ANICK'S CONJECTURE AND INFINITIES IN THE MINIMAL MODELS OF SULLIVAN[1]

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Abstract. An *elliptic* space is one whose rational homotopy and rational cohomology are both finite dimensional. David Anick conjectured that any simply connected finite CW-complex S can be realized as the k-skeleton of some elliptic complex as long as $k > \dim S$, or, equivalently, that any simply connected finite Postnikov piece S can be realized as the base of a fibration $F \to E \to S$ where E is elliptic and F is k-connected, as long as $k > \dim S$. This conjecture is only known in a few cases, and here we show that in particular if the Postnikov invariants of S are decomposable, then the Anick conjecture holds for S. We also relate this conjecture with other finiteness properties of rational spaces.

§**1. Introduction.** A topological space is *elliptic* if its rational homotopy and rational cohomology are both finite dimensional. This class includes many interesting and well-known spaces which enjoy important structural properties ([H1], [FHT]). Elliptic spaces are very special, as the generic space is not elliptic, even amongst those satisfying one of the finiteness conditions. However, Anick conjectured the following:

Conjecture (Anick). *Any simply connected finite CW-complex S can be approximated arbitrarily closely on the right by an elliptic space. That is, for each natural number n there is an elliptic space E_n and an n-equivalence $S \to E_n$.*

In [JM], we showed that in the category of simply connected spaces with the homotopy type of a CW complex, this is equivalent to

Conjecture*(Anick). *Any simply connected finite Postnikov piece S can be approximated arbitrarily closely on the left by an elliptic space. That is, for each natural number n there is an elliptic space E_n and an n-equivalence $E_n \to S$.*

The Anick conjecture is most naturally viewed in the full subcategory $\mathcal{Q}_1$ of simply connected *rational* spaces with finite type (rational) homology (where we shall work henceforth), and in this setting it is known to be true in several cases. Elliptic spaces, which abound, themselves satisfy the conjecture. For example, if $\pi_*(S) = \pi_{\mathrm{odd}}(S)$ has finite dimension, then S is already elliptic and so Anick's conjecture is trivially true. Dually, if $H_*(S) = H_{\mathrm{odd}}(S)$ has finite dimension, then

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S is a finite wedge of odd spheres and so any finite Postnikov piece is elliptic. Moreover, as we shall see as a consequence of Theorem 2, if $H^*(S)$ is Noetherian, the conjecture holds and the approximation $E_n \to S$ can be chosen so its homotopy fibre is, rationally, a product of Krull-dim $H^*(S)$ odd spheres.

In [JM], we also showed that Anick's conjecture holds for spaces essentially built from those where $\pi_*(S) = \pi_{\mathrm{odd}}(S)$ by amplifying [W, P. 427] by an odd-dimensional cohomology class.

Here, we establish Anick's conjecture for spaces with decomposable Postnikov invariants. To be precise, first note that if $S^{(n)}$ denotes the n^{th} Postnikov piece of S, then the Postnikov invariants of S, $k_n\colon S^{(n-1)} \to K(\pi_n S, n+1)$, are cohomology classes of $H^{n+1}(S^{(n-1)}; \pi_n S) \cong H^{n+1}(S^{(n-1)}; \mathbf{Q}) \otimes \pi_n S$. We prove

Theorem 1. *Suppose that S is a finite Postnikov piece such that its even Postnikov invariants are decomposable, i.e.,*

$$k^n(S) \in H^+(S^{(n-1)}; \mathbf{Q}) \cdot H^+(S^{(n-1)}; \mathbf{Q}) \otimes \pi_n S, \quad \textit{for all even } n.$$

Then, Anick's conjecture holds for S.

We also characterize those finite Postnikov pieces for which the Anick conjecture is true "in one step":

Theorem 2. *If S is a finite Postnikov piece, then there is a fibration $F \to E \to S$ in which E is elliptic and F is a product of (rational) odd spheres if and only if $H^*(S)$ is finitely generated as an algebra over $\mathbf{Q}$.*

Finally, we also relate the Anick conjecture with other classical problems in rational homotopy concerning finiteness properties of spaces (see theorems 9 and 10). Of special interest is a generalization of a deep result of S. Halperin which characterizes non elliptic spaces [H2].

The principal tool we shall use is the Sullivan minimal model, and a basic reference is [FHT]. For our purposes, we note that to any 1-connected space S there corresponds, in a contravariant way, a commutative differential graded algebra $(\Lambda X, d)$, called the minimal model of S, which is unique up to isomorphism and algebraically models the rational homotopy type of the space. By ΛX we mean the free commutative graded algebra generated by the graded vector space X, i.e., $\Lambda X = TX/I$ where TX denotes the tensor algebra over X and I is the ideal generated by $x \otimes y - (-1)^{|y||x|} y \otimes x$, $x, y \in X$. The differential d of any element of X is a polynomial in ΛX with no linear term. This fact is equivalent to the existence of a basis of X (hereafter called a *KS-basis*) $\{x_i\}_{i \geq 0}$ for which $dx_i \in \Lambda X_{<i}$, where $X_{<i}$ denotes the subspace of X generated by $\{x_j\}_{j<i}$. This correspondence yields an equivalence between the homotopy categories of 1-connected rational spaces of finite type and that of 1-connected rational commutative graded differential algebras of finite type.

Moreover, to any fibration $F \to E \to B$ of 1-connected spaces is associated a relative Sullivan algebra (hereafter called a *KS-extension*)

$$(\Lambda X, d) \to (\Lambda X \otimes \Lambda Y, d) \to (\Lambda Y, \bar{d})$$

in which $(\Lambda Y, \bar{d})$ is the quotient $(\Lambda X \otimes \Lambda Y, d)/I$, being I the differential ideal generated by $\Lambda^+ X$, and $(\Lambda X, d)$ (resp. $(\Lambda Y, \bar{d})$) is the minimal model of the base B (resp. the fibre F). If $(\Lambda X \otimes \Lambda Y, d)$ is itself a minimal model we shall call it a minimal KS-extension.

Hence, the Anick conjecture above can be expressed as follows.

Conjecture A. *Given a minimal model* $(\Lambda X, d)$ *in which* $\dim X < \infty$*,ss and a fixed integer* N*, there is an elliptic minimal KS-extension* $(\Lambda X \otimes \Lambda Y, d)$ *in which* $Y = Y^{\geq N}$*.*

The paper is organized as follows. In the next section we prove theorem 2. Then in section §3 we present another approach to the Anick conjecture, prove theorem 1 and give two examples. Finally, in section §4 we present some finiteness properties of rational spaces related to these results.

§2. Elliptic closures and Almost elliptic spaces

If $(\Lambda X, d)$ is a Sullivan model with $\dim X < \infty$, a minimal KS-extension $(\Lambda X \otimes \Lambda Y, d)$ is said to be an *elliptic closure* of $(\Lambda X, d)$ if $(\Lambda X \otimes \Lambda Y, d)$ is elliptic, and in this case we call $(\Lambda Y, \bar{d})$ the fibre of the closure. Here, we characterize those spaces whose elliptic closures are just "one step away". To be precise, we call a Sullivan model $(\Lambda X, d)$ with $\dim X < \infty$ *almost elliptic* if it is not elliptic, but has an elliptic closure of the form

$$(\Lambda X, d) \to (\Lambda X \otimes \Lambda Y, d) \to (\Lambda Y, 0)$$

where $d : Y \to \Lambda X$. We remark that if $(\Lambda X, d)$ is almost elliptic, then by part 2 of Theorem 4 in the next section, any such elliptic closure $(\Lambda X \otimes \Lambda Y, d)$ as above necessarily satisfies $Y = Y^{\mathrm{odd}}$.

Theorem 2 is equivalent to:

Theorem 2′. *If* $(\Lambda X, d)$ *is a Sullivan model with* $\dim X < \infty$*, then the following are equivalent:*

A. $H(\Lambda X, d)$ *is finitely generated as an algebra over* $\mathbf{Q}$*.*

B. $(\Lambda X, d)$ *is almost elliptic.*

We begin with a lemma.

Lemma 3. *Let* $(H, 0)$ *be finitely generated as an algebra over* $\mathbf{Q}$*. Consider a 1-step extension of the form* $(H \otimes \Lambda Y, d)$ *with* $Y = Y^{\mathrm{odd}} < \infty$*. Then*

$$\dim H^*(H \otimes \Lambda Y, d) < \infty \iff \dim H/(dY) < \infty.$$

(Remark: it suffices to assume that $\ker d$ *is a finitely generated* H*-module, instead of assuming* H *Noetherian.)*

Proof. Suppose that $\dim H^*(H \otimes \Lambda Y, d) < \infty$. Because the differential is zero in H, the spectral sequence obtained from the filtration $F^p = H \otimes \Lambda^{\leq p} Y$ collapses at the E_2 term. Thus, $E_2^0 = H/(dY)$ is a sub-algebra of $H^*(H \otimes \Lambda Y, d)$, and hence is also finite dimensional.

Now suppose that $\dim H/(dY) < \infty$. Since $Y = Y^{\mathrm{odd}} < \infty$, $H \otimes \Lambda Y$ is a finitely generated H-module. Since H is Noetherian, $\ker d$ is a finitely generated H-module. Therefore, $H^*(H \otimes \Lambda Y, d)$ is also finitely generated over H, and, as $(dY) \cdot H^*(H \otimes \Lambda Y, d) = 0$, it is also finitely generated over $H/(dY)$. Since $\dim H/(dY) < \infty$, $\dim H^*(H \otimes \Lambda Y, d) < \infty$. ☐

Proof of Theorem 2′. (A $\Rightarrow$ B) This argument is implicit in the proof of Proposition 2 of [H1]. Let $H = H^*(\Lambda X, d)$, and let $(\Lambda X, d) \xrightarrow{\simeq} (\Lambda Z, d)$ be the bigraded model of

$(\Lambda X, d)$ [HS]. Recall that $Z = \sum_{p\geq 0,q} Z_p^q$ is bigraded, and that $Z_0 \cong H^+/(H^+\cdot H^+)$ has as basis a set of generators of H and is therefore finite dimensional. For each z_i in a basis of Z_0^{even}, add a generator y_i with $dy_i = z_i^N$. Let $Y = \text{span}\{y_i\}$ and consider the extension $(\Lambda Z \otimes \Lambda Y, d)$. Note that this lifts to an extension $(\Lambda X \otimes \Lambda Y, d) \xrightarrow{\simeq} (\Lambda Z \otimes \Lambda Y, d)$.

Now filter $(\Lambda Z \otimes \Lambda Y, d)$ by $F^p = \Lambda Z \otimes \Lambda^{\leq p} Y$, yielding a convergent spectral sequence. Then $(E_1, d_1) = (H(\Lambda Z) \otimes \Lambda Y, d_1)$ with $d_1 y_i = [z_i]^N$. It clearly suffices to show that $H(E_1, d_1)$ is finite dimensional. By lemma 3 above, this is equivalent to showing that $H(\Lambda Z)/(dY) \cong H(\Lambda X)/(dY)$ is finite dimensional. However, $H(\Lambda X) \cong H(\Lambda Z)$ is a finitely generated H^{even}-module, indeed, the classes of ΛZ_0^{odd} are generators. Therefore, $H(\Lambda Z)/(dY)$ is a finitely generated $H^{\text{even}}/(dY)^{\text{even}}$-module. But the latter is a quotient of $\Lambda Z_0^{\text{even}}/(dY) \otimes \Lambda^{\text{even}} Z_0^{\text{odd}}$, which is finite dimensional by construction. Hence, $H(\Lambda Z)/(dY)$ is finite dimensional, and so by lemma 3, $H(E_1, d_1)$ is finite dimensional. Thus, $H^*(\Lambda X \otimes \Lambda Y, d) < \infty$, and the extension clearly satisfies the other conditions.

(B $\Rightarrow$ A) Suppose that $(\Lambda X, d)$ is almost elliptic, so that there is a minimal extension

$$(\Lambda X, d) \to (\Lambda X \otimes \Lambda Y, d) \to (\Lambda Y, 0)$$

where $(\Lambda X \otimes \Lambda Y, d)$ *is* elliptic. We will consider $H := H(\Lambda X, d)$ as a module over the subalgebra A generated by 1 and dY. Now let $\bar{Y}$ be a graded vector space with $\bar{Y}^{d+1} = Y^d$, and consider the extension

$$(\Lambda \bar{Y}, 0) \to (\Lambda \bar{Y} \otimes \Lambda X \otimes \Lambda Y, D) \to (\Lambda X \otimes \Lambda Y, d), \tag{1}$$

where we define $Dy_i = -\bar{y}_i + dy_i$ on basis elements y_i of Y. Note that $(\Lambda \bar{Y} \otimes \Lambda X \otimes \Lambda Y, D)$ is a $\Lambda \bar{Y}$-module, so that $H \cong H(\Lambda \bar{Y} \otimes \Lambda X \otimes \Lambda Y, D)$ is one as well. Moreover, since $\bar{y}_i \cdot [\alpha] = [dy_i - Dy_i][\alpha] = [dy_i][\alpha]$, this coincides with the structure of H over A mentioned above.

The E_2 term of the Serre spectral sequence for the fibration (1) is

$$E_2 = \Lambda \bar{Y} \otimes H(\Lambda X \otimes \Lambda Y, d)$$

which is a finitely generated $\Lambda \bar{Y}$-module, since $\dim H(\Lambda X \otimes \Lambda Y, d) < \infty$. Thus, $E_\infty \cong G(H)$, the associated-graded algebra of H, is also a finitely generated $\Lambda \bar{Y}$-module. Since the filtration is finite in each degree, this shows that H is a finitely generated A-module. But A is also finitely generated over $\mathbf{Q}$, so H is a finitely generated algebra over $\mathbf{Q}$. This concludes the proof of the theorem. $\square$

§3 Tickling Anick's conjecture

We first collect some important (known) results bearing on this conjecture in theorem 4 below. In the next section, we will generalize some of what follows when the homotopy is not necessarily finite dimensional.

In the following, $\text{cat}(\Lambda X, d)$ denotes the Lusternik-Schnirelmann category of any rational space with minimal model $(\Lambda X, d)$, which can be characterized in terms of models as follows [FH]. Consider the commutative diagram

$$\begin{tikzcd}
(\Lambda X, d) \arrow[r, "q"] \arrow[dr, hook, "i"] & (\Lambda X/\Lambda^{>m} X, d) \\
 & (\Lambda X \otimes (\mathbf{Q} \oplus M), \delta) \arrow[u, "p", "\simeq"'] \arrow[ul, dotted, bend left, "\rho"]
\end{tikzcd}$$

in which $(\Lambda X, d) \to (\Lambda X \otimes (\mathbf{Q} \oplus M), \delta)$ is a a semi-free extension of $(\Lambda X, d)$-differential modules which is a *model* of the quotient map q, meaning that $p \circ i = q$ and that p is a quasi-isomorphism. Then, $\mathrm{cat}(\Lambda X, d)$ is the smallest m such that i admits a retraction, that is, a map ρ as above of $(\Lambda X, d)$-differential modules with $\rho i = \tau id_{\Lambda X}$ [FH, HL, He].

Theorem 4 ([H1], [H2], [FH]). *If $(\Lambda X, d)$ is a minimal model with $\dim X < \infty$, then the following are equivalent:*

1. $\dim H(\Lambda X, d) < \infty$.
2. *Given a KS-basis $\{x_1, \ldots, x_n\}$, for $1 \le i \le n$, each $[x_i]$ is nilpotent in the cohomology of the fibre of*
$$(\Lambda X_{<i}, d) \to (\Lambda X, d) \to (\Lambda X_{\ge i}, \bar{d}).$$
3. $\mathrm{cat}(\Lambda X, d) < \infty$.
4. *There are no non-trivial morphisms (over $\mathbb{C}$) $(\Lambda X, d) \to (\Lambda a, 0)$, where $|a| = 2$.*

One method of attack on Anick's conjecture would be to attempt an induction on $\dim X$. In this approach, one views $(\Lambda X, d) = (\Lambda(x_1, \ldots, x_n), d)$ as a series of extensions
$$(\Lambda(x_1, \ldots, x_{i-1}), d) \to (\Lambda(x_1, \ldots, x_i), d) \to (\Lambda x_i, 0).$$
For each i, we then add spaces of odd generators Z_i so that
$$(A_i, d_i) := (\Lambda(x_1, \ldots, x_i) \otimes \Lambda(Z_1 \oplus \cdots \oplus Z_i), d_i)$$
is elliptic, and consider the extension
$$(A_i \otimes \Lambda x_{i+1}, d),$$
which we then wish to render elliptic. Using theorem 4 above, we see that this step is equivalent to the existence of a finite dimensional extension
$$(A_i \otimes \Lambda x_{i+1} \otimes \Lambda Z_{i+1}, d)$$
in which the class $[x_{i+1}]$ is nilpotent in the fibre of
$$(A_i, d_i) \to (A_i \otimes \Lambda x_{i+1} \otimes \Lambda Z_i, d) \to (\Lambda x_{i+1} \otimes \Lambda Z_i, \bar{d}),$$

The induction begins very easily. If $|x_1|$ is odd, we can set $Z_1 = 0$, since $(\Lambda x_1, 0)$ is elliptic in that case, and if $|x_1|$ is even, we set $Z_1 = \langle z \rangle$ with $dz = x_1^N$. To proceed, if $|x_{i+1}|$ is odd, we can set $Z_{i+1} = 0$, since $[x_{i+1}]^2 = 0$ then. However, if $|x_{i+1}|$ is even, we are faced with the special case of Anick's conjecture stated below, which, as our discussion has just shown, is in fact equivalent to the full conjecture.

Conjecture A′. *Suppose that $N \in \mathbf{N}$, $(\Lambda Y, d)$ is elliptic and that $|x|$ is even in the extension $(\Lambda Y \otimes \Lambda x, d)$. Then, there is an extension*
$$(\Lambda Y, d) \to (\Lambda Y \otimes \Lambda x \otimes \Lambda Z, d) \to (\Lambda x \otimes \Lambda Z, \bar{d})$$
for which the class $[x]$ is nilpotent in $H^*(\Lambda x \otimes \Lambda Z, \bar{d})$, and in which $Z = Z^{\ge N}$ is finite dimensional.

Another interesting question related to this conjecture through theorem 4 is

Conjecture A$''$. *Let $(\Lambda X, d)$ be a minimal model in which both X^{even} and $H^{even}(\Lambda X, d)$ are finite dimensional. Then, given a KS-basis $\{x_n\}_{n\geq 0}$ of $(\Lambda X, d)$, each $[x_n]$ is nilpotent in the cohomology algebra $H^*(\Lambda X_{\geq n}, \overline{d})$.*

Proposition 5. *Conjecture A$''$ implies the Anick conjecture.*

Proof. Let $(\Lambda X, d)$ be a minimal model with $\dim X < \infty$ and let N be a fixed integer. Introduce new generators Z of odd degree greater than N so that in the minimal model $(\Lambda X \otimes \Lambda Z, d)$ the even cohomology in degrees greater than N vanishes. Hence, by conjecture A$''$, every x_i of even degree in a KS-basis of $(\Lambda X, d)$ is such that its cohomology class is nilpotent in $H^*(\Lambda X_{\geq i} \otimes \Lambda Z, \overline{d})$. Therefore, since there are a finite number of even generators there is a finite dimensional subspace $\overline{Z} \subset Z$ such that the same property also holds in $(\Lambda X \otimes \Lambda \overline{Z}, d)$. By theorem 4, this minimal model is elliptic. $\square$

We remark that conjecture A$'$ is clearly true in the case where $[dx] \in H(\Lambda Y, d)$ is *spherical*, since then we may change the KS-basis $\{y_1, \ldots, y_n\}$ in $(\Lambda Y, d)$ so that $dx = y_i$ for some i. Then,

$$(\Lambda Y \otimes \Lambda x, d) \cong (\Lambda(y_i, x) \otimes \Lambda(y_1, \ldots, \hat{y_i}, \ldots, y_n); d) \cong (\Lambda(y_1, \ldots, \hat{y_i}, \ldots, y_n); \bar{d}),$$

where $(\Lambda(y_1, \ldots, \hat{y_i}, \ldots, y_n); \bar{d})$ is the fibre of

$$(\Lambda y_i; 0) \to (\Lambda(y_1, \ldots, y_i, \ldots, y_n); d) \xrightarrow{\rho} (\Lambda(y_1, \ldots, \hat{y_i}, \ldots, y_n); \bar{d}).$$

Then, the Mapping theorem [FH] applied to ρ yields $\mathrm{cat}(\Lambda(y_1, \ldots, \hat{y_i}, \ldots, y_n); \bar{d}) < \infty$, so by theorem 4, $(\Lambda Y \otimes \Lambda x, d)$ is already elliptic.

A principal result of this paper is that conjecture A$'$ holds at the other end of the spectrum, when $[dx]$ is actually decomposable in the algebra $H(\Lambda Y)$. In this case, we are able to establish the conjecture even when $\dim Y = \infty$, as long as $\mathrm{cat}(\Lambda Y, d) < \infty$. This result together with its corollary clearly implies theorem 1.

Theorem 6. *Suppose that $\mathrm{cat}(\Lambda Y, d) < \infty$, that $|x|$ is even in a minimal extension $(\Lambda Y \otimes \Lambda x, d)$, and that $[dx] \in H^+(\Lambda Y) \cdot H^+(\Lambda Y)$. Then, for any $N \in \mathbf{N}$, there is a minimal extension*

$$(\Lambda Y \otimes \Lambda x, d) \to (\Lambda Y \otimes \Lambda x \otimes \Lambda U, d) \to (\Lambda U, \bar{d})$$

for which the class $[x]$ is nilpotent in $H^*(\Lambda x \otimes \Lambda U, \bar{d})$, and in which $U = U^{\mathrm{odd}} = U^{\geq N}$ is finite-dimensional.

Proof. Suppose that $dx = \sum_{i=1}^n \alpha_i \beta_i + d\epsilon$, where $d\alpha_i = d\beta_i = 0$, and α_i, β_i and $\epsilon \in \Lambda^+ Y$. Then, $y \mapsto y$ and $x \mapsto x' - \epsilon$ defines an isomorphism of $(\Lambda Y \otimes \Lambda x, d)$ with $(\Lambda Y \otimes \Lambda x', d')$, where $d'y = dy$ and $d'x' = \sum_{i=1}^n \alpha_i \beta_i$. Hence, we will assume without loss of generality that

$$dx = \sum_{i=1}^n \alpha_i \beta_i$$

for cycles $\alpha_i, \beta_i \in \Lambda^+ Y$. Since $|x|$ is even, we may suppose that each $|\alpha_i|$ is even and each $|\beta_i|$ is odd.

We next replace $(\Lambda Y, d)$ by an algebra where "long" cohomology classes actually vanish. First, choose

$$K = \max\{\mathrm{cat}(\Lambda Y, d), \frac{N+3-|x|}{2}\}.$$

This choice of K will also guarantee that the new generators we add are of degree at least N.

Denote by $(\Lambda Y, d) \xrightarrow{q} (\Lambda Y/\Lambda^{>K}Y, D)$ the canonical quotient map and let

$$(\Lambda Y/\Lambda^{>K}Y, D) \to (\Lambda Y/\Lambda^{>K}Y \otimes \Lambda x, D)$$

be the pullback of $(\Lambda Y, d) \to (\Lambda Y \otimes \Lambda x, d)$ over q. With the reader's indulgence, we will continue to denote the images of α_i and β_i under q by the same symbols, and will denote $\Lambda Y/\Lambda^{>K}Y$ by A.

Now consider the vector space $\mathcal{P}_k$ of homogeneous polynomials of degree k in n commuting variables $a_1, \ldots, a_n$, so that for $m > K$,

$$p \in \mathcal{P}_m \Rightarrow p(\alpha_1, \ldots, \alpha_n) = 0 \quad \text{in } A.$$

For $0 \leq j \leq K$, let U_j be a vector space of the same dimension as $\mathcal{P}_j$, and let $u_j : \mathcal{P}_j \to U_j$ be an isomorphism. Choose, for each $K \geq j > 0$, a basis $\mathcal{B}_j$ of $\mathcal{P}_j$ which consists of monomials and such that $a_i p \in \mathcal{B}_{j+1}$ whenever $p \in \mathcal{B}_j$. Set $\mathcal{B}_0 = \{1\}$ and define an extension $(A \otimes \Lambda x \otimes \Lambda U, d)$ as follows. For convenience, let $x^{(i)}$ denote $\frac{x^i}{i!}$.

For each $p \in \mathcal{B}_K$, define

$$du_K(p) = xp(\alpha_1, \ldots, \alpha_n). \tag{1}$$

Use this equation to determine the topological degree of $u_K(p)$, for each $p \in \mathcal{B}_K$, so that d is of topological degree $+1$. This is clearly possible since each such p is a monomial, and this makes U_K a graded vector space. (If we wished, we could extend d by linearity to all of U_K, where it would still satisfy (1).) This defines d on U_K.

We recursively extend d to $U = U_K \oplus U_{K-1} \oplus \cdots \oplus U_0$ as follows: for $K > j \geq 0$ and for all $p \in \mathcal{B}_j$,

$$du_j(p) = x^{(K-j+1)}p(\alpha_1, \ldots, \alpha_n) - \sum_{i=1}^{n} u_{j+1}\big(\alpha_i p(\alpha_1, \ldots, \alpha_n)\big)\beta_i. \tag{2}$$

Note that at step j, $u_{j+1}\big(\alpha_i p(\alpha_1, \ldots, \alpha_n)\big)$ makes sense and its derivative has already been defined. As in the case when $j = K$, we use this equation to determine the topological degree of each $u_j(p)$, for $p \in \mathcal{B}_j$, so that d is of topological degree $+1$. This makes each U_j a graded vector space, with $U_j = U_j^{\mathrm{odd}}$, and the choice of K ensures that $U_j = U_j^{>N}$, for $0 \leq j \leq K$.

We now prove by (descending) induction that $d^2 U_j = 0$ for $K \geq j \geq 0$. To begin, we note that $p \in \mathcal{P}_K$ implies that

$$d\big(xp(\alpha_1, \ldots, \alpha_n)\big) = (dx) \cdot p(\alpha_1, \ldots, \alpha_n) = \big(\sum_{i=1}^{n} \alpha_i \beta_i\big) \cdot p(\alpha_1, \ldots, \alpha_n) = 0$$

because each term $\alpha_i \cdot p(\alpha_1, \ldots, \alpha_n)$ in the sum is zero in A, as we remarked above. Hence, $d^2 U_K = 0$.

Now suppose that we have shown that $d^2 U_j = 0$ for $K \geq j > m$. For a monomial $p \in \mathcal{B}_m$, we let p denote $p(\alpha_1, \ldots, \alpha_n)$ for convenience and compute:

$$\begin{aligned}
d^2 u_m(p) &= d\big(x^{(K-m+1)}p - \sum_{i=1}^{n} u_{m+1}\big(\alpha_i p\big)\beta_i\big) \\
&= \sum_{i=1}^{n} x^{(K-m)}\alpha_i p \beta_i - \sum_{i=1}^{n} du_{m+1}\big(\alpha_i p\big)\beta_i \\
&= \sum_{i=1}^{n} x^{(K-m)}\alpha_i p \beta_i - \sum_{i=1}^{n} \Big(x^{(K-m)}\alpha_i p\beta_i - \sum_{k=1}^{n} du_{m+2}\big(\alpha_k\alpha_i p\big)\beta_k\beta_i\Big) \\
&= \sum_{i,k=1}^{n} du_{m+2}\big(\alpha_k\alpha_i p\big)\beta_k\beta_i \\
&= 0.
\end{aligned}$$

The last equality holds because of the degrees of the α's and β's: $u_{m+2}\big(\alpha_k\alpha_i p\big)$ is symmetric in i and k, while $\beta_k\beta_i$ is anti-symmetric in these indices. This closes the induction and we see that $(\Lambda Y/\Lambda^{>K}Y \otimes \Lambda x \otimes U, D)$ is a well-defined extension for which $[x]^{K+1} = \bar{D}u_0(1)$ in the fibre $(\Lambda x \otimes U, \bar{D})$.

Now let $p\colon (\Lambda Y \otimes \Lambda W, d) \xrightarrow{\simeq} (\Lambda Y/\Lambda^{>K}Y, D)$ be the Sullivan relative model of $(\Lambda Y, d) \xrightarrow{q} (\Lambda Y/\Lambda^{>K}Y, D)$. Since p is onto and induces an isomorphism in cohomology, we can use it to obtain an extension $(\Lambda Y \otimes \Lambda W, d) \to (\Lambda Y \otimes \Lambda W \otimes \Lambda x \otimes \Lambda U, d')$ so that we have the commutative diagram

$$\begin{tikzcd}
(\Lambda Y \otimes \Lambda W, d) \arrow[r] \arrow[d, "\simeq"', "p"] & (\Lambda Y \otimes \Lambda W \otimes \Lambda x \otimes \Lambda U, d') \arrow[r] \arrow[d, "\simeq"', "p'"] & (\Lambda x \otimes \Lambda U, \bar{d}') \arrow[d, equal] \\
(\Lambda Y/\Lambda^{>K}Y, D) \arrow[r] & (\Lambda Y/\Lambda^{>K}Y \otimes \Lambda x \otimes \Lambda U, D) \arrow[r] & (\Lambda x \otimes \Lambda U, \bar{D}).
\end{tikzcd}$$

Lastly, since $K \geq \mathrm{cat}(\Lambda Y, d)$, there is a retraction $r\colon (\Lambda Y \otimes \Lambda W, d) \to (\Lambda Y, d)$ satisfying $r(y) = y$ for $y \in Y$. To conclude, let

$$(\Lambda Y \otimes \Lambda x, d) \to (\Lambda Y \otimes \Lambda x \otimes \Lambda U, d) \to (\Lambda x \otimes \Lambda U, \bar{d})$$

be the pushout of the first row of the diagram above over r. Then, the equation $[x]^{K+1} = \bar{d}u_0(1)$ holds in the fibre $(\Lambda x \otimes U, \bar{d})$, showing that $[x]$ is nilpotent there. $\square$

Since $\dim H(\Lambda Y, d) < \infty \Rightarrow \mathrm{cat}(\Lambda Y, d) < \infty$, this immediately yields

Corollary 7. *Conjecture A' holds if $[dx]$ is decomposable in the algebra $H(\Lambda Y, d)$.*

Example a. Let S be any simply connected elliptic space, and n any integer. Then, if $\alpha \in H^n(S, \mathbf{Q})$ is homogeneous, as in [W, P. 427] the *amplification* of S by α is a space $S(\alpha)$ obtained as the pullback of the path space fibration $\mathcal{P}K(\mathbf{Q}, n) \to K(\mathbf{Q}, n)$ over any map $S \to K(\mathbf{Q}, n)$ representing α. If $|\alpha|$ is even, $S(\alpha)$ will always be elliptic, but if $\alpha \neq 0$ and $|\alpha|$ is odd, $\dim H^*(S(\alpha), \mathbf{Q}) = \infty$. However, if α is decomposable in the algebra $H^*(S, \mathbf{Q})$, corollary 7 shows that, given any integer N, there is an elliptic space E and a map $E \to S(\alpha)$ which is an isomorphism on rational homotopy groups in dimensions less than N, i.e., the Anick conjecture holds for $S(\alpha)$.

Example b. A simple example of a case not covered by Theorem 1 is the 4-Postnikov stage $(\mathbf{S}^2 \vee \mathbf{S}^2)^{(4)}$ of $\mathbf{S}^2 \vee \mathbf{S}^2$: its minimal model is

$$(\Lambda(a, b, u, v, y, c, e), d)$$

with $|a| = |b| = 2$, $du = a^2$,$dv = b^2$, $dy = ab$, $dc = ay - bu$, and $de = by - av$. If $\{c^*, e^*\}$ is the basis of $\pi_4(S^2 \vee \mathbf{S}^2) \otimes \mathbf{Q}$ which is dual to $\{c, e\}$, the Postnikov invariant

$$k^4[(\mathbf{S}^2 \vee \mathbf{S}^2)^{(3)}] = [ay - bu] \otimes c^* + [by - av] \otimes e^*$$

is not decomposable, since $[ay - bu]$ is not decomposable in the algebra $H^*((S^2 \vee \mathbf{S}^2)^{(3)}, \mathbf{Q})$. (Neither is $[by - av]$, of course.)

However, we may proceed as follows. We focus on the extension $(\Lambda(a, b, u, v, y, c), d)$, the case with both generators c and e being completely analogous. Let N be given.

First, we replace the model above with the quasi-isomorphic model

$$(A, d) = (\Lambda(a, b, y)/(a^2, b^2) \otimes \Lambda c, d),$$

where $|a| = |b| = 2$, $dy = ab$, and $dc = ay$,. We now add odd generators $u_i \in U^{>N}$ to make the class of $dc^{\{N+1\}} = c^{\{N\}}ay$ decomposable in $H(A \otimes \Lambda U, d)$: Let $z^{\{k\}}$ denote the divided power $\frac{z^k}{k!}$ to minimize cumbersome coefficients.

If we set $du_0 = c^{\{N\}}a$, then $d(c^{\{N\}}y - u_0 b) = 0$, and $d(au_0) = 0$, and we see that

$$dc^{\{N+1\}} = c^{\{N\}}ay = a \cdot (c^{\{N\}}y - u_0 b) + b \cdot (au_0)$$

is decomposable in $H(A \otimes \Lambda u_0, d)$.

Now, set $x = c^{\{N+1\}}$, so that

$$dx^{\{2\}} = xdx = x(a(c^{\{N\}}y - u_0 b) + bau_0) = x(a\beta_1 + b\beta_2),$$

where we denote $c^{\{N\}}y - u_0 b = \beta_1$ and $au_0 = \beta_2$. As in the proof of theorem 6, we now set $du_1 = xa$ and $du_2 = xb - y\beta_1$.

Then,

$$dx^{\{2\}} = x(a\beta_1 + b\beta_2) = (xa - y\beta_2)\beta_1 + (xb - y\beta_1)\beta_2 = d(u_1\beta_1 + u_2\beta_2)$$

This shows that $dc^{\{2N+2\}} = \frac{(N+1)!^2}{(2N+2)!} d(u_2\beta_1 + u_3\beta_2)$ is a boundary, and if we set $du_3 = c^{\{2N+2\}} - \frac{(N+1)!^2}{(2N+2)!}(u_2\beta_1 + u_3\beta_2)$, we see that

$$\dim H(A \otimes \Lambda(u_0, u_1, u_2, u_3), d) < \infty.$$

We remark that we could replace u_1 by xu_0, since $d(xu_0) = du_1$.

The proof of theorem 6 and the same kind of argument used in the preceding example establishes:

Theorem 8. *Let* $(\Lambda Y \otimes \Lambda x, d)$ *be a minimal model in which* $\mathrm{cat}(\Lambda Y, d) < \infty$ *and* $|x|$ *is even. Suppose that there is a* $k \in \mathbf{N}$ *and a minimal extension* $(\Lambda Y, d) \xrightarrow{j} (\Lambda Y \otimes \Lambda x \otimes \Lambda U, d)$ *in which* $U = U^{\mathrm{odd}} = U^{\geq N}$ *is finite-dimensional,* $[dx^k] = \sum_i \alpha_i.\beta_i$ *is decomposable in* $H^*(\Lambda Y \otimes \Lambda x \otimes \Lambda U, d)$*, with each even class* α_i *having a representative in the ideal generated by* $\Lambda^+ Y$*. Then, for any* $N \in \mathbf{N}$*, there is a minimal extension*

$$(\Lambda Y \otimes \Lambda x, d) \to (\Lambda Y \otimes \Lambda x \otimes \Lambda V, d) \to (\Lambda V, \bar{d})$$

for which the class $[x]$ *is nilpotent in* $H^*(\Lambda x \otimes \Lambda V, \bar{d})$*, with* $V = V^{\mathrm{odd}} = V^{\geq N}$ *finite-dimensional.*

§4. Infinities in a minimal model

The results above indicate the importance of theorem 4 in these questions. Here we obtain results in the same spirit in a general model, where we do not assume finiteness of the rational homotopy. Clearly, there is no direct generalization of theorem 4 to a general model. For example, in an infinite wedge of odd spheres, all generators of a minimal model are of odd degree and therefore parts (2) and (4) of theorem 4 are satisfied, whereas the cohomology is infinite dimensional. It is true however, via the Mapping theorem, that the existence of a non trivial morphism $(\Lambda X, d) \to (\Lambda a, 0)$ implies $\dim H(\Lambda X, d) = \infty$. Here we continue this discussion with the following results.

Theorem 9. *If* $(\Lambda X, d)$ *is any Sullivan minimal model, then either*

1. *There is a non-trivial morphism (over* $\mathbf{C}$*)* $\varphi\colon (\Lambda X, d) \to (\Lambda a, 0)$*, or*
2. *Each* $[x_i]$ *in a KS-basis* $\{x_i\}_{i \in \mathbf{N}}$ *of* X *is nilpotent in* $(\Lambda X_{\geq i}, \bar{d})$*.*

Proof. Let $(\Lambda X, d)$ be a minimal model and assume there is no non-trivial morphism $(\Lambda X, d) \to (\Lambda a, 0)$. Then, for each $i \in \mathbf{N}$ there is no non-trivial morphism $(\Lambda X_{\geq i}, \bar{d}) \to (\Lambda a, 0)$, since pre-composing such a morphism with the projection $(\Lambda X, d) \to (\Lambda X, d)/(\Lambda^+ X_{<i}, d) \cong (\Lambda X_{\geq i}, \bar{d})$ would yield a non zero morphism $(\Lambda X, d) \to (\Lambda a, 0)$. Hence, it is enough to prove that if x is the first element of a KS-basis of X, then $[x]$ is nilpotent in $H^*(\Lambda X, d)$. We suppose that x is of even degree and introduce some notation:

Let $\{x\} \cup \{x_n \mid n \geq 1, n \in \mathbf{N}\}$ (resp. $\{y_k \mid k \geq 1, k \in \mathbf{N}\}$) be a KS-basis for X^{even} (resp. a KS-basis for X^{odd}). Also write

$$dy_k = p_k + q_k, \qquad p_k \in \Lambda X^{\mathrm{even}}, \quad q_k \in \Lambda^+ X^{\mathrm{odd}} \cdot \Lambda X,$$

and observe that the for each k, the polynomial p_k in the variables $\{x_n\}$ is precisely $d_0 y_k$ where d_0 denotes the differential in the associated pure model $(\Lambda X, d_0)$ (see [H1]).

A map of graded algebras $\varphi\colon \Lambda X \to \Lambda a$ must satisfy $\varphi(y_k) = 0$, and $\varphi(x_n) = \lambda_n a_2^{|x_n|/2}$, $\lambda_n \in \mathbf{C}$ and $k, n \in \mathbf{N}$. On the other hand, since

$$\varphi d y_k = \varphi p_k = d \varphi y_k = 0,$$

the existence of a non-trivial morphism is equivalent to finding a non-trivial solution of the system

$$p_k = 0, \qquad k \geq 1.$$

Now let $I = (p_k \mid k \in \mathbf{N})$ be the ideal of $\mathbf{C}[X^{\text{even}}]$ generated by these polynomials. As usual, $Z(I)$ denotes the zero set of I, i.e. $Z(I) = \{\lambda \in \mathbf{C}^{\aleph_0} \mid p_k(\lambda) = 0, \forall k \in \mathbf{N}\}$. Then, since $|\mathbf{C}| > \aleph_0$, the infinite form of Hilbert's Nullstellensatz [L], implies that $Z(I) = \{0\}$ iff $\sqrt{I} = \mathbf{C}^+[X^{\text{even}}]$. Since we are assuming there are no non-trivial solutions to the system above, there exists $N \in \mathbf{N}$ and $f_j \in \mathbf{C}[X^{\text{even}}]$ such that $x^N = \sum_j f_j p_j$. This implies that $x^N = \sum_j f_j d_0 y_j = d_0(\sum_j f_j y_j) = d_0\Phi$, where Φ denotes $\sum_j f_j y_j$. Thus, $d\Phi = x^N + \Omega$, where $\Omega \in \Lambda^+ X^{\text{odd}} \cdot \Lambda X$, and so $[x^N] = [\Omega]$, where the latter is clearly nilpotent.

Another approach to a generalization of theorem 4 is

Theorem 10. *Let $(\Lambda X \otimes \Lambda Y, d)$ be a minimal model with*

1. $\dim X < \infty$,
2. $\dim H(\Lambda X \otimes \Lambda Y, d) = \infty$, *and*
3. $\dim H(\Lambda Y, \bar{d}) < \infty$.

Then, there exists a non-trivial morphism $\varphi\colon (\Lambda X \otimes \Lambda Y, d) \to (\Lambda a, 0)$ which is non-zero when restricted to $(\Lambda X, d)$.

Proof. We proceed by induction on $\dim X$. To begin, assume $\dim X = 1$ and let x be a generator of X. If the degree of x were odd the complex $(\Lambda x \otimes \Lambda Y, d)$ would have finite dimensional cohomology in view of the Serre spectral sequence, so x is of even degree. Suppose there is no non-trivial morphism $(\Lambda x \otimes \Lambda Y, d) \to (\Lambda a, 0)$. By theorem 9 above, $x^M = d\phi$ for some ϕ. Consider the extension $(\Lambda x \otimes \Lambda Y \otimes \Lambda u, d)$ with $du = x^M$. Then, the Serre spectral sequence with base $(\Lambda x \otimes \Lambda u, d)$ has $E_2 = H(\Lambda x \otimes \Lambda u, d) \otimes H(\Lambda Y, \bar{d})$, which is finite dimensional because both factors are. Thus, $\dim H(\Lambda x \otimes \Lambda Y \otimes \Lambda u, d) < \infty$. Moreover, the map

$$(\Lambda x \otimes \Lambda Y, d) \otimes (\Lambda v, 0) \to (\Lambda x \otimes \Lambda Y \otimes \Lambda u, d)$$

defined as the identity on $\Lambda x \otimes \Lambda Y$ and sending $v \to u - \phi$ is an isomorphism of differential algebras, so

$$H(\Lambda x \otimes \Lambda Y \otimes \Lambda u, d) \cong H(\Lambda x \otimes \Lambda Y, d) \otimes H(\Lambda v, 0),$$

implying that $H(\Lambda x \otimes \Lambda Y, d) < \infty$ which is a contradiction. This establishes the theorem in the case $\dim X = 1$. We note that for any non-trivial morphism $\varphi\colon (\Lambda x \otimes \Lambda Y, d) \to (\Lambda a, 0)$ we must have $\varphi(x) \neq 0$, since otherwise, we could define a non-zero $\tilde{\varphi} : (\Lambda Y, \bar{d}) \to (\Lambda a, 0)$, which would contradict $\dim H(\Lambda Y, \bar{d}) < \infty$.

Now assume the result is true for $\dim X < k$, and suppose $\{x_1, \ldots, x_k\}$ is a KS-basis for $(\Lambda X, d)$. Write $\Lambda X \otimes \Lambda Y$ as the extension $\Lambda x_1 \otimes \Lambda(x_2, \ldots, x_k) \otimes \Lambda Y$.

If $H(\Lambda(x_2, \ldots, x_k) \otimes \Lambda Y, d) = \infty$, then by the induction hypothesis, there is a non-trivial homomorphism $\phi : \Lambda(x_2, \ldots, x_k) \otimes \Lambda Y \to \mathbf{C}[a_2]$, which is necessarily non-zero on $\Lambda(x_2, \ldots, x_k)$. We then define $\varphi : \Lambda x_1 \otimes \Lambda(x_2, \ldots, x_k) \otimes \Lambda Y \to \mathbf{C}[a_2]$ using the projection $\Lambda x_1 \otimes \Lambda(x_2, \ldots, x_k) \otimes \Lambda Y \to \Lambda(x_2, \ldots, x_k) \otimes \Lambda Y$.

On the other hand, if $H(\Lambda(x_2, \ldots, x_k) \otimes \Lambda Y, d) < \infty$, use the $\dim X = 1$ case to define the desired non-trivial homomorphism. This closes the induction and completes the proof. $\square$

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