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FREE SUBMODULES FOR THE CENTRAL REPRESENTATION IN THE COHOMOLOGY OF LIE ALGEBRAS

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Abstract. If Z is the centre of the Lie algebra L, its cohomology $H^*(L)$ is naturally a module over the exterior algebra ΛZ. Under suitable hypotheses on L, motivated by recent work by Pouseele and Tirao, we find free summands in $H^*(L)$ for this module structure, thus establishing the Toral Rank Conjecture for a new class of Lie algebras.

1. Introduction

We consider finite-dimensional complex Lie algebras L. Recall that for such an algebra L, the Toral Rank Conjecture (TRC) [4] states that

$$\dim H^*(L) \geq 2^{\dim Z},$$

where Z is the centre of L, and $H^*(L)$ denotes the cohomology with trivial coefficients. The TRC is known to hold for nilpotent Lie algebras of dimension at most 14 [1]. It holds for two-step nilpotent Lie algebras (see [6] and [1]) and more generally for positively graded Lie algebras where the centre is the summand of highest grading (see [3] and [7]). Recently Hannes Pouseele and Paulo Tirao gave a remarkably simple result, which establishes the TRC for a class of Lie algebras that includes algebras of large nilpotency class that are not positively graded [5]. The aim of this paper is to show that the argument in [5] can be extended to a larger class of algebras and that the conclusion of their theorem can also be strengthened.

The key idea is to note that the cohomology $H^*(L)$ is naturally a module over the exterior algebra ΛZ and to look for free summands in $H^*(L)$:

Theorem. *Suppose that the Lie algebra L is a direct sum of non-trivial subalgebras A, B, C, where C is central and A, B and L are unimodular. Then the cohomology $H^*(L)$, as a ΛC-module, contains a free module on two generators.*

This extends [5, Theorem 1], which is the case where it is assumed that A and B are abelian and that $B \oplus C$ is an ideal of L. Note that the unimodularity hypothesis on A, B and L is satisfied when L is nilpotent, for example.

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Corollary. *The cohomology $H^*(L)$ has dimension at least $2^{\dim(C)+1}$.*

This strengthens [5, Corollary 2], which gives $\dim H^*(L) \geq 2^{\dim(C)}$. We give examples later to show that C may indeed be the centre Z of L. We also use the theorem to indicate how one may, for example, analyse the ΛZ structure of the cohomology of the free two-step algebras on m generators, where (except for $m = 2$), there are no free submodules.

2. Preliminaries

If L is a Lie algebra, and L^* denotes the dual, its cohomology is obtained as follows: let $d : L^* \to \Lambda^2 L^*$ be the transpose of the bracket $[\ ,\ ] : \Lambda^2 L \to L$, and extend it to a derivation of the Koszul complex ΛL^* of degree 1. The Jacobi identity is equivalent to $d^2 = 0$, and the cohomology $H^*(L) = H^*(\Lambda L^*, d)$ is the graded algebra defined as $H^*(L) = \ker d / \operatorname{im} d$.

If $x \in L$, i_x denotes the derivation of ΛL^* extending the natural map $x \in (L^*)^*$ to a derivation of ΛL^* of degree -1, and, when extended using the cap product $\Lambda Z \otimes \Lambda L^* \to \Lambda L^*$, makes ΛL^* a module over ΛZ. The Lie derivative $\mathcal{L}_x = i_x d + d i_x$ is the extension of the transpose of $ad(x) : L \to L$ to a derivation of degree 0 of ΛL^*, and x belongs to the centre $Z(L) \iff \mathcal{L}_x = 0$. Thus if $x \in Z(L)$, i_x induces a derivation of the algebra $H^*(L)$, and the ΛZ-module structure on ΛL^* induces one on $H^*(L)$. In [2], the homomorphism $\Lambda Z \to \operatorname{End}(H^*(L))$ defining this module structure is called the *central representation*.

A Lie algebra is unimodular if $\operatorname{trace} ad(x) = 0$ for all $x \in L$, and it is easy to show that this is equivalent to $d : \Lambda^{\dim L - 1} L^* \to \Lambda^{\dim L} L^*$ being zero. For L as in the statement of the theorem, we choose bases for A, B and C, and relative to the resulting basis for L, we define the Hodge star $\star : \Lambda^k L^* \to \Lambda^{\dim L - k} L^*$ in the usual manner. It is straightforward to show that if L is unimodular, $H^*(L)$ is isomorphic to the space of harmonic forms (see [2], for example); recall that a form α is harmonic if $d\alpha = 0$ and $d \star \alpha = 0$.

The approach used in [2] (and independently in [5]) can be interpreted as follows ([5] uses homology whereas we will use cohomology in this paper): suppose there exists a closed p-form $\alpha \in \Lambda L^*$ such that the submodule $\Lambda Z \cdot [\alpha]$ of $H^*(L)$ generated by α is free. This occurs if and only if the $(p-k)$-form $i_{z_1} i_{z_2} \dots i_{z_k} \alpha$ is not exact, where $\{z_1, \dots, z_k\}$ is any basis for the centre Z. Then, if $\{z_{j_1}, z_{j_2}, \dots, z_{j_l}\}$ is any subset of $\{z_1, \dots, z_k\}$, the classes $[i_{z_{j_1}} i_{z_{j_2}} \dots i_{z_{j_l}} \alpha]$ are all linearly independent, and hence $\dim H^*(L) \geq 2^{\dim Z}$. In this case, the central representation is faithful.

The result in [5] is obtained by taking suitable hypotheses on an ideal I of L so that such an α is given by the pullback to L of a nonzero form in $\Lambda^{\dim L/I} L/I$. (As noted in [2], there are many examples of Lie algebras where the central representation is not faithful; [2] shows that $H^*(L)$ is actually a module over a much larger algebra containing ΛZ and begins the study of that module structure with the TRC as a goal.) The result [5] and our theorem above give examples where the central representation is faithful.

3. Proof of the theorem

Proof. Let L^* denote the dual of L, and define subspaces $U = (B \oplus C)^\perp$, $V = (A \oplus C)^\perp$ and $W = (A \oplus B)^\perp$ of L^*. The Koszul complex of L can then be written as $(\Lambda U \otimes \Lambda V \otimes \Lambda W; d)$. The fact that A and B are subalgebras and that C is central implies that

$$d : U \to \Lambda^2 U \oplus (U \otimes V), \tag{1}$$
$$d : V \to \Lambda^2 V \oplus (U \otimes V), \text{and} \tag{2}$$
$$d : W \to U \otimes V. \tag{3}$$

Now let $\sigma, \varepsilon, \tau$ be nonzero elements in $\Lambda^{\dim U} U, \Lambda^{\dim V} V$ and $\Lambda^{\dim W} W$ respectively. Thus $\sigma\varepsilon\tau$ is a nonzero element in $\Lambda^{\dim L} L$.

We shall show that the unimodularity assumptions imply that $d\sigma = 0$, and that the class $[\sigma] \in H^{\dim U}(L)$ is nonzero. If $\{z_1, \ldots, z_k\}$ is a basis for C and $z = z_1 \ldots z_k \in \Lambda^k C$, we will then have $[i_z \sigma\tau] = \pm[\sigma] \neq 0$, and so the ΛC module generated by $[\sigma\tau]$ is free. An identical argument will show that the ΛC module generated by $[\tau\varepsilon]$ is also free.

To show that $d\sigma = 0$, first note that the unimodularity of the Lie algebra L is equivalent to the condition

$$0 = \mathcal{L}_x(\sigma\varepsilon\tau), \quad \forall x \in L.$$

As C is central, $\mathcal{L}_x \tau = 0$ and so

$$0 = (\mathcal{L}_x \sigma)\tau\varepsilon + \sigma(\mathcal{L}_x \tau)\varepsilon, \quad \forall x \in L.$$

Now let $x \in B$, write $d_{|V} = \bar{d}_V + \theta$, with $\bar{d}_V : V \to \Lambda^2 V$ and $\theta : V \to U \otimes V$, and denote $\overline{\mathcal{L}}_x = i_x \bar{d}_V + \bar{d}_V i_x$. The unimodularity of B then gives $\sigma\varepsilon(\mathcal{L}_x \tau) = \sigma\varepsilon(\overline{\mathcal{L}}_x \tau) = 0$ as well. Hence, for $x \in B$, we have

$$0 = (\mathcal{L}_x \sigma)\tau\varepsilon = ((i_x d + d i_x)\sigma)\tau\varepsilon = (i_x d\sigma)\tau\varepsilon.$$

Since $\sigma \in \Lambda^{\dim U} U$, (1) implies $d\sigma = \sigma \otimes v$ for some $v \in V$. Thus $0 = (i_x d\sigma)\tau\varepsilon$ gives $0 = (i_x d\sigma)$, and so $v = 0$. Hence, $d\sigma = 0$.

Hence by (3), $d(\sigma\tau) = (d\sigma)\tau = 0$. Now, since the hypotheses are symmetric in U and V, a similar argument shows that $d\varepsilon = 0$ and $d(\varepsilon\tau) = 0$. We also know that $\sigma = \pm\star\varepsilon\tau$, where $\star$ denotes the Hodge star map, so σ is harmonic and thus $[\sigma] \neq 0$. By the same reasoning, $[\varepsilon] \neq 0$. Hence, the ΛC modules generated by $[\sigma\tau]$ and $[\varepsilon\tau]$ are free. $\square$

4. Example

Let $\mathfrak{f}_n$ denote the $(m+1)$-dimensional standard filiform algebra with basis $\{x_0, \ldots, x_n\}$ and relations $[x_0, x_{i-1}] = x_i$, $2 \leq i \leq n$, and let $\mathfrak{h}_m$ denote the $(2m+1)$-dimensional Heisenberg algebra with basis $\{y_1, \ldots, y_m, z_1, \ldots, z_m, w\}$ and relations $[y_i, z_i] = w$, $1 \leq i \leq m$. Consider the extension of $\mathfrak{f}_n \oplus \mathfrak{h}_m$ defined by introducing the symbols

$$a_1, \ldots, a_m, b_1, \ldots, b_{n-1}, c_0, \ldots, c_m,$$

and defining relations as follows:

$$\begin{aligned} [x_0, z_i] &= a_i, \quad 1 \le i \le m, \\ [x_0, y_1] &= -b_1, \\ [x_0, b_{i-1}] &= -b_i, \quad 2 \le i \le n-1, \\ [x_n, y_1] = [x_j, b_{n-j}] &= c_0, \quad 1 \le j \le n-1, \\ [x_0, w] = [y_i, a_i] &= c_1, \quad 1 \le i \le m, \text{ and} \\ [x_0, y_i] &= c_i, \quad 2 \le i \le m. \end{aligned}$$

Now let $A = \mathfrak{f}_n \oplus \langle a_i \mid 1 \le i \le m \rangle$, $B = \mathfrak{h}_m \oplus \langle b_i \mid 1 \le i \le n-1 \rangle$, $C = \langle c_i \mid 0 \le i \le m \rangle$ and $L = A \oplus B \oplus C$, with the products above. Note that L is a nilpotent algebra of nilpotency n, A and B are nonabelian subalgebras of L, and $Z(L) = C$, so $\dim Z(L) = m+1$. Moreover, the derived algebra $[L, L]$ is nonabelian, and hence L is not of the form treated in [5], but satisfies the hypotheses of our theorem.

5. Application to free two-step algebras

Suppose $F_m = \mathbb{C}^m \oplus \mathbb{C}^{\binom{m}{2}}$ is the free two-step algebra on m generators where $Z(F_m) = \mathbb{C}^{\binom{m}{2}}$. It is not difficult to show that for $m > 2$, the cohomology of $H^*(F_m)$ does not contain any free ΛZ summands. However, using the theorem, we can see that $H^*(F_m)$ does contain many free ΛC summands, for certain $C \subset Z$, as follows. Write $m = k + l$ for k, l positive integers, and decompose F_m as a direct sum of subalgebras as follows:

$$F_m = F_k \oplus F_l \oplus \mathbb{C}^{\binom{m}{2}-\binom{k}{2}-\binom{l}{2}} = \left(\mathbb{C}^k \oplus \mathbb{C}^{\binom{k}{2}}\right) \oplus \left(\mathbb{C}^l \oplus \mathbb{C}^{\binom{l}{2}}\right) \oplus \mathbb{C}^{\binom{m}{2}-\binom{k}{2}-\binom{l}{2}}. \tag{4}$$

Here, the last summand is the repository for brackets of elements in $\mathbb{C}^k$ and $\mathbb{C}^l$. The decomposition (4) allows an application of the theorem with $A = F_k, B = F_l$ and $C = \mathbb{C}^{\binom{m}{2}-\binom{k}{2}-\binom{l}{2}}$, and thus guarantees two free $\Lambda\mathbb{C}^{\binom{m}{2}-\binom{k}{2}-\binom{l}{2}}$ submodules in $H^*(F^m)$. Combining these for different k, l (valid when the co-generators are independent) actually yields the TRC for the algebras F_m for $m \le 5$, and gives an explicit method of constructing nontrivial cohomology classes in $H^*(F^m)$.

It is interesting to note that for $m \le 5$, computer calculations show that the theorem predicts the maximal dimension of central subspaces C for which there are free ΛC summands in $H^*(F_m)$. We conjecture that this will hold for all m.

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