i.e. where $\dim \pi_*(T) \otimes \mathbf{Q} + \dim H^*(T;\mathbf{Q}) < \infty$) is the same as the rational Moore – Toomer invariant, $\mathrm{e}_0 T$, which is the largest integer p such that in the spectral sequence of Milnor and Moore, $E_\infty^{p,\star} T \neq 0$ [T]. This reduces the calculation of $\mathrm{cat}_0 T$ to the problem of finding a "longest" representative of the top cohomology class. However, even if one has (rationally) complete algebraic data such as a Sullivan minimal model, considerable obstacles remain, and much effort continues to be spent in obtaining estimates. (See [C],[CFJP],[CJ],[GJ], [JS] and [J3], for example.)

Here we consider the case motivated by the example of a homogenous space G/H, where G is a compact, connected Lie group and H is either an embedded $n-$torus or $(SU_2)^n$. The value of $\mathrm{cat}_0 = e_0$ may be found as follows. If X is a graded vector space, let ΛX denote the (graded-commutative) algebra $\mathbf{Q}[X^{\mathrm{even}}] \otimes \Lambda X^{\mathrm{odd}}$, where the second factor is an exterior algebra). A Sullivan minimal model for G/H is of the form

$$(1) \qquad (\Lambda(x_1, ..., x_n, y_1, ...y_r), d)$$

where $|x_i| = 2$ if H is an embedded $n-$torus or $|x_i| = 4$ if $H = (SU_2)^n$, the y_j are of odd degree, r is the rank of G and dy_j is a homogeneous polynomial in the cocyles x_i [GHV, Chp. XI]. It is a *model* of G/H in the sense that (in particular), $H^*(G/H;\mathbf{Q}) \cong H^*(\Lambda(x_1, ..., x_n, y_1, ...y_r), d)$ as algebras. If $\Lambda^k X$ denotes the subspace generated by monomials of word length k, then for any class $\beta \in H^*(G/H)$, define the *length* of β to be

$$e_0(\beta) = \max\left\{k \mid \exists\, b \in \Lambda^{\geq k}(x_1, ..., x_n, y_1, ...y_r) \text{ with } [b] = \beta\right\},$$

Then, $e_0(G/H)$ is simply the maximum of all the lengths of non-zero cohomology classes [FH], and since $H^*(G/H)$ is a Poincaré duality algebra, it is clear that this occurs for the top class α, whose representatives lie in $\mathbf{Q}[x_1, ..., x_n] \otimes \Lambda^{r-n}(y_1, ...y_r)$ [GHV, p. 78].

In the special case when rank H = rank G, i.e., $r = n$, all representatives of α are in the subalgebra $\Lambda(x_1, ..., x_r)$, and so for degree reasons they will all have the same length, which we can compute by noting [H] that $\dim G/H = n(1-d) + \sum_{i=1}^n |y_i|$:

$$(2) \qquad e_0(G/H) = \frac{1}{d} \dim G/H = -n + \sum_i D_i,$$

where $D_i = \frac{|y_i|+1}{2}$ is the degree of the polynomials dy_i, and $d = 2$ or 4 respectively when H is an $n-$ torus or $(SU_2)^n$. In particular, when rank H = rank G, $e_0(G/H)$ depends only on the graded vector space $\pi_*(G/H) \otimes \mathbf{Q}$, i.e., the degrees of the generators in a minimal model[1].

When rank H is not maximal, the situation is more complicated, since then the top class has its representatives in $\mathbf{Q}[x_1, ..., x_n] \otimes \Lambda^{r-n}(y_1, ...y_r)$ and, as the degrees of the y_j may not be the same, different representatives of α may have different lengths[2]. We can however

[1] This simplicity is not totally unexpected, since in this case, G/H is a *formal* space, meaning that its complete rational homotopy type (and hence its model) is determined by the algebra $H^*(G/H)$.

[2] Suppose $dy_1 = x_1^2, dy_2 = x_2^4$ and $dy_3 = x_1x_2^3$. Then $a = x_1x_2^3y_3 - x_2^5y_1$ and $a' = x_1^2x_2y_2 - x_2^5y_1$ are representatives of the top class of different lengths.

always assume that some subsequence $dy_{i_1}, ..., dy_{i_n}$ is a regular sequence in the polynomial ring $\mathbf{Q}(x_1, ..., x_n)$[3][J1, Lemma 3.3].

In particular, when $r = n+1$, there is an j such that $dy_1, \ldots, \widehat{dy_j}, \ldots, dy_{n+1}$ is a regular sequence, and dy_j is a zero divisor in the quotient $\mathbf{Q}[x_1, ..., x_n]/(dy_1, \ldots, dy_{j-1})$. One may also assume that $|y_i| \leq |y_{i+1}|$ for all i. Straightforward attempts [J1] to compute e_0 in this case lead one to pose a natural algebraic question (conjecture 2.3), which implies the following formula for e_0.

Conjecture 1.1 *Suppose a space T has a minimal model of the form (1), but where the x_i may now have any (fixed) even topological degree. Let D_i denote the degree of dy_i as a polynomial*[4]*. If $dy_1, \ldots, \widehat{dy_j}, \ldots, dy_r$ is a regular sequence in $\mathbf{Q}[x_1, ..., x_{r-1}]$ and dy_j is a zero divisor in the quotient by $(dy_1, \ldots, dy_{j-1})$, then*

$$e_0(T) = 2 - r + \sum_{i \neq j} D_i. \tag{3}$$

This is known [J1] to be true when $j = 1, 2, r-1$ or r, and hence in general for $r \leq 4$.

The principal result of this note is

Theorem 1.2 *Conjecture 1.1 holds for $j = 3$ (and hence for $r \leq 5$) when dy_1 or dy_2 has a factor of degree 2.*

In the case of a homogeneous space G/H, one knows that dy_1 is of degree 2, since $|y_1| = 3$ and dy_1 is essentially the restriction of the Killing form of G to H [GHV, Chp. XI].

This paper is organized as follows. In the next section we establish that conjecture 1.1 follows from the purely algebraic conjecture 2.3 below. We then deduce theorem 1.2 from proposition 2.6, which yields some special cases of 2.3. In section 3, we give the proof of proposition 2.6, and in the last section we give further evidence for conjecture 2.3 and end with two examples.

2 Reduction to commutative algebra

In the sequel, we will let $r = n + 1$ for convenience. Since all the above formulae remain valid if one works over $\mathbf{C}$ rather than $\mathbf{Q}$ [J2, Thm 4], we shall do this henceforth, and will denote $\mathbf{C}[x_1, \ldots, x_n]$ by R.

As above, we suppose that a space T has a minimal model of the form

[3]This is not always possible when the degrees of the x_i are not the same, as in the example $\Lambda(x_1, x_2, y_1, y_2, y_3; d)$ with $|x_1| = 6$, $|x_2| = 8$, $dy_1 = x_1(x_1^4 + x_2^3), dy_2 = x_2(x_1^4 + x_2^3)$ and $dy_3 = x_1^3 x_2^2$, where the computation of e_0 is more difficult. Estimates from [CJ] show that $\mathrm{e}_0 \geq 8$.

[4]We may assume that each $dy_i \neq 0$ because e_0 is additive on products [T].

$$(\Lambda X \otimes \Lambda Y; d) = (R \otimes \Lambda(y_1, \dots, y_{n+1}); d),$$

where $dy_i = f_i \in R$ and $|f_j| \leq |f_{j+1}|$. We bigrade this model by defining $(R \otimes \Lambda Y)^n_j := (R \otimes \Lambda^j Y)^n$, and, since the differential is homogeneous of bidegree $(1, -1)$ (in the order: topological, lower), this induces a bigradation on the cohomology which we will write as $H^p_* = \sum_j H^p_j$. A key fact for us is that $H^*_1 \neq 0$, and $H^*_{>1} = 0$ [GHV, p. 78].

Lemma 2.1 *(cf.[J1, P. 51]) Suppose that $f_1, ..., \widehat{f_i}, ..., f_{n+1}$ is a regular sequence in R. Then,*

$$e_0(T) \leq 1 - n + \sum_{j \neq i} D_j.$$

Proof. Let $(\Lambda U, d)$ denote the (formal) model $(R \otimes \Lambda(y_1, \dots, \widehat{y_i}, \dots, y_{n+1}); d)$. By analogy with the case of maximal rank, we know that

$$\mathrm{cat}_0(\Lambda U, d) = \mathrm{e}_0(\Lambda U, d) = -n + \sum_{j \neq i} D_j.$$

Since the model of T is just $(\Lambda U \otimes \Lambda y_i, d)$, by [FH, Lemma 6.6],

$$\mathrm{e}_0(T) = \mathrm{cat}_0(T) \leq \mathrm{cat}_0(\Lambda U, d) + 1 = 1 - n + \sum_{j \neq i} D_j. \quad \square$$

Now suppose that $f_1, \dots, \widehat{f_i}, \dots, f_{n+1}$ is a regular sequence in R and that f_i is a zero divisor in $R/(f_1, \dots, f_{i-1})$. Since any re-ordering of the (homogeneous) elements in a regular sequence in R is still a regular sequence, we may assume that $|f_j| \leq |f_{j+1}|$, $j = 1, \dots, n$, and that $i < j \Rightarrow D_j > D_i$.

We now show that conjecture 1.2 is equivalent to conditions I and II in the

Lemma 2.2 *Suppose $f_1, \dots, \widehat{f_i}, \dots, f_{n+1}$ is a regular sequence in R and that f_i is a zero divisor in $R/(f_1, \dots, f_{i-1})$. Then*

$$e_0(T) = 1 - n + \sum_{j \neq i} D_j$$

iff there is $h \in R$ such that

I. $hf_i \in (f_1, \dots, f_{i-1})$, and

II. $h \notin (f_1, \dots, \widehat{f_i}, \dots, f_{n+1})$.

Proof. Suppose that $\mathrm{e}_0(T) = 1 - n + \sum_{j \neq i} D_j$. A straightforward degree argument shows that there is a representative of the top class of the form $\alpha = [hy_i + \sum_{j<i} \beta_j y_j]$, where h and $\beta_1, \dots, \beta_{i-1}$ are homogeneous polynomials in the x_i. The fact that this is a cycle shows that (I) is true. To see that (II) holds, let $(\Lambda U, d)$ denote the (formal) model $(R \otimes$

$\Lambda(y_1, \ldots, \widehat{y_i}, \ldots, y_{n+1}); d)$, as in lemma 2.1. The Gysin sequence associated to the fibration $(\Lambda U, d) \to (\Lambda U \otimes \Lambda y_i, d) \to (\Lambda y_i, 0)$ is of the form

$$\cdots \to H^N(\Lambda U, d) \xrightarrow{q} H^N(\Lambda U \otimes \Lambda y_i, d) \xrightarrow{p} H^{N-|y_i|}(\Lambda U, d) \to H^{N+1}(\Lambda U, d) \to \cdots,$$

where q is induced by the inclusion and $p([\varphi + \psi y_i]) = [\psi]$, for $\varphi, \psi \in \Lambda U$. If $h \in (f_1, ..., \widehat{f_i}, ..., f_{n+1})$, then $p(\alpha) = h = 0$ in $H^*(\Lambda U, d) = R/(f_1, ..., \widehat{f_i}, ..., f_{n+1})$, so by exactness, $\alpha = q([\beta])$ for some polynomial $\beta \in R$. But the differential is homogeneous in the lower degree, and $0 \neq \alpha \in H_1$, so this is impossible. Hence, (II) holds.

Now suppose that there is a h satisfying (I) and (II). Lemma (2.1) then shows that it suffices to show there is $\alpha \neq 0$ with $e_0(\alpha) \geq 1 - n + \sum_{j \neq i} D_j$. However, (I) implies that that there are homogeneous polynomials $h, \beta_1, \ldots, \beta_{i-1}$ such that $d(hy_i + \sum_{j<i} \beta_j y_j) = 0$. If we let $\gamma = hy_i + \sum_{j<i} \beta_j y_j$, to see that $[\gamma] \neq 0$, simply note that $p[\gamma] = [h]$, which is non-zero in $H^*(\Lambda U, d)$, by (II). Using the Poincaré duality in $H^*(T)$, we may now multiply γ up to a representative α of the top class. A straightforward degree and length counting argument shows that $\mathrm{e}_0(\alpha) \geq 1 - n + \sum_{j \neq i} D_j$, completing the proof of the lemma. □

As usual, for an ideal $\mathfrak{a}$ and a polynomial g, we denote $(\mathfrak{a} : g) = \{h \mid hg \in \mathfrak{a}\}$. With lemma 2.2 in mind, we now make the promised algebraic

Conjecture 2.3 *Let $g_1, \ldots, g_n$ be a regular sequence of homogeneous polynomials in the ring $\mathbf{C}[x_1, \ldots, x_n]$, written in order of increasing degree, and consider the ideals $\mathfrak{a} = (g_1, \ldots, g_i)$ and $\mathfrak{b} = (g_{i+1}, \ldots, g_n)$, where $1 \leq i \leq n$. Suppose further that $\deg g_i < \deg g_{i+1}$. If $g \notin \mathfrak{a}$ is any polynomial satisfying $\deg g_i \leq deg\, g < \deg g_{i+1}$, such that $(\mathfrak{a} : g) \neq \mathfrak{a}$, then there exists a polynomial h such that*

I. $h \in (\mathfrak{a} : g)$

II. $h \notin \mathfrak{a} + \mathfrak{b}$

For completeness we state the following proposition, which is an obvious consequence of lemma 2.2 (upon noting that the r and j of 1.1 are, respectively, $n + 1$ and $i - 1$ in the notation of 2.3).

Proposition 2.4 *Conjecture 2.3 implies conjecture 1.1.*

With the obvious modifications to hypotheses and conclusions for $i = 0$ and n understood, it is known that conjecture 2.3 holds for $i = 0, 1, n - 1$, and n [J1].

We now make the following important reduction:

Lemma 2.5 *If Conjecture 2.3 holds for a fixed regular sequence and some fixed i with $g_1 = u$, then, for any homogeneous v such that $uv, g_2, \ldots, g_n$ is also a regular sequence, the conclusion of Conjecture 2.3 holds for the same i. In particular, it suffices to prove Conjecture 2.3 in the case where each of $g_1, \ldots, g_i$ is irreducible.*

Proof. Suppose g is a non-trivial zero divisor modulo $(uv, g_2, \ldots, g_i)$. If $g \in (u, g_2, \ldots, g_i)$, then $vg \in (uv, g_2, \ldots, g_i)$. Moreover, $v \notin (uv, g_2, \ldots, g_n)$, otherwise $v \in (g_2, \ldots, g_i)$ for degree reasons, and so the sequence $uv, g_2, \ldots, g_n$ would not be regular.

If $g \notin (u, g_2, \dots, g_i)$ then g is a non-trivial zero divisor modulo the latter ideal and by hypothesis we get $h' \notin (u, g_2, \dots, g_n)$ such that $h'g \in (u, g_2, \dots, g_i)$, and so $h = h'v$ clearly satisfies $hg \in (uv, g_2, \dots, g_i)$. If $h \in (uv, g_2, \dots, g_n)$, then for some polynomial a, $(h' - au)v \in (g_2, \dots, g_n)$. However, v is not a zero divisor modulo this ideal, because of the regularity of the sequence $uv, g_2, \dots, g_n$, and the fact that any re-ordering of a regular sequence (of homogeneous polynomials) is still a regular sequence. This yields the contradiction $h' \in (u, g_2, \dots, g_n)$, so we must have $h \notin (uv, g_2, \dots, g_n)$ as required.

To see that it suffices to prove Conjecture 2.3 in the case where each of $g_1, \dots, g_i$ is irreducible, simply note that $u, g_2, \dots, g_n$ is a regular sequence whenever $uv, g_2, \dots, g_n$ is, and use the first part of the lemma. □

We will now show that Theorem 1.2 follows from the following

Proposition 2.6 *Conjecture 2.3 is true for $i = 2$ if g_1 is irreducible of degree two.*

Proof of theorem 1.2. Using lemma 2.5 and proposition 2.6, conjecture 2.3 holds when $i = 2$, if g_1 has an irreducible factor of degree 2. If g_1 has a reducible factor of degree two then it has a linear factor and so one can suppose g_1 is linear. We are then reduced to the case of one less variable, and $i = 1$, where conjecture 2.3 holds by [J1]. Moreover, since the order of the generators for $\mathfrak{a}$ or $\mathfrak{b}$ is irrelevant in conjecture 2.3, it is clear that proposition 2.6 implies that conjecture 2.3 is true if $i = 2$ and either of g_1 or g_2 has a factor of degree 2.

Finally, lemma 2.1 now shows that conjecture 1.1 holds for $i = 3$ if either dy_1or dy_2 has a factor of degree 2. □

3 Proof of Proposition 2.6

In this section we will prove Proposition 2.6, and we keep the notation of conjecture 2.3. The requisite commutative algebra may be found in [Ha],or [AM].

Since the g_j form a regular sequence, the ring $S_i = \mathbf{C}[x_1, \dots, x_n]/(g_1, \dots, g_i)$ is of pure dimension $n-i$ and the zero ideal has no embedded prime ideals. It follows that all associated prime ideals of the annihilator $(0 : g)$ have height zero in S_i, or, equivalently, that all associated prime ideals of $(g_i : g)$ have height one in $S := S_{i-1}$. One can then see that conjecture 2.3 is equivalent to showing that $(g_i : g) \not\subset (g_i, g_{i+1}, \dots g_n)$, in S. In particular, the conjecture intrinsically concerns the ideal $(g_i : g)$ in S and not the explicit polynomial g. When S is a normal domain (see definition 3.2), any ideal in S having only height one associated primes, is in fact of the form $(g_i : g)$ for some pair g, g_i and, in this case, the conjecture concerns all ideals in $I_1(S)$ (see after 3.3). We shall use this fact to reformulate proposition 2.6 in the form of 3.1 below, but first we fix some notation.

Let J^k denote the elements of degree k of a homogeneous ideal J. Since

$$(g_i, g_{i+1}, \dots, g_n)^{\deg g_i} = \langle g_i \rangle$$

for reasons of degree, if one can show that in the ring S one has

$$\dim(g_i : g)^{\deg g_i} \geq 2,$$

then the conclusion of conjecture 2.3 follows.

We now specialize to the case $i = 2$. In view of the above discussion, Proposition 2.6 now follows from

Proposition 3.1 *Let $g_1, g_2 \in \mathbf{C}[x_1, \dots, x_n]$ be a regular sequence with g_1 a quadratic form of rank ≥ 3 (hence irreducible), and g_2 of degree $d \geq 2$. If $S = \mathbf{C}[x_1, \dots, x_n]/(g_1)$, then for any homogeneous ideal $\mathfrak{a} \subset S$, all of whose associated primes have height one and which strictly contains the ideal (g_2), $\dim_{\mathbf{C}} \mathfrak{a}^d \geq 2$.*

Though we only work with the quotient of a polynomial ring by a quadratic form, we will now recall some results from commutative algebra for normal domains.

Definition 3.2 *A normal domain is a Noetherian integral domain which is integrally closed in its field of quotients.*

Remarks 3.3

1. Being normal is a local property (see [AM, 5.13]). Krull has shown that the normality of A is equivalent to the conjunction of the following two properties:

1. *$A_{\mathfrak{p}}$ is a PID for all height one prime ideals $\mathfrak{p}$ of A (i.e. is regular)*
2. *If $f \in A$ is neither zero nor a unit then every associated prime of the ideal (f) has height one.*

2. A Noetherian ring is a UFD if and only if every height one prime ideal is principal ([Ma] p.141), and every UFD is normal.

3. The ring $\mathbf{C}[x_1, \dots, x_n]/(x_1^2 + \dots + x_r^2)$ is normal for $r \geq 3$ and a UFD for $r \geq 5$ ([Ha] ch 2, §6, ex 6.5).

Now let $S = \oplus_{k \geq 0} S^k$ be a graded normal domain and let $I_1(S)$ be the set of homogeneous ideals in S, all of whose associated prime ideals have height one. Note that these associated primes are themselves homogeneous. The primary decomposition of any $\mathfrak{a} \in I_1(S)$ is then unique and has the form

$$\mathfrak{a} = \mathfrak{p}_1^{(n_1)} \cap \dots \cap \mathfrak{p}_s^{(n_s)}$$

where the $\mathfrak{p}_i$ are the associated primes of $\mathfrak{a}$ and $\mathfrak{p}^{(n)}$ is the nth symbolic power of $\mathfrak{p}$, i.e. the contraction of $\mathfrak{p}^n S_{\mathfrak{p}}$ in S. As usual, $\mathfrak{p}^{(0)} = S$.

It is well known that the ideals $\mathfrak{a}, \mathfrak{b} \in I_1(S)$ are isomorphic as graded ideals, if and only if $f\mathfrak{a} = g\mathfrak{b}$ for some $f, g \in S^k$ with $f \neq 0 \neq g$. Note that $\mathfrak{a} \subseteq \mathfrak{b}$ and $\mathfrak{a} \simeq \mathfrak{b}$ together imply that $\mathfrak{a} = \mathfrak{b}$. Since $(f) \simeq (g)$ for any 2 non-zero $f, g \in S^k$, we define a graded vector space

$S(-k)$ by $S(-k)^l = S^{l-k}$, and when we write $\mathfrak{a} \simeq S(-k)$, we shall mean that $\mathfrak{a} \simeq (f)$ for some f with $\deg f = k$. In particular, $\mathfrak{a} \simeq S(-k)$ will imply that $\mathfrak{a} \cong S(-k)$ as graded vector spaces.

Further facts we will need are as follows:

Remarks 3.4

1. For any $f \in S$ that is neither zero or a unit, the principal ideal (f) is in $I_1(S)$.

2. If $\mathfrak{a} \in I_1(S)$ and $f \in S$ is neither zero or a unit, then $f\mathfrak{a} \in I_1(S)$.

3. If $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{a}\mathfrak{b}$ are all in $I_1(S)$ with primary decompositions $\mathfrak{a} = \mathfrak{p}_1^{(n_1)} \cap \cdots \cap \mathfrak{p}_s^{(n_s)}$ and $\mathfrak{b} = \mathfrak{p}_1^{(m_1)} \cap \cdots \cap \mathfrak{p}_s^{(m_s)}$ respectively (where some n_i or m_i may be zero) then $\mathfrak{a}\mathfrak{b} = \mathfrak{p}_1^{(n_1+m_1)} \cap \cdots \cap \mathfrak{p}_s^{(n_s+m_s)}$.

4. If S is a UFD, then every $\mathfrak{a} \in I_1(S)$ is principal, and so $\mathfrak{a} \simeq S(-m)$ for some positive integer m. Moreover, if $f \in S^k$ is homogeneous of degree k and $S_f := S[\frac{1}{f}]$ is a UFD, then every ideal $\mathfrak{a} \in I_1(S)$ is isomorphic as a graded ideal to an ideal of the form $\mathfrak{p}_1^{(n_1)} \cap \cdots \cap \mathfrak{p}_s^{(n_s)}$, where $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ are the associated primes of the principal ideal (f). (These follow from standard facts about the (homogeneous) divisor class group of a normal domain; see for example [Ha] ch 2, §6.)

Proof of proposition 3.1

We treat three cases. Recall that $S = \mathbf{C}[x_1, \ldots, x_n]/(g_1)$ and let $r = \text{rank } g_1$. Note that $n \geq r$, so that $\dim S^k \geq 2$ whenever $k > 0$.

Case (1)
For $r \geq 5$, S is a UFD by 3.3 (3), so that by 3.4 (4), every ideal $\mathfrak{a} \in I_1(S)$ is isomorphic to $S(-m)$ for some $m > 0$. If an ideal $\mathfrak{a} \simeq S(-m)$ strictly contains a principal ideal $(g_2) \simeq S(-d)$, then $d > m$ and so $\mathfrak{a}^d$ has dimension $\dim(S^{d-m}) \geq 2$.

Case (2)
When $r = 4$, S is normal by 3.3 (3), and we may write g_1 as $x_1x_2 - x_3x_4$ after a change of basis.

Since $S(-1) \simeq (x_1) = (x_1, x_3) \cap (x_1, x_4)$, the ideals $\mathfrak{p}_1 = (x_1, x_3)$ and $\mathfrak{p}_2 = (x_1, x_4)$ are prime of height one and $\mathfrak{p}_1 \cap \mathfrak{p}_2 = (x_1)$. Since $S_{x_1} \simeq \mathbb{C}[x_1, x_3, x_4, 1/x_1]$ is clearly a UFD, by 3.4 (4), any ideal $\mathfrak{a} \in I_1(S)$ is isomorphic to one of the form $\mathfrak{p}_1^{(m_1)} \cap \mathfrak{p}_2^{(m_2)}$ where $m_1, m_2 \geq 0$ and $m_1 + m_2 > 0$.

If $m_1 = m_2 = m$, then $\mathfrak{p}_1^{(m)} \cap \mathfrak{p}_2^{(m)} = (x_1^m) \simeq S(-m)$ by 3.4 (2,3), and if this strictly contains a principal ideal isomorphic to $S(-d)$, then $d > m$ and $\dim_{\mathbb{C}}[\mathfrak{p}_1^{(m)} \cap \mathfrak{p}_2^{(m)}]^d = \dim S^{d-m} \geq 2$.

By symmetry it remains to consider the ideals

$$x_1^m \mathfrak{p}_1^{(n)} = \mathfrak{p}_1^{(n+m)} \cap \mathfrak{p}_2^{(m)} \subset \mathfrak{p}_1^{(n+m)}$$

for $n > 0$ and $m \geq 0$. Clearly $\dim_{\mathbb{C}}[\mathfrak{p}_1]^1 = 2$ so that $\dim_{\mathbb{C}}[\mathfrak{p}_1]^k \geq 2$ for $k \geq 1$. Also, $x_1^m[\mathfrak{p}_1]^1 \subset \mathfrak{p}_1^{(m+1)}$ so that $\dim_{\mathbb{C}}[\mathfrak{p}_1^{(m+1)}]^d \geq 2$ for $d \geq m+1$. Then, $\dim_{\mathbb{C}}[x_1^m \mathfrak{p}_1^{(n)}]^d = \dim_{\mathbb{C}}[\mathfrak{p}_1^{(n)}]^{d-m} \geq 2$ for $d \geq n+m$. However, if $\mathfrak{p}_1^{(n+m)} \cap \mathfrak{p}_2^{(m)}$ contains a principal ideal isomorphic to $S(-d)$ then $d \geq n+m$, completing the proof of this case.

Case (3)
If $r = 3$, S is again normal by 3.3 (3) and we can write g_1 as $x_1^2 - x_2x_3$. Clearly $\mathfrak{p} = (x_1, x_2)$ is prime of height one in S and $\mathfrak{p}^{(2)} = (x_2) \simeq S(-1)$. Since S_{x_2} is a UFD, by (3.4 (4)), every ideal $a \in I_1(X)$ is either isomorphic to $(x_2^{m+1}) \simeq S(-(m+1))$ or to $\mathfrak{p}^{(2m+1)} \simeq \mathfrak{p}(-m) = \mathfrak{p}(-m)$, where $m \geq 1$. Clearly $\dim_k(\mathfrak{p}^{(2m)})^d = \dim_k S^{d-m} \geq 2$ for $d > m$. As in the previous case we also have $\dim_k(\mathfrak{p}^{(2m+1)})^d \geq \dim_k(\mathfrak{p})^1 = 2$ for $d > 2m+1$. If $p^{(2m)}$ (resp. $p^{(2m+1)}$) strictly contains a principal ideal $\simeq S(-d) \simeq \mathfrak{p}^{(2d)}$ then $m > d$ (resp. $m \geq d$) and consequently $\dim_k(\mathfrak{a})^d \geq 2$. □

4 Remarks and Examples

4.1 Comments on conjecture 2.3

In the terminology of the conjecture, let $S = \mathbf{C}[x_1, \ldots, x_n]/(g_1, \ldots, g_n)$. As we have already said this is a complete intersection ring, hence Gorenstein, and hence Cohen-Macaulay. If $S/(0 : g)$ is a Cohen-Macaulay ring, then, $\deg S/(0 : g) + \mathfrak{b} < \deg S/\mathfrak{b}$, where the degree is that of a graded ring of finite type over $\mathbb{C}$, so that in the ring S we have $(0, g) \neq \mathfrak{b}$ and the conjecture holds for this g.

If all the g_i are general then the conjecture holds trivially, because $i = n$ in this case. If we just consider the case $i = 2$, then for $n \geq 5$ and a general g_1, $\mathbb{C}[x_1, \ldots, x_n]/(g_1)$ is a UFD by the Grothendieck-Lefschetz theorem for hypersurfaces. In this case one can apply the same argument as in the first case of the proof of proposition 3.1. A similar result holds for general g_1 when $n = 4$ provided the degree of g_1 is ≥ 4 (Noether-Lefschetz theorem).

4.2 Examples

We present 3 examples. In the first, we see that previous lower bounds for cat_0 are sharp, but the formula of this paper is much easier to use. The second is an example where the formula is applicable and for which known results are not good enough to determine cat_0. The third gives evidence that conjecture 2.3 is true without the hypotheses of theorem 1.2. All are spaces with model $(\Lambda(x_1, \cdots, x_4, y_1, \cdots, y_5); d)$ where a straightforward calculation shows that dy_1, dy_2, dy_4, dy_5 is a regular sequence in $\mathbf{Q}[x_1, \cdots, x_4]$, and that dy_3 is a non-trivial zero divisor in $\mathbf{Q}[x_1, \cdots, x_4]/(dy_1, dy_2)$.

Example 1. Suppose $dy_1 = x_1x_2^2$, $dy_2 = x_3x_4^3$, $dy_3 = x_1x_3^4$, $dy_4 = x_1^6 + x_2^6$ and $dy_5 = x_3^6 + x_4^6$. The best lower bound for cat_0 obtainable from previously known results is 16, and is found by applying [CJ, Theorem 1] to the fibration with $(\Lambda x_3; 0)$ as base, where the 2-holonomy is weakly trivial. One then computes cat_0 of the fibre using the additivity of cat_0 on products and the equal rank formula 2 of the introduction. Formula 3 of this paper (i.e., the $r = 5$, $i = 3$ case of conjecture 1.1) quickly yields $\mathrm{cat}_0 = 16$, showing this lower bound to be sharp.

Example 2. Here we consider $dy_1 = x_1x_2 + x_3x_4$, $dy_2 = x_1x_3 - x_4^2$, $dy_3 = x_2(x_3^2 - x_2x_4)$, $dy_4 = x_1^4 + x_2^2x_3^2$ and $dy_5 = x_2^4 + x_3^4$. The lower bounds of [CJ] are not applicable, as the holonomy is non-trivial for all choices of base, and the best estimate for cat_0 previously available is $8 \leq \mathrm{cat}_0 \leq 9$. The lower bound is found by applying the Mapping Theorem to the fibration with $(\Lambda x_1, x_4; 0)$ as base, and then computing cat_0 of the fibre using [J1, Theorem 3.2]). The upper bound is a consequence of lemma 2.1. The formula of this paper immediately shows that $\mathrm{cat}_0 = 9$.

Example 3. Here, $dy_1 = x_1^2x_2 + x_3x_4^2$, $dy_2 = x_1x_2x_3 - x_4^3$, $dy_3 = x_4^4$, $dy_4 = x_1^5 + x_2^5$ and $dy_5 = x_3^5$. A lower bound of 13 may be found by applying [GJ, Theorem 1] to the fibration with $(\Lambda x_4; 0)$ as base, and then proceeding as in example 2. Again, lemma 2.1 shows this bound to be sharp. This shows that Conjecture 2.3 is true in this case, though neither dy_1 nor dy_2 has a factor of degree 2.

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