RATIONAL LUSTERNIK-SCHNIRELMANN CATEGORY OF FIBRATIONS

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Abstract. If $F \to E \to B$ is a fibration, a classical result of Varadarajan asserts that $\mathrm{cat}\, E \le \mathrm{cat}\, F + \mathrm{cat}\, B(\mathrm{cat}\, F + 1)$, where $\mathrm{cat}\, S$ denotes the Lusternik-Schnirelmann category of S. We give improved upper bounds in the rational case of the form

$$\mathrm{cat}_0 E \le \mathrm{cat}_0 F + \mathrm{cat}_0 B(\mathrm{cat}_0 F + 2 - r_0 F),$$

where $r_0 F$ is a new invariant, namely the *rational retraction index* of F satisfying

$$\mathrm{depth}\, F \le r_0 F \le \mathrm{cat}_0 F,$$

so that we recover the classical formula when $r_0 F = 1$. However, the retraction index is often larger than 1, and in particular, we prove that if $H^*(F;\mathbb{Q})$ is a Poincaré duality algebra with at least 2 generators, then $r_0 F \ge 2$, giving the bound of [11] without their dimension hypothesis. Moreover, if F is coformal, then $r_0 F = \mathrm{cat}_0 F$, which yields the much lower estimate

$$\mathrm{cat}_0 E \le \mathrm{cat}_0 F + 2\mathrm{cat}_0 B.$$

Introduction

Let S be a simply connected CW complex of finite type. The Lusternik - Schnirelmann category of S, $\mathrm{cat}\, S$, is the least integer n such that S can be covered by $n+1$ open sets, each contractible in S [12]. In general, the computation of the category is very difficult, even for compact Lie groups, where some recent progress has been made by James and Singhoff [11]. We present here a new upper bound for the category of the total space of a fibration that considerably improves the classical one in many cases.

The main ingredient is a new homotopy invariant related to the category, the *rational retraction index*, $r_0 S$. This integer lies between the rational category (i.e. the category of the rationalization S_0) and the depth of the rational loop space homology

$$\mathrm{depth}\, H_*(\Omega S;\mathbb{Q}) \le r_0 S \le \mathrm{cat}_0 S.$$

For example, if S is an H-space of finite dimension, so that $\pi_{\mathrm{even}}(S) \otimes \mathbb{Q} = 0$ and $\dim \pi_{\mathrm{odd}}(S) \otimes \mathbb{Q} < \infty$, then $r_0 S = \mathrm{cat}_0 S$.

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The integer r_0S and the rational category $\operatorname{cat}_0 S$ can be obtained from the Sullivan minimal model of the space [13], as follows. Denote by $(\Lambda V, d)$ a minimal model of S, and consider the commutative diagram

$$\begin{tikzcd}
(\Lambda V, d) \arrow[r, "q"] \arrow[dr, hook, "i"] & (\Lambda V/\Lambda^{>m}V, d) \\
& (\Lambda V \otimes (\mathbb{Q} \oplus M_m), \delta) \arrow[u, "p"', "\simeq"] \arrow[ul, dotted, bend left, "\rho"]
\end{tikzcd}$$

in which $(\Lambda V, d) \to (\Lambda V \otimes (\mathbb{Q} \oplus M_m), \delta)$ is a a semi-free extension of $(\Lambda V, d)$-differential modules which is a *model* of the quotient map q, meaning that $p \circ i = q$ and that p is a quasi-isomorphism. The *rational category of* S, $\operatorname{cat}_0 S$, is the smallest m such that i admits a retraction, that is, a map ρ as above of $(\Lambda V, d)$-differential modules with $\rho i = \mathrm{id}_{\Lambda V}$ [2, 7, 9].

Since p is surjective, by changing the basis of the free ΛV-module $\Lambda V \otimes (\mathbb{Q} \oplus M_m)$, if necessary, we may suppose that $p(M_m) = 0$.

Definition. The *rational retraction index* r_0S, is the largest integer r, not exceeding $\operatorname{cat}_0 S$, such that in any semi-free model with $p(M_{\operatorname{cat}_0 S}) = 0$, we have $\rho(M_{\operatorname{cat}_0 S}) \subset \Lambda^{\geq r} V$.

We show why r_0S is a homotopy invariant in section 2. Some other important properties of r_0 are given by Propositions 1-4 below:

Proposition 1. *If $S = S_1 \times \cdots \times S_m$ is a product, then $r_0 S \geq r_0 S_1 + \cdots + r_0 S_m \geq m$.*

Proposition 2. *If $\mathbf{S}^n$ denotes the n-sphere, $r_0(S \times \mathbf{S}^n) = r_0 S + 1$.*

Proposition 3. *If S is coformal, then $r_0 S = \operatorname{cat}_0 S$.*

Proposition 4. *If S is a Poincaré duality complex whose rational cohomology algebra requires at least two generators, then $r_0 S \geq 2$.*

The retraction index turns out to be an important tool for estimating the rational category of the total space of a fibration $F \to E \to B$. The classical formula for an upper bound for the category, due to Varadarajan [15], is

$$\operatorname{cat} E \leq \operatorname{cat} F + \operatorname{cat} B\,(\operatorname{cat} F + 1).$$

Our principal result is to use the invariant $r_0 S$ in order to improve this formula. In the following, we suppose that B is simply connected and that all spaces have the homotopy type of a CW complex of finite type. We obtain

Theorem 1. *If $F \to E \to B$ is a fibration, then*

$$\operatorname{cat}_0 E \leq \operatorname{cat}_0 F + \operatorname{cat}_0 B(\operatorname{cat}_0 F + 2 - r_0 F).$$

In particular, for fibrations with a Poincaré duality complex as fibre, we obtain a rational version of the James-Singhoff theorem [11] that avoids their dimension hypothesis:

Corollary 1. *Let $F \to E \to B$ be a fibration. Suppose that F is a Poincaré duality complex whose cohomology algebra requires at least two generators. Then*

$$\mathrm{cat}_0 E \leq \mathrm{cat}_0 F + \mathrm{cat}_0 B \mathrm{cat}_0 F.$$

From the relation between retraction index and depth, we deduce from the theorem the following weak version of our upperbound theorem:

Corollary 2. *If $F \to E \to B$ is a fibration, then*

$$\mathrm{cat}_0 E \leq \mathrm{cat}_0 F + \mathrm{cat}_0 B(\mathrm{cat}_0 F + 2 - \mathrm{depth}\, H_*(\Omega F; \mathbb{Q})).$$

When the inclusion of the fibre $F \hookrightarrow E$ induces an injection in rational homotopy, the upper bound for $\mathrm{cat}_0 E$ can be reduced by $\mathrm{cat}_0 B$:

Theorem 2. *If $F \xrightarrow{i} E \to B$ is a fibration, and $\pi_* i \otimes \mathbb{Q}$ is injective, then*

$$\mathrm{cat}_0 E \leq \mathrm{cat}_0 F + \mathrm{cat}_0 B(\mathrm{cat}_0 F + 1 - r_0 F)$$

The hypothesis on $F \hookrightarrow E$ is satisfied for instance when the fibration admits a section, or if the centre of the rational homotopy Lie algebra of F is trivial. Together, Theorem 2 and Proposition 3 yield

Corollary 3. *If $F \xrightarrow{i} E \to B$ is a fibration, $\pi_* i \otimes \mathbb{Q}$ is injective, and F is coformal, then*

$$\mathrm{cat}_0 E \leq \mathrm{cat}_0 F + \mathrm{cat}_0 B$$

We note that the conclusion of Theorem 1 is also valid when $\mathrm{cat}_0 B$ is replaced by $\mathrm{cat}_0(E \to B)$.

This paper is organized as follows. In the next section, we prove Propositions 1 to 4, using Sullivan minimal models. The relationship between the depth and the retraction index is proved in section 3, and the main theorems are established in section 4. In section 5, we give examples to show that the predicted bound of the theorem is sharp, and then conclude with some open questions.

In what follows, all spaces will have the homotopy type of simply connected CW complexes of finite type. We refer the reader to [4] for the necessary background in rational homotopy theory.

2. Properties of $r_0 F$

In this section we establish some properties of the retraction index.

We begin by recalling some rather technical but ultimately very useful facts about the models we will work with. Suppose $(\Lambda Y, d)$ is a connected minimal model and $d = d_2$ is the quadratic part of the differential. Recall [3] that for a positive integer q we have a commutative diagram of bigraded $(\Lambda Y, d_2)$-differential modules

$$\begin{tikzcd}
(\Lambda Y, d_2) \arrow[r] \arrow[dr, "i_q"'] & (\Lambda Y / \Lambda^{>q} Y, d_2) \\
 & (\Lambda Y \otimes (\mathbb{Q} \oplus M_q), \delta_2) \arrow[u, "p_q", "\simeq"']
\end{tikzcd}$$

where $(\Lambda Y, d_2) \to (\Lambda Y \otimes (\mathbb{Q} \oplus M_q), \delta_2)$ is a semi-free extension in which Y^i (in topological degree i) has bigradation $(1, i)$, $\mathbb{Q} = \mathbb{Q}^{0,0}$, and $M_q = M_q^{q,*}$. As mentioned above, we may suppose that $p_q(M_q) = 0$, and in this case, all maps in the diagram are bigraded maps of bidgree $(0, 0)$ (Indeed, it for this we add the mysterious condition $p(M_{\operatorname{cat}_0 S}) = 0$ in the definition of $r_0 S$.)

The differential δ_2 may then be perturbed by a standard technique (relying on the fact that $H^{>q,*}(\Lambda Y/\Lambda^{>q} Y, d_2) = 0$) to a differential δ such that we have a commutative diagram of filtered $(\Lambda Y, d)$-differential modules

$$\begin{tikzcd}
(\Lambda Y, d) \arrow[r] \arrow[dr, hook, "i_q"'] & (\Lambda Y/\Lambda^{>q} Y, d) \\
& (\Lambda Y \otimes (\mathbb{Q} \oplus M_q), \delta) \arrow[u, "p_q", "\simeq"']
\end{tikzcd}$$

in which the filtration in any module A is $F^r A = \Lambda^{\geq r} Y.A$. Note that p_q is unchanged, and $\delta - \delta_2 \colon (\Lambda Y \otimes (\mathbb{Q} \oplus M_q))^{i,*} \to (\Lambda Y \otimes (\mathbb{Q} \oplus M_q))^{\geq i+2,*}$. In particular, $\delta : M_q \to \Lambda^{>q} Y \oplus \Lambda^+ Y \otimes M_q$.

We will have occasion to lift filtered maps over filtered maps, and the lift will always be able to be constructed, as above, as a perturbation of a bigraded map, say of bidegree (r, d), and then the lift will be a filtered map of filtration degree r. Taking account of the preceeding comments, one may now establish the homotopy invariance of $r_0 S$ by applying the standard lifting lemma in its filtration preserving version.

Before beginning the proofs, we need a lemma that shows that the retraction index does not 'decay' in higher Ganea fibrations. We refer the reader to the diagram above for notation.

Lemma 1. *Suppose* $\operatorname{cat}_0(\Lambda Y, d) = n$ *and* $r_0(\Lambda Y, d) = r$. *Then, for any* $q \geq n$, *there is a retraction* ρ_q *of* i_q *such that* $\rho_q(M_q) \subset \Lambda^{\geq q-n+r} Y$.

Proof. The natural inclusion $\Lambda^{>q+1} Y \to \Lambda^{>q} Y$ of differential ideals induces a commutative diagram of bigraded $(\Lambda Y, d_2)$-differential modules

$$\begin{tikzcd}
(\Lambda Y/\Lambda^{>q+1} Y, d_2) \arrow[r] & (\Lambda Y/\Lambda^{>q} Y, d_2) \\
(\Lambda Y \otimes (\mathbb{Q} \oplus M_{q+1}), \delta_2) \arrow[u, "p_{q+1}", "\simeq"'] \arrow[r, "\mu_0"] & (\Lambda Y \otimes (\mathbb{Q} \oplus M_q), \delta_2) \arrow[u, "p_q", "\simeq"']
\end{tikzcd}$$

in which $\mu_0(M_{q+1}) \subset \Lambda^{q+1} Y \oplus (Y \otimes M_q)$. By a perturbation argument similar to that used above, we can perturb δ_2 to δ and simultaneously μ_0 to $\lambda_{q+1} = \mu_0 + \mu_1 + \cdots$, with $\mu_i \colon M_{q+1} \to \Lambda^{q+i+1} Y \oplus (\Lambda^{i+1} Y \otimes M_q)$, in such a way as to obtain a commutative diagram of filtered $(\Lambda Y, d)$-differential modules

$$\begin{tikzcd}
(\Lambda Y/\Lambda^{>q+1} Y, d) \arrow[r] & (\Lambda Y/\Lambda^{>q} Y, d) \\
(\Lambda Y \otimes (\mathbb{Q} \oplus M_{q+1}), \delta) \arrow[u, "p_{q+1}", "\simeq"'] \arrow[r, "\lambda_{q+1}"] & (\Lambda Y \otimes (\mathbb{Q} \oplus M_q), \delta) \arrow[u, "p_q", "\simeq"']
\end{tikzcd}$$

in which $\lambda_{q+1}(M_{q+1}) \subset \Lambda^{\geq q+1} Y \oplus \Lambda^{\geq 1} Y \otimes M_q$.

Now supppose that $(\Lambda Y\otimes(\mathbb{Q}\oplus M_n),\delta)\xrightarrow{\rho}(\Lambda Y,d)$ is a retraction guaranteed by $\operatorname{cat}_0(\Lambda Y,d)=n$ and $r_0(\Lambda Y,d)=r$, and define ρ_q to be the composition

$$\rho_q=\rho\circ\lambda_{n+1}\circ\cdots\circ\lambda_{q-1}\circ\lambda_q.$$

The properties of the λ_k's assure us that

$$\begin{aligned}\rho_q(M_q)&\subset\rho(\Lambda^{\geq q}Y\oplus\Lambda^{\geq q-n}Y\otimes M_n)\\&\subset\Lambda^{\geq q}Y+\Lambda^{\geq q-n}Y.\Lambda^{\geq r}Y\\&\subset\Lambda^{\geq q-n+r}Y,\end{aligned}$$

since $-n+r\leq 0$. This completes the proof the lemma. □

We now proceed with the proofs of the Propositions.

Proposition 1. *If $S=S_1\times\cdots\times S_m$ is a product, then $r_0S\geq r_0S_1+\cdots+r_0S_m\geq m$.*

Proof. For $1\leq k\leq m$, let $(\Lambda X_k,d)$ be a minimal model of S_k, and denote $\operatorname{cat}_0 S_k$ by n_k, r_0S_k by r_k, $\sum_k n_k$ by N and $\sum_k r_k$ by R. By hypothesis, there are commutative diagrams

$$\begin{tikzcd}(\Lambda X_k,d) \arrow[r] \arrow[dr, hook, "i_k"'] & (\Lambda X_k/\Lambda^{>n_k}X_k,d) \\ & (\Lambda X_k\otimes(\mathbb{Q}\oplus M_{n_k}),\delta) \arrow[u, "q", "\simeq"']\end{tikzcd}$$

with retractions ρ_k of i_k satisfying $\rho_k(M_{n_k})\subset\Lambda^{\geq r_k}X_k$.

The tensor product map $\bigotimes_{k=1}^m(\Lambda X_k,d)\to\bigotimes_{k=1}^m(\Lambda X_k/\Lambda^{>n_k}X_k,d)$ factors through $\bigotimes_{k=1}^m(\Lambda X_k)/\Lambda^{>N}(X_1\oplus\cdots\oplus X_m)$; Since q is surjective, the homotopy lifting property yields a map τ making the following diagram commutative

$$\begin{tikzcd}\bigotimes\Lambda X_k \arrow[r] \arrow[dr, hook, "i"'] & \bigotimes(\Lambda X_k)/\Lambda^{>N}(X_1\oplus\cdots\oplus X_m) \arrow[r] & \bigotimes\Lambda X_k/\Lambda^{>n_k}X_k \\ & \bigotimes\Lambda X_k\otimes(\mathbb{Q}\oplus M) \arrow[u, "\simeq"'] \arrow[r, "\tau"] & \bigotimes\Lambda X_k\otimes(\bigotimes\mathbb{Q}\oplus M_{n_k}). \arrow[u, "\simeq"']\end{tikzcd}$$

Moreover, $\tau i=\bigotimes_k i_k$, and since τ is the lift of a filtration preserving map over another, we may assume that τ is as well. Therefore, a retraction of i is given by $\rho=(\bigotimes_k\rho_k)\circ\tau$. Moreover, this will satisfy

$$\rho(M)\subset(\bigotimes_k\rho_k)(\bigotimes\Lambda X_k\otimes(\bigotimes\mathbb{Q}\oplus M_k))^{\geq N,*},$$

and a short computation using the fact that $\rho_k(\Lambda X_k\otimes(\mathbb{Q}\oplus M_k))^{\geq n_k,*}\subset\Lambda^{\geq r_k}X_k$ shows that $(\bigotimes_k\rho_k)(\bigotimes\Lambda X_k\otimes(\bigotimes\mathbb{Q}\oplus M_k))^{\geq N,*}\subset\Lambda^{\geq R}(X_1\oplus\cdots\oplus X_m)$, as required.□

The proof of Proposition 2 follows that of the Ganea conjecture for $Mcat_0$ of [10].

Proposition 2. *If* $\mathbf{S}^n$ *denotes the* n*-sphere,* $r_0(S\times\mathbf{S}^n)=r_0S+1$.

Proof. We give the proof for an even sphere, that of the odd sphere being even easier. Suppose $(\Lambda X,d)$ is a minimal model for S and $(\Lambda(y,u),d)$ with $dy=0$ and $du=y^2$ a minimal model for the sphere $\mathbf{S}^{|y|}$. Let m denote $\operatorname{cat}_0 S$, and $r+1$ denote $r_0(S\times\mathbf{S}^{|y|})$. In view of Proposition 1, all we need to show is that $r_0S\geq r$. Consider the (ΛX)-module maps

$$\begin{aligned}&\mu:(\Lambda X,d)\to(\Lambda X\otimes\Lambda(y,u),d)\quad\text{defined by }\ \alpha\mapsto\alpha\cdot y,\qquad\text{and}\\&\varphi:(\Lambda X\otimes\Lambda(y,u),d)\to(\Lambda X,d)\quad\text{defined by }\ \alpha_0+y\alpha_1+y^2\beta+u\gamma\mapsto\alpha_1\end{aligned}$$

for $\alpha,\alpha_0,\alpha_1\in\Lambda X$ and $\beta,\gamma\in\Lambda X\otimes\Lambda(y,u)$. A quick check shows that both are also maps of differential modules, and that $\varphi\circ\mu=\mathrm{id}_{\Lambda X}$. Note that with the canonical filtration $F^rA=\Lambda^{\geq r}X.A$, both μ and φ are filtration preserving.

Let $\Lambda(X,y,u)$ denote $\Lambda X\otimes\Lambda(y,u)$.

Choose a semi-free model $\Lambda(X,y,u)\xrightarrow{j}\Lambda(X,y,u)\otimes(\mathbb{Q}\oplus N^{m+1,*})$ of the projection $\Lambda(X,y,u)\to\Lambda(X,y,u)/\Lambda^{>m+1}(X,y,u)$ and a retraction

$$\rho:\Lambda(X,y,u)\otimes(\mathbb{Q}\oplus N^{m+1,*})\to\Lambda(X,y,u)$$

with $\rho(N^{m+1,*})\subset\Lambda^{\geq r+1}(X,y,u)$. The map μ induces a map $\bar\mu$ and the homotopy lifting property yields a commutative diagram of $(\Lambda X,d)$-modules

$$\begin{tikzcd}
(\Lambda X,d) \arrow[r] \arrow[dr, hook, "i"'] & (\Lambda X/\Lambda^{>m}X,d) \arrow[r, "\bar\mu"] & (\Lambda X,y,u)/\Lambda^{>m+1}(X,y,u) & (\Lambda X,y,u) \arrow[l] \arrow[dl, hook, "j"] \\
& \Lambda X\otimes(\mathbb{Q}\oplus M^{m,*}) \arrow[u, "\simeq"'] \arrow[r, "\bar{\bar\mu}"] & \Lambda(X,y,u)\otimes(\mathbb{Q}\oplus N^{m+1,*}) \arrow[u, "\simeq"'] &
\end{tikzcd}$$

in which we may choose $\bar{\bar\mu}$ to be filtration preserving, since it is the lift of such a map over another. Because φ also preserves the filtration, the $(\Lambda X,d)$-module map $\varphi\circ\rho\circ\bar{\bar\mu}$ shows that $r_0S\geq r$.□

An almost identical argument establishes

Proposition 5. *If* $\mathbf{CP}^n$ *denotes complex projective* n*-space,*

$$r_0(S\times\mathbf{CP}^n)\leq r_0S+n.$$

We note however than this is probably not a sharp bound, since in the last section, we present the example $r_0(\mathbf{CP}^2\times\mathbf{CP}^2)=2=r_0(\mathbf{CP}^2)+r_0(\mathbf{CP}^2)<r_0(\mathbf{CP}^2)+2$.

Proposition 3 follows directly from the more general result Proposition 6 below. Recall [2] that the Milnor-Moore spectral sequence $\operatorname{Ext}_{H_*(\Omega S;\mathbb{Q})}(\mathbb{Q},\mathbb{Q})\implies H^*(S;\mathbb{Q})$ coincides from the E_2-term on with the spectral sequence obtained by filtering a Sullivan minimal model $(\Lambda X,d)$ of S by the word length.

The Ginsburg invariant $l_0(S)$ is the least n such that $\mathrm{E}^{n+1}=\mathrm{E}^{\infty}$, and we always have $l_0(S)\leq\operatorname{cat}_0 S$ [6]. For example, when S is a coformal space, $l_0(S)=1$, though the following example shows that the converse is not true: If S is the homotopy fibre of the map

$$K(\mathbb{Q}^3,4)\times K(\mathbb{Q},6)\to K(\mathbb{Q}^3,8)\times K(\mathbb{Q},12)$$

represented by the elements $x_1^2, x_2^2, x_3^2, x_4^2 - (x_1 + x_2 + x_3)^3$ in $H^*(K(\mathbb{Q}^3, 4) \times K(\mathbb{Q}, 6)) = \mathbb{Q}[x_1, x_2, x_3, x_4]$, where $|x_1| = |x_2| = |x_3| = 4$ and $|x_4| = 6$, then $l_0(S) = 1$ but S is not coformal.

An easy consequence of the definition is that if in a Sullivan minimal model $(\Lambda X, d)$ of S we have $d\alpha \in \Lambda^{\geq l_0(S)+m}X$, then there is $\alpha' \in \Lambda^{\geq m}X$ with $d\alpha = d\alpha'$.

Proposition 6. *If* $l_0(S) = 1$, *then* $r_0 S = \operatorname{cat}_0 S$.

Proof. Denote by $(\Lambda X, d)$ a minimal model of S, and suppose $\operatorname{cat}_0 S = n$.We then have a commutative diagram

$$\begin{tikzcd}
(\Lambda X, d) \arrow[r] \arrow[dr, "i"'] & (\Lambda X/\Lambda^{>n}X, d) \\
& (\Lambda X \otimes (\mathbb{Q} \oplus M^{n,*}), \delta) \arrow[u, "p", "\simeq"']
\end{tikzcd}$$

where $\delta(M^{n,*}) \subset \Lambda^{>n}X \oplus (\Lambda^+ X \otimes M^{n,*})$ increases (length) filtration degree by at least one.

We will prove by induction on the topological degree that we can choose a retraction $\rho : (\Lambda X \otimes (\mathbb{Q} \oplus M^{n,*}), \delta) \to (\Lambda X, d)$ with $\rho(M^{n,*}) \subset \Lambda^{\geq n}X$.

Suppose $m \in M$ is an element of least topological degree. Then $\delta m = dm = \beta$ is a cocyle in $\Lambda^{>n}X$. Because $\operatorname{cat}_0 S = n$, $H^* i$ is injective, so $\beta = d\alpha$, and, as mentioned above, we may choose $\rho(m) = \alpha \in \Lambda^{\geq n}X$ because $l_0(S) = 1$. Now assume ρ has been defined on $M^{n,<k}$ with $\rho(M^{n,<k}) \subset \Lambda^{\geq n}X$, and let $m \in M^{n,k}$. Then, since ρ is filtation preserving on $M^{n,<k}$, and $\delta m \in \Lambda^{>n}X \oplus (\Lambda^+ X \otimes M^{n,<k})$, $\rho(\delta m)$ is a cocyle in $\Lambda^{>n}X$, which we can again choose to be the coboundary of an element $\rho(m) = \alpha \in \Lambda^{\geq n}X$. $\square$

Remark. Note that this actually proves that $l_0(S) = 1 \implies r_0 S = e_0 S = \operatorname{cat}_0 S$, where $e_0 S$ is the Toomer-Moore invariant which always satisfies $e_0 S \leq \operatorname{cat}_0 S$ [14]

We prove now a slightly more general result that will have Proposition 4 as a corollary [8, 1].

Proposition 7. *If S and T are complexes with* $\dim T < p$ *such that*

1. $S = T \cup e^p$,
2. $T \hookrightarrow S$ *induces an injection* $H^*(S; \mathbb{Q}) \to H^*(S; \mathbb{Q})$,
3. $\operatorname{cat}_0 S = \operatorname{cat}_0 T + 1$, *and*
4. $\pi_*(T) \otimes \mathbb{Q} \to \pi_*(S) \otimes \mathbb{Q}$ *is surjective,*

then $r_0(S) \geq 2$.

Proof. Suppose $\operatorname{cat}_0 S = n$ and that $(\Lambda X, d)$ is a Sullivan minimal model for S. If $P \oplus (\ker d)^p = (\Lambda X)^p$, we see that

$$(\Lambda X, d) \xrightarrow{\simeq} (\Lambda X/(\Lambda^{>p}X \oplus P), d) =: (A, d)$$

Now $A^{>p} = 0$, $A^p = (\ker d)^p \ni \omega$ with $[\omega] \neq 0$, and $A^{<p} = (\Lambda X)^{<p}$, so $(A/\langle\omega\rangle, d)$ is quasi-isomorphic to a model for T.

This can be seen as follows: Let $\mathcal{M}_S \xrightarrow{\varphi} \mathcal{M}_T$ represent $T \hookrightarrow S$ and as we did above for S, choose a surjective quasi-isomorphism $\mathcal{M}_T \xrightarrow{\psi} C$ with $C^{>p-1} =$

0. Since $\psi\varphi(\Lambda^{>p}X \oplus P \oplus \langle\omega\rangle) = 0$, it factors through A and $A/\langle\omega\rangle$ to yield a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\bar{\varphi}"] \arrow[dr] & C \\ & A/\langle\omega\rangle \arrow[u, "f"'] \end{tikzcd} .$$

Since $H^m\varphi$ is an isomorphism for $m < p$, so is $H^m\bar{\varphi}$. Moreover, $H^{\geq p}A = \langle\omega\rangle$ and $H^{\geq p}C = 0$, so f is a quasi-isomorphism.

Thus, $A \to A/\langle\omega\rangle$ is a model for $T \hookrightarrow S$. However, we may replace $A/\langle\omega\rangle$ by a quasi-isomorphic model $(A \oplus \mathbb{Q}v, d)$ where A is a differential subalgebra, $dv = w$ and $v \cdot A^+ = 0$. Now choose an extension $\Lambda X \otimes \Lambda Y \xrightarrow{\simeq} A \oplus \mathbb{Q}v$. Since this is also a model of T, we have a model $\Lambda X \otimes \Lambda Y \otimes (\mathbb{Q} \oplus F) \xrightarrow{q} \Lambda(X \oplus Y)/\Lambda^{>n}(X \oplus Y)$ with $q(F) = 0$. Because $\operatorname{cat}_0 T = n - 1$ and $r_0 T \geq 1$, by Lemma 1, there is a retraction $\rho : \Lambda X \otimes \Lambda Y \otimes (\mathbb{Q} \oplus F) \to \Lambda X \otimes \Lambda Y$ with $\rho(F) \subset \Lambda^{\geq 2}(X \oplus Y)$.

Now suppose $\tilde{q}\colon \Lambda X \otimes (\mathbb{Q} \oplus G) \xrightarrow{\simeq} \Lambda X/\Lambda^{>n}X$ with $\tilde{q}(G) = 0$, let W denote $X \oplus Y$ and consider the commutative diagram

$$\begin{tikzcd} \Lambda X \otimes (\mathbb{Q} \oplus G) \arrow[r, "\tilde{q}"] \arrow[d, dash, "\psi"'] & \Lambda X/\Lambda^{\geq n}X \arrow[d, dash] \\ \Lambda W \otimes (\mathbb{Q} \oplus F) \arrow[r, "q"] & \Lambda W/\Lambda^{\geq n}W \end{tikzcd}$$

Note that since $0 = \tilde{q}(G)$, $q\psi(G) = 0$, $\psi(G) \subset \ker q = \Lambda^{\geq n}W \oplus (\Lambda W \otimes F)$, and so in particular, $\rho\psi(G) \subset \Lambda^{\geq 2}W$.

We also have the following commutative diagram

$$\begin{tikzcd} A & \Lambda X \arrow[l, "\simeq"'] \arrow[r] \arrow[d, dash] & \Lambda X \otimes (\mathbb{Q} \oplus G) \arrow[d, dash, "\psi"] \\ A \oplus \mathbb{Q}v \arrow[u, dash] & \Lambda W \arrow[l, "\sigma"', "\simeq"] & \Lambda W \otimes (\mathbb{Q} \oplus F) \arrow[l, "\rho"'] \end{tikzcd}$$

where

$$\sigma\rho\psi(G) \subset \sigma\rho(\Lambda^{\geq n}W \oplus (\Lambda W \otimes F)) \subset (A^+ \oplus \mathbb{Q}v) \cdot (A^+ \oplus \mathbb{Q}v),$$

from which we may conclude that $\sigma\rho\psi(G) \subset A$. Thus, $\sigma\rho\psi$ factors through A to yield $\tilde{\sigma} : \Lambda X \otimes (\mathbb{Q} \oplus G) \to A$ which we can lift over the surjective quasi-isomorphism $\Lambda X \xrightarrow{\simeq} A$ to a map $\tilde{\rho} : \Lambda X \otimes (\mathbb{Q} \oplus G) \to \Lambda X$ of $(\Lambda X, d)$-modules, which is thus a retraction. We now complete the proof of the proposition with

Lemma 2. *$\tilde{\rho}(G) \subset \Lambda^{\geq 2}X$.*

Proof of Lemma 2. From our lift of $\sigma\rho\psi$, we obtain the diagram

$$\begin{tikzcd} \mathbb{Q} \oplus X \cong \Lambda X/\Lambda^{>1}X \arrow[d, dash, "\nu"] & \Lambda X \arrow[l, "\tilde{\tau}"'] \arrow[d, dash, "j"] & \Lambda X \otimes (\mathbb{Q} \oplus G) \arrow[l, "\tilde{\rho}"'] \arrow[d, dash, "\psi"] \\ \mathbb{Q} \oplus X \oplus Y \cong \Lambda W/\Lambda^{>1}W & \Lambda W \arrow[l, "\tau"'] & \Lambda W \otimes (\mathbb{Q} \oplus F) \arrow[l, "\rho"'] \end{tikzcd}$$

where the left-hand square commutes and the right hand square commutes up to homotopy (because $\sigma\rho\psi = \sigma j\tilde{\rho}$ and σ is a quasi-isomorphism.) Moreover, since $\rho\psi(G) \subset \Lambda^{\geq 2}W$, $\tau\rho\psi(G) = 0$.

Since $T \hookrightarrow S$ is surjective in rational homotopy, and $(\Lambda X \otimes \Lambda Y, d)$ is a minimal extension ($dY \subset \Lambda^+ X \oplus (\Lambda^+ X \otimes \Lambda Y) \oplus (\Lambda X \otimes \Lambda^{>1} Y)$), we can conclude that the differential in $\mathbb{Q} \oplus X \oplus Y$ is zero.

The homotopy $\tau\rho\psi \simeq \tau\tilde{j}\rho$ means that we have a ΛX-module map

$$h : \Lambda X \otimes (\mathbb{Q} \oplus G) \to \mathbb{Q} \oplus X \oplus Y,$$

with $h(1) = 0$, $h(G) \subset X \oplus Y$ (for degree reasons) and

$$\tau\tilde{j}\rho - \tau\rho\psi = dh + hd = hd.$$

Applying this to G, we find

$$\tau\tilde{j}\rho(G) = hd(G) \subset h(\Lambda^{>n} X \oplus (\Lambda^+ X \otimes G)) = 0,$$

since $\Lambda^{>n} X \cdot h(1) = 0$ and $\Lambda^+ X \cdot (X \oplus Y) = 0$. Hence, $\tilde{\rho}(G) \subset \Lambda^{\geq 2} X$. This concludes the proof of the lemma and the Proposition.□

3. The retraction index and the depth

We prove here the promised relation between the retraction index and the homotopy Lie algebra.

Theorem 3. *If S is a space,*

$$\text{depth } H_*(\Omega S; \mathbb{Q}) \leq r_0 S.$$

Proof. The proof will imitate that of Lemma 2.9 of [3].

Recall that the depth of a graded augmented $\mathbb{Q}$-algebra A is the least n such that $\text{Ext}_A^n(\mathbb{Q}, A) \neq 0$. Denote by $(\Lambda V, d)$ the Sullivan minimal model of S, and let $(\Lambda V \otimes (\mathbb{Q} \oplus M_m), d) \xrightarrow{\simeq} (\Lambda V/\Lambda^{>m} V, d)$ be a semi-free model of the quotient $(\Lambda V, d) \xrightarrow{q} (\Lambda V/\Lambda^{>m} V, d)$ of $(\Lambda V, d)$-differential modules as in the introduction. Since, depth $H_*(\Omega S; \mathbb{Q}) \leq \text{cat}_0 S = n$ [3], we know that a retraction

$$\rho_1 : (\Lambda V \otimes (\mathbb{Q} \oplus M_n), d) \to (\Lambda V, d)$$

of $(\Lambda V, d)$-differential modules exists. Suppose that depth $H_*(\Omega S; \mathbb{Q}) = r$. We contruct a retraction ρ_r with $\rho_r(M_n) \subset \Lambda^{\geq r} V$ by induction as follows.

For degree reasons, we can always assume that $\rho_1(M_n) \subset \Lambda^{\geq 1} V$. Now assume that for $1 \leq i - 1 \leq r$ we have a map of $(\Lambda V, d)$-differential modules

$$\rho_{i-1} : (\Lambda V \otimes (\mathbb{Q} \oplus M_n), d) \to (\Lambda V, d)$$

such that $\rho_{i-1}(v \otimes 1) = v$ for all $v \in V$, and $\rho_{i-1}(M_n) \subset \Lambda^{\geq i} V$. Write $\rho_{i-1} \mid_{M_n} = \alpha + \beta$ with $\text{Im}\, \alpha \subset \Lambda^{i-1} V$ and $\text{Im}\, \beta \subset \Lambda^{\geq i} V$. Then, if d_2 is the quadratic part of the differential d, we have $d_2 \alpha = \alpha \delta_1$ where $\delta_1 \colon M_n \to V \otimes M_n$.

Thus the map α is a cocycle in $\text{Hom}_{\Lambda V}^{i-1}((\Lambda V \otimes (\mathbb{Q} \oplus M_n), \delta_1), (\Lambda V, d_2))$. By [3,1.16], $[\alpha] \in \text{Ext}_{H_* \Omega S; \mathbb{Q}}^{i-1}(\mathbb{Q}, \text{Hom}\,(M_n, \mathbb{Q}))$, and from [3, 2.5], we know that $\text{Hom}\,(M_n, \mathbb{Q})$ is an n^{th} syzygy of $\mathbb{Q}$. Just as in the proof of [3, 2.9] we find that $\text{Ext}_{H_*(\Omega S; \mathbb{Q})}^{i-1}(\mathbb{Q}, \text{Hom}\,(M_n, \mathbb{Q})) = 0$. Hence, there is a $(\Lambda V, d)$-linear map $\sigma \colon \Lambda V \otimes M_n \to \Lambda V$ of total degree -1 such that $\sigma(M_n) \subset \Lambda^{i-2}$ and $\alpha = d_2 \sigma + \sigma \delta_1$. Now extend σ to $\Lambda V \otimes (\mathbb{Q} \oplus M_n)$ by setting $\sigma(\Lambda V \otimes \mathbb{Q}) = 0$, and define $\rho_i = \rho_{i-1} - d\sigma + \sigma d$. □

4. Proof of Theorems 1 and 2

Let $F \to E \to B$ be a fibration, and $(\Lambda X, d) \to (\Lambda X \otimes \Lambda Y, D) \to (\Lambda Y, \bar{D})$ its relative Sullivan model. We denote $\operatorname{cat}_0 B = n$, $\operatorname{cat}_0 F = m$ and $r_0 F = r_0$, and for convenience of notation, set $\varepsilon = m - r_0 + 2$, which we note is always at least 2.

Our goal is to show that $(\Lambda X \otimes \Lambda Y, D)$ is a homotopy retract of

$$((\Lambda X \otimes \Lambda Y)/\Lambda^{>m+n\varepsilon}(X \oplus Y), D),$$

as $(\Lambda X \otimes \Lambda Y, D)$-modules. To accomplish this, it suffices to find a $(\Lambda X \otimes \Lambda Y, D)$-module Γ_0, and a surjection $(\Lambda X \otimes \Lambda Y, D) \to \Gamma_0$ such that

1. $\Lambda^{>m+n\varepsilon}(X \oplus Y).\Gamma_0 = 0$, and
2. $(\Lambda X \otimes \Lambda Y, D)$ is a homotopy retract of Γ_0.

To begin, we define $(\Lambda X \otimes \Lambda Y, D)$-modules Γ_k by

$$\begin{aligned} \Gamma_{n+1} &= (\Lambda X \otimes \Lambda Y)/(\Lambda^{>n} X \otimes \Lambda Y), \quad \text{and} \\ \Gamma_k &= \Gamma_{k+1}/(\Lambda^k X \otimes \Lambda^{>m+(n-k)\varepsilon} Y), \quad \text{for } 0 \le k \le n. \end{aligned}$$

We thus have a sequence of surjective maps

$$(\Lambda X \otimes \Lambda Y, D) \to \Gamma_{n+1} \to \Gamma_n \to \Gamma_{n-1} \to \cdots \to \Gamma_0,$$

all of which, as we shall show, admit homotopy retractions.

Before building the retractions, we show that Γ_0 satisfies (1) above. Note that $\Gamma_0 = \Lambda X \otimes \Lambda Y/(I \oplus I_0 \oplus I_1 \oplus \cdots \oplus I_n)$ where

$$\begin{aligned} I &= \Lambda^{>n} X \otimes \Lambda Y, \qquad \text{and} \\ I_k &= \Lambda^{n-k} X \otimes \Lambda^{>m+k\varepsilon} Y. \end{aligned}$$

An element γ of Γ_0^+ therefore has a representative of the form $\gamma = \sum_{i=0}^n \gamma_i$, with $\gamma_i \in \Lambda^i X \otimes \Lambda^{\ge 1-i} Y$. Hence, an element of $\Lambda^{>m+n\varepsilon}(X \oplus Y).\Gamma_0$ is represented by a sum of terms of the form $\gamma_0^{j_0}\gamma_1^{j_1}\dots\gamma_n^{j_n}$ with $\sum_{i=0}^n j_i = m + n\varepsilon + 1$, which we denote here by s. Each term $\gamma_0^{j_0}\gamma_1^{j_1}\dots\gamma_n^{j_n}$ belongs to $\Lambda^{j_1+2j_2+\cdots+nj_n} X \otimes \Lambda^{\ge j_0} Y$. Now $\sum_{i=1}^n ij_i \ge \sum_{i=1}^n j_i = s - j_0$, so $\gamma_0^{j_0}\gamma_1^{j_1}\dots\gamma_n^{j_n} \in \Lambda^{\ge s-j_0} X \otimes \Lambda^{\ge j_0} Y$. But either $j_0 \le m$ or $m + k\varepsilon < j_0 \le m + (k+1)\varepsilon$, for some k with $0 \le k \le n$. If $j_0 \le m$, $s - j_0 \ge n\varepsilon + 1 \ge n + 1$ so that $\gamma_0^{j_0}\gamma_1^{j_1}\dots\gamma_n^{j_n} \in I$. In the other case, $s - j_0 \ge (n-k-1)\varepsilon + 1 \ge (n-k-1)+1 = n-k$, and so $\gamma_0^{j_0}\gamma_1^{j_1}\dots\gamma_n^{j_n} \in I_k \oplus \cdots \oplus I_n \oplus I$. Hence, in either case, $\Lambda^{>m+n\varepsilon}(X \oplus Y).\Gamma_0 = 0$.

We begin the retractions with the first map. Since $\operatorname{cat}_0 B = n$, the inclusion i_n in the following commutative diagram admits a retraction ρ

$$\begin{tikzcd} (\Lambda X, d) \arrow[r] \arrow[dr, "i_n"] & (\Lambda X/\Lambda^{>n} X, d) \\ & (\Lambda X \otimes (\mathbb{Q} \oplus N_n), \delta) \arrow[u, "p_n", "\simeq"'] \arrow[ul, dotted, bend left, "\rho"] \end{tikzcd} .$$

In particular, $(\Lambda X \otimes \Lambda Y, D)$ is a retract of the $(\Lambda X \otimes \Lambda Y, D)$-differential graded module

$$(\Lambda X \otimes (\mathbb{Q} \oplus N_n) \otimes \Lambda Y, D) := (\Lambda X \otimes (\mathbb{Q} \oplus N_n), \delta) \otimes_{(\Lambda X, d)} (\Lambda X \otimes \Lambda Y, D),$$

which is quasi-isomorphic to $(\Lambda X/\Lambda^{>n}X \otimes \Lambda Y, D) = \Gamma_{n+1}$. This gives us the first homotopy retraction.

We continue with the contruction of the retractions $\Gamma_k \to \Gamma_{k+1}$, for $0 \le k \le n$, and we will facilitate this by replacing Γ_k by another module Γ'_k, which is quasi-isomorphic to Γ_k, and which contains Γ_{k+1} as a submodule.

First, we recall that by Lemma 1, for each $q \ge m$,the map i_q in the diagram

$$\begin{tikzcd}
(\Lambda Y, \bar{D}) \arrow[r] \arrow[dr, "i_q"] & (\Lambda Y/\Lambda^{>q}Y, d) \\
& (\Lambda Y \otimes (\mathbb{Q} \oplus M_q), \bar{D}_q) \arrow[u, "p_q", "\simeq"'] \arrow[ul, dotted, bend left, "\rho_q"]
\end{tikzcd}$$

of $(\Lambda Y, \bar{D})$-modules admits a retraction ρ_q satisfying $\rho_q(M_q) \subset \Lambda^{\ge q-m+r}Y$.

Now note that Γ_k is isomorphic as a graded vector space to

$$\oplus_{i=0}^{k-1}(\Lambda^i X \otimes \Lambda Y) \oplus_{i=k}^{n} (\Lambda^i X \otimes \Lambda Y/\Lambda^{>m+(n-i)\varepsilon}Y).$$

We define $(\Lambda X \otimes \Lambda Y, D)$-module Γ'_k by

$$\Gamma'_k = (\Gamma_{k+1} \oplus (\Lambda^k X \otimes \Lambda Y \otimes \bar{M}), d), \quad \text{where } \bar{M} = M_{m+(n-k)\varepsilon}\,.$$

The summand Γ_{k+1} is included as a $(\Lambda X \otimes \Lambda Y, D)$-differential submodule, and the $(\Lambda X \otimes \Lambda Y)$-module structure and the differential on $\Lambda^k X \otimes \Lambda Y \otimes \bar{M}$ are defined by

$$\begin{aligned}
x \cdot (\alpha \otimes \beta \otimes \bar{m}) &= 0, \\
y \cdot (\alpha \otimes \beta \otimes \bar{m}) &= (-1)^{|\alpha||y|}\alpha \otimes (y \cdot \beta) \otimes \bar{m}, \quad \text{and} \\
d(\alpha \otimes \beta \otimes \bar{m}) &= (-1)^{|\alpha|}\alpha \otimes \bar{D}_q(\beta \otimes \bar{m}),
\end{aligned}$$

where $x \in X$, $y \in Y$, $\alpha \otimes \beta \otimes \bar{m} \in \Lambda^k X \otimes \Lambda Y \otimes \bar{M}$, and we have denoted $m+(n-k)\varepsilon$ by q. We extend d as a derivation of the module, but we need to check that $d^2 = 0$ on $\Lambda^k X \otimes \Lambda Y \otimes \bar{M}$.

Note that we can write $\bar{D}_q = \bar{D}'_q + \bar{D}''_q$, where $\bar{D}'_{q|Y} = \bar{D}_{q|Y} = \bar{D}$, $\bar{D}_{q|\bar{M}} = \bar{D}'_q + \bar{D}''_q$, and where $\bar{D}'_q\bar{M} \subset \Lambda^{>q}Y$ and $\bar{D}''_q\bar{M} \subset \Lambda^+ Y \otimes \bar{M}$. Hence,

$$\begin{aligned}
d^2(\alpha \otimes \beta \otimes \bar{m}) &= d((-1)^{|\alpha|}\alpha \otimes \bar{D}_q(\beta \otimes \bar{m})) \\
&= d((-1)^{|\alpha|}\alpha \otimes (\bar{D}\beta \otimes \bar{m} + (-1)^{|\beta|}\beta \cdot \bar{D}'_q\bar{m} + (-1)^{|\beta|}\beta \cdot \bar{D}''_q\bar{m})) \\
&= \alpha \otimes \bar{D}_q(\bar{D}\beta \otimes \bar{m}) + d((-1)^{|\alpha|}\alpha \otimes (-1)^{|\beta|}\beta \cdot \bar{D}'_q\bar{m}) + \\
&\qquad + \alpha \otimes \bar{D}_q((-1)^{|\beta|}\beta \cdot \bar{D}''_q\bar{m}) \\
&= \alpha \otimes \bar{D}_q(\bar{D}\beta \otimes \bar{m}) + (-1)^{|\alpha|+|\beta|}D\alpha \otimes \beta \cdot \bar{D}'_q\bar{m} \\
&\qquad + \alpha \otimes \bar{D}((-1)^{|\beta|}\beta \cdot \bar{D}'_q\bar{m})) + \alpha \otimes \bar{D}_q((-1)^{|\beta|}\beta \cdot \bar{D}''_q\bar{m}).
\end{aligned}$$

But, $(-1)^{|\alpha|+|\beta|}D\alpha \otimes \beta \cdot \bar{D}'_q\bar{m} \in \Lambda^{\ge k+1}X \otimes \Lambda^{>q}Y = 0$ in Γ_{k+1}, so

$$\begin{aligned} d^2(\alpha\otimes\beta\otimes\bar m) &= \alpha\otimes\bar D_q(\bar D\beta\otimes\bar m)\\ &\qquad +\alpha\otimes\bar D((-1)^{|\beta|}\beta\cdot\bar D'_q\bar m))+\alpha\otimes\bar D_q((-1)^{|\beta|}\beta\cdot\bar D''_q\bar m)\\ &= \alpha\otimes\bar D_q(\bar D\beta\otimes\bar m)+\alpha\otimes\bar D_q((-1)^{|\beta|}\beta\cdot\bar D_q\bar m))\\ &= \alpha\otimes\bar D_q^2(\beta\otimes\bar m)\\ &= 0. \end{aligned}$$

We next show that (Γ_k, D_k) and (Γ'_k, d) are quasi-isomorphic. Recall the surjection

$$\tau:\Gamma_{k+1}\to\Gamma_k=\Gamma_{k+1}/(\Lambda^kX\otimes\Lambda^{>q}Y),$$

and the surjective quasi-isomorphism $p_q\colon(\Lambda Y\otimes(\mathbb{Q}\oplus M_q),\bar D_q)\to(\Lambda Y/\Lambda^{>q}Y,\bar D)$, and define $\pi\colon\Gamma'_k\to\Gamma_k$ by mapping $\pi_{|\Gamma_{k+1}}=\tau$ and $\pi(\alpha\otimes\beta\otimes\bar m)=0$.

To see that π is a map of $(\Lambda X\otimes\Lambda Y,\bar D)$-modules, it suffices to check that $\pi d(\alpha\otimes\beta\otimes\bar m)=0$. But $\pi d(\alpha\otimes\beta\otimes\bar m)=\pi(\alpha\otimes\bar D_q(\beta\otimes\bar m))=\pm\alpha.\beta\bar D'_q\bar m\in\Lambda^kX\otimes\Lambda^{>q}Y$ and so is zero in Γ_k.

Since π is surjective, to show that it is a quasi-isomorphism, it suffices to show that $H^*(\ker\pi)=0$. But $\ker\pi=(\Lambda^kX\otimes\Lambda^{>q}Y)\oplus(\Lambda^kX\otimes\Lambda Y\otimes\bar M)=\Lambda^kX\otimes\ker p_q$, and the differential in $\ker\pi$ is zero on Λ^kX, and $\bar D_q$ on $\ker p_q$. Thus, $H^*(\ker\pi)=\Lambda^kX\otimes H^*(\ker p_q)=0$, because p_q is a surjective quasi-isomorphism.

Finally, we complete the proof of the theorem by proving that Γ_{k+1} is a retract of Γ'_k.

Define $\sigma_k:\Gamma'_k\to\Gamma_{k+1}$ as the identity on Γ_{k+1} and, using the retraction ρ_q, by

$$\sigma_k(\alpha\otimes\beta\otimes\bar m)=\alpha\otimes\rho_q(\beta\otimes\bar m)$$

on $\Lambda^kX\otimes\Lambda Y\otimes M$.

To see that σ_k is a map of $\Lambda X\otimes\Lambda Y$-modules, it suffices to check that $\sigma_k(x.\alpha\otimes\beta\otimes\bar m)=x.\sigma_k(\alpha\otimes\beta\otimes\bar m)$ for $x.\alpha\otimes\beta\otimes\bar m\in\Lambda^kX\otimes\Lambda Y\otimes M$. The left hand side is zero because $x.\alpha\otimes\beta\otimes\bar m=0$ in Γ'_k, while $x.\sigma_k(\alpha\otimes\beta\otimes\bar m)\in\Lambda^{k+1}X\otimes\Lambda^{\geq q-m+r_0}Y$. However, $q=m+(n-k)\varepsilon$ and $\varepsilon=m-r_0+2$, so $q-m+r_0=(n-k)\varepsilon+r_0=(n-k-1)\varepsilon+m+2$, so $x.\sigma_k(\alpha\otimes\beta\otimes\bar m)\in\Lambda^{k+1}X\otimes\Lambda^{>m+(n-k-1)\varepsilon}Y=I_{n-(k+1)}$, which is zero in Γ_{k+1}.

To conclude, we show that $\sigma_kd=D_{k+1}\sigma_k$, and remark that this only needs to be checked on $\Lambda^kX\otimes\Lambda Y\otimes\bar M$:

$$\begin{aligned}\sigma_kd(\alpha\otimes\beta\otimes\bar m)&=(-1)^{|\alpha|}\alpha\otimes\rho_q\bar D_q(\beta\otimes\bar m)\\ &=(-1)^{|\alpha|}\alpha\otimes\bar D\rho_q(\beta\otimes\bar m)\\ &=(-1)^{|\alpha|}\alpha\otimes\bar D(\beta\rho_q(\bar m)).\end{aligned}$$

On the other hand,

$$\begin{aligned}D_{k+1}\sigma_k(\alpha\otimes\beta\otimes\bar m)&=D_{k+1}(\alpha\otimes\beta\rho_q(\bar m))\\ &=D_{k+1}\alpha\otimes\beta\rho_q(\bar m))+(-1)^{|\alpha|}\alpha\cdot D_{k+1}(\beta\rho_q(\bar m))\\ &=(-1)^{|\alpha|}\alpha\cdot D_{k+1}(\beta\rho_q(\bar m)).\end{aligned}$$

But, $D_{k+1}=\bar D+D^{\#}$, where $D^{\#}(\Lambda Y)\subset\Lambda^+X\otimes\Lambda Y$, so $\sigma_kd=D_{k+1}\sigma_k$.

The proof of Theorem 2 follows exactly the same lines. In that case, we are able to decrease ε by one to $\operatorname{cat}_0F-r_0F+1$ because when $F\to E$ is injective in rational homotopy, the differential in the model of the fibration $(\Lambda X\otimes\Lambda Y,D)$ can be chosen to satisfy $DY\subset\Lambda X\otimes\Lambda^+Y\oplus(\Lambda^+X\otimes\Lambda Y)$. □

5. Examples and Open problems

We present here some examples to show that the estimates of the theorems and the corollaries, are sharp, and end by posing some open questions.

Example 1. Consider the fibration whose minimal model is

$$\Lambda(h, w; d) \to \Lambda(h, w, a, b, x, y; d) \to \Lambda(a, b, x, y; \bar{d})$$

where $|a| = |b| = 2, |x| = |y| = 3, |h| = 4, |w| = 7$, $da = db = dh = 0$, $dx = a^2 + h$, $dy = b^2 + h$, and $dw = h^2$.

Here, the base space is $\mathbf{S}^4_{\mathbb{Q}}$ with $cat_0\mathbf{S}^4_{\mathbb{Q}} = 1$, and the fibre is $\mathbf{S}^2_{\mathbb{Q}} \times \mathbf{S}^2_{\mathbb{Q}}$, with $cat_0(\mathbf{S}^2_{\mathbb{Q}} \times \mathbf{S}^2_{\mathbb{Q}}) = 2$, so the estimate of Theorem 1, using Proposition 2 (or 3 or 4) is

$$\mathrm{cat}_0 E \leq \mathrm{cat}_0 F + \mathrm{cat}_0 B(\mathrm{cat}_0 F + 2 - r_0 F) = 2 + 1(2 + 2 - 2) = 4.$$

Moreover, $\Lambda(a, b, x, w; d)$ is a pure tower, its formal dimension is 8, and $|a| = |b| = 2$, so the top class is a homogeneous polynomial of degree 4 in a and b. Using $\mathrm{e}_0 = \mathrm{cat}_0$, we see that $\mathrm{cat}_0\Lambda(a, b, x, w; d) = 4$.

This example shows that the estimates of Theorem 1, Corollaries 1 and 2, and Propositions 1 and 4 are sharp, and that some extra hypothesis on $F \to E \to B$ is necessary for the estimate of Theorem 2 to hold.

Example 2. Here we present examples with $r_0 = 1$ where the bound of Theorem 1 is attained. Let $n \geq 1$ and $k \geq 2$ and consider the space with model $A_k(n) = \Lambda(a, x; d)$ with $da = 0$ and $dx = a^{n+1}$, and $|a| = 2k$. We then have a fibration

$$A_{k(n+1)}(j) \leftarrow A_k((n + 1)(j + 1) - 1) \leftarrow A_k(n)$$

with model

$$\Lambda(a, x; d) \to \Lambda(a, x, b, y; d) \to \Lambda(b, y; \bar{d})$$

where $da = 0 = db$, $dx = a^{j+1}$, $dy = b^{n+1} + a$, and $|a| = 2k(n + 1)$, which is the model of a fibration. The base space B satisfies $\mathrm{cat}_0 A_{k(n+1)}(j) = j$, the fibre F has $\mathrm{cat}_0 A_k(n) = n$, while the total space E has

$$\mathrm{cat}_0 A_k((n + 1)(j + 1) - 1) = (n + 1)(j + 1) - 1 = \mathrm{cat}_0 F + \mathrm{cat}_0 B(\mathrm{cat}_0 F + 1).$$

By Theorem 1, we see that $r_0 F = 1$ and so the upper bound is attained.

Example 3. If we take the product of the total space and the fibre of example 2 with $(\mathbf{S}^2_{\mathbb{Q}})^p = \Pi^p_{i=1} S^2_{\mathbb{Q}}$ we then have a fibration

$$A_{k(n+1)}(j) \leftarrow A_k((n + 1)(j + 1) - 1) \times (\mathbf{S}^2_{\mathbb{Q}})^p \leftarrow A_k(n) \times (\mathbf{S}^2_{\mathbb{Q}})^p$$

where $\mathrm{cat}_0 B = j$, $\mathrm{cat}_0 F = n + p$, $\mathrm{cat}_0 E = (n + 1)(j + 1) + p - 1$, and $r_0 F = p + 1$, showing that the bound of Theorem 1 can be attained for any value of $r_0 F$.

Example 4. Consider the rational fibration $F \to E \to B$ with model

$$\Lambda(a, b, x, y; d) \to \Lambda(a, b, x, y, z; d) \to \Lambda(z; 0)$$

where $da = db = 0$, $dy = a^3$, $dx = b^3$ and $dz = a^2b$. Using the lower bounds of [5,Cor. 2], we find that $\mathrm{cat}_0 E \geq 5$. Since the fibre is an odd sphere, which is coformal, and $F \to E$ is injective in rational homotopy, Corollary 3 guarantees

$$\mathrm{cat}_0 E \leq \mathrm{cat}_0 F + \mathrm{cat}_0 B = 1 + \mathrm{cat}_0(\mathbf{CP}^2_{\mathbb{Q}} \times \mathbf{CP}^2_{\mathbb{Q}}) = 5,$$

which is sharp as well.

Example 5. Consider the rational fibration $F \to E \to B$ with model

$$\Lambda(a, x; d) \to \Lambda(a, b, c, x, y, z, w; d) \to \Lambda(b, c, y, z, w; \bar{d})$$

where $|b| = |c| = 2$, $da = db = dc = 0$, $dx = a^2$, $dy = b^2$ $dz = bc$, and $dw = c^2 + a$. Since the fibre is coformal and the base is $\mathbf{S}^4_{\mathbb{Q}}$, Theorem 1 and Proposition 3 together show that

$$\mathrm{cat}_0 E \leq \mathrm{cat}_0 F + 2\mathrm{cat}_0 B = 3 + 2 = 5.$$

A minimal model of E is $\Lambda(b, c, x, y, z; \tilde{d})$ with $|b| = |c| = 2$, $\tilde{d}b = \tilde{d}c = 0$, $\tilde{d}x = c^4$, $d\tilde{y} = b^2$, and $\tilde{d}z = bc$. Hence, the Mapping Theorem applied to

$$\Lambda(b; 0) \to \Lambda(b, c, x, y, z; \tilde{d}) \to \Lambda(c, x, y, z; d')$$

yields $\mathrm{cat}_0 E \geq \mathrm{cat}_0(\mathbf{CP}^3 \times \mathbf{S}^3 \times \mathbf{S}^3) = 5$, showing again that the estimate of Theorem 1 is sharp, even when the fibre is not formal.

Example 6. Here we will show that $r_0(\mathbf{CP}^2 \times \mathbf{CP}^2) = r_0(\mathbf{CP}^2) + r_0(\mathbf{CP}^2)$. We first compute $r_0(\mathbf{CP}^2)$. Let $\Lambda(a, x; d)$ be a model for $\mathbf{CP}^2$, so that $|a| = 2$, $da = 0$ and $dx = a^3$. Since $\mathrm{cat}_0(\mathbf{CP}^2) = 2$, we first consider the usual diagram

$$\begin{tikzcd}
\Lambda(a,x;d) \arrow[r] \arrow[dr, hook, "i"'] & (\Lambda(a,x)/\Lambda^{>2}(a,x), \bar{d}) \\
& (\Lambda(a,x;d) \otimes (\mathbb{Q} \oplus M), \delta) \arrow[u, "p"', "\simeq"]
\end{tikzcd}$$

where $p(M) = 0$. Note that $M = M^{2, \geq 5}$, and that $(\Lambda(a, x))^{\geq 6} \subset \Lambda^{\geq 2}(a, x)$, so in order to see that $r_0(\mathbf{CP}^2) = 1$, it suffices show that no retraction ρ of i can satisfy $\rho(M^5) \subset \Lambda^{\geq 2}(a, x)$.

Because $H^*(\Lambda(a, x)/\Lambda^{>2}(a, x)) = \langle [1], [a], [x], [a^2], [ax] \rangle$, H^*i is an isomorphsim in cohomology in degrees less than 5, and there must be an element $m \in M^5$ to make H^5p an isomorphism. In order that $p(m) = 0$, we must have $dm = a^3$, so that $H^*p(x - m) = [y]$. Then, any differential retraction ρ must send m to x, so $r_0(\mathbf{CP}^2) = 1$.

Now consider the analagous diagram for $\mathbf{CP}^2 \times \mathbf{CP}^2$:

$$\begin{tikzcd}
\Lambda(a,b,x,y;d) \arrow[r, "q"] \arrow[dr, hook, "i"'] & \Lambda(a,b,x,y)/\Lambda^{>4}(a,b,x,y) \\
& (\Lambda(a,b,x,y;d) \otimes (\mathbb{Q} \oplus T), \delta) \arrow[u, "p"', "\simeq"]
\end{tikzcd}$$

where $|b| = 2$, $db = 0$ and $dy = b^3$.

It is straightforward that H^*q is an isomorphsim in cohomology in degrees less than 9, so $T = T^{4,\geq 9}$. (From now on, a single superscript on T will indicate topological degree.) Moreover,

$$(\Lambda(a,b,x,y))^9 \oplus (\Lambda(a,b,x,y))^{\geq 11} \subset \Lambda^{\geq 3}(a,b,x,y),$$

so it suffices to study retractions on $T^{\leq 10}$.

We compute what we need of $T^{\leq 10}$ as follows: Let $(\Lambda X, d)$ be the minimal model above for $\mathbf{CP}^2 \times \mathbf{CP}^2$, and suppose $(\Lambda X, d) \to \mathbb{Q}$ has minimal model $(\Lambda X \otimes \Lambda \bar{X}, d)$. Then $\bar{X} = \langle \bar{a}, \bar{b}, \bar{x}, \bar{y} \rangle$ where $d\bar{a} = a$, $d\bar{b} = b$, $d\bar{x} = x - a^2\bar{a}$, and $d\bar{y} = y - b^2\bar{b}$. The short exact sequence

$$0 \to \Lambda^{>4} X \otimes \Lambda \bar{X} \to \Lambda X \otimes \Lambda \bar{X} \to \Lambda X / \Lambda^{>4} X \otimes \Lambda \bar{X} \to 0$$

and the isomorphism $(\mathbb{Q} \oplus T)^* \cong H^*(\Lambda X/\Lambda^{>4} X \otimes \Lambda \bar{X})$ together yield the isomorphism

$$(\mathbb{Q} \oplus T)^n \cong H^{n+1}(\Lambda^{>4} X \otimes \Lambda \bar{X}, d)$$

which we will also use to determine the differential on M^9. Note that

$$H^{10}(\Lambda^{>4} X \otimes \Lambda \bar{X}, d) = \operatorname{span}\{a^j b^{5-j} \mid 0 \leq j \leq 5\}.$$

Thus, we have $T^9 = \operatorname{span}\{m_j \mid 0 \leq j \leq 5\}$ with $\delta t_j = a^j b^{5-j}$. Note that since $Z(\Lambda X)^9 = 0$, the restriction of any retraction to T^9 is uniquely determined. In particular, for any retraction ρ, we must have $\rho(t_j) = a^j b^{2-j} y$ for $0 \leq j \leq 2$, and $\rho(t_j) = a^{j-3} b^{5-j} x$ for $3 \leq j \leq 5$.

Now, the cycle $at_2 - bt_3 \in (\Lambda(a,b,x,y) \otimes T)^{5,11}$ must be a boundary because $H^{>4,*}(\Lambda^{>4} X \otimes, d) = 0$. Hence there is $t \in T^{10}$ with $\delta t = at_2 - bt_3$. Then, $\rho(\delta t) = a^3 y - b^3 x$, and the only solution to $d\rho(t) = a^3 y - b^3 x$ is $\rho(t) = xy \in \Lambda^2(a,b,x,y)$. Hence, $r_0(\mathbf{CP}^2 \times \mathbf{CP}^2) = 2$.

Open Problems.

1. If we denote by $G_m(S) \xrightarrow{p_m} S$ the m^{th} Ganea fibration of S and suppose that $\operatorname{cat} S = n$, then p_n admits a section σ. If we define $r(S)$ to be the largest $m \leq n$ such that $\sigma \circ p_{m-1}$ is homotopic to the injection $G_{m-1}(S) \hookrightarrow G_n(S)$, then clearly $r_0 S \geq r(S)$. Does equality hold when S is a rational space?
2. Are Theorems 1 and 2 true with $\operatorname{cat} S$ and $r(S)$ replacing $\operatorname{cat}_0 S$ and $r_0 S$ respectively?
3. We know from Propositions 1 and 2 that $r_0(S \times T) \geq r_0 S + r_0 T$ and that r_0 satisfies its own 'Ganea'-type conjecture: $r_0(S \times \mathbf{S}^n) = r_0 S + r_0(\mathbf{S}^n)$, and example 6 above shows that $r_0(\mathbf{CP}^2 \times \mathbf{CP}^2) = r_0(\mathbf{CP}^2) + r_0(\mathbf{CP}^2)$. Is $r_0(S \times T) = r_0 S + r_0 T$, for all spaces S and T?
4. Does $l_0 S > 1 \implies r_0 S < \operatorname{cat}_0 S$?
5. If S is formal, what invariant of the algebra $H^*(S, \mathbb{Q})$ is $r_0 S$?

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