Faculty of Science »
Department of mathematics and statistics

MAT 1341C Final Exam, 2011

26-April, 2011.

Instructor - Barry Jessup

Family Name:____________________

First Name:____________________

Student number:____________________

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Some Advice

Take 5 minutes to read the entire paper before you begin to write, and read each question carefully. The multiple choice questions are only worth 1 point and the others are worth more.
Make a note of the questions you feel confident you can do, and try those first: you do not have to try the questions in the order given.

Instructions

1. You have 3 hours to complete this exam.
2. This is a closed book exam, and no notes of any kind are allowed. **The use of calculators**, cell phones, pagers or any text storage or communication device **is not permitted.**
3. Questions 1 to 10 are multiple choice. These questions are worth 1 point each and no part marks will be given. Please record your answers in the spaces opposite.
4. Questions 11 – 15 require a complete solution, and are worth 6 points each, so spend your time accordingly. Answer these questions in the space provided, and use the backs of pages if necessary.
   Question 16 is a bonus question and should only be attempted after all other questions have been completed and checked.
5. **The correct answer requires justification written legibly and logically: you must convince me that you know why your solution is correct.**
6. Where it is possible to check your work, do so.

Good luck! Bonne chance!

1. Let $W = \{(x, y, z, w) \in \mathbf{R}^4 \mid xy + zw \geq 0\}$. Then,

A. $(0, 0, 0, 0) \in W$ but W is not closed under multiplication by scalars.

B. $(0, 0, 0, 0) \notin W$ but W is closed under addition.

C. W is closed under addition but W is not closed under multiplication by scalars.

D. W is closed under addition and W is closed under multiplication by scalars.

E. W is not closed under addition but W is closed under multiplication by scalars.

F. None of the other statements is true.

2. Which of the following statements are true?

I. A set $\{u, v, w\}$ of vectors is linearly independent iff for scalars $a, b, c \in \mathbf{R}$, $au + bv + cw = 0$ implies $a = b = c = 0$.

II. A set $\{u, v, w\}$ of vectors is linearly independent iff for scalars $a, b, c \in \mathbf{R}$, $au + bv + cw = 0$ if $a = b = c = 0$.

III. A set $\{u, v, w\}$ of vectors is linearly independent if u is not a multiple of v, and u is not a multiple of w.

IV. $\{(1, 0, 1), (0, 1, 0), (1, 1, 1)\}$ spans $\mathbf{R}^3$.

V. $\{(1, -1), (1, 1)\}$ is linearly independent in $\mathbf{R}^2$.

A. I & II

B. II & IV

C. I & IV

D. III & V

E. III & II

F. I & V

3. If three $n \times n$ matrices $A,\ B$ and C satisfy $AB - BA = C$, then ABA is **always** equal to :

A. $A^2B - C$

B. $A^2B - CA$

C. $BA^2 + CA$

D. A^2B

E. $A^2B + AC$

F. $A^2B + BC$

4. Complete the following phrase to make a true statement:

"A homogeneous linear system of 2011 linear equations in 1231 unknowns..."

A. ... is always consistent.

B. ... always has a unique solution.

C. ... may be inconsistent.

D. ... which is consistent always has a unique solution.

E. ... which is consistent never has a unique solution.

F. ... is never consistent.

5. Compute $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{2011}$

A. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2011 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2011 \end{bmatrix}$ C. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ E. $\begin{bmatrix} 0 & 0 & 2011 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ F. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2011 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

6. The dimension of $S = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}_{22}(\mathbf{R}) \mid a + d = 0\}$ is:

A. 0
B. 1
C. 2
D. 3
E. 4
F. S is not a vector space, so we cannot speak of its dimension.

7. The matrix $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ is not diagonalizable over the reals. Why?

A. Because A does not have any real eigenvectors

B. Because A does not have any real eigenvalues..

C. Because A does not have three distinct eigenvalues.

D. Because A does not have three independent eigenvectors.

E. Because A is lower triangular.

F. Because 'diagonalizable' is too difficult to say very quickly.

8. The set of vectors $\{(1,1,1),(1,-1,0),(1,1,-2)\}$ is an orthogonal basis of $\mathbf{R}^3$. Find (c_1,c_2,c_3) such that $(1,0,0) = c_1(1,1,1) + c_2(1,-1,0) + c_3(1,1,-2)$.

A. $(-\frac{1}{2}, -\frac{1}{3}, -\frac{1}{6})$

B. $(\frac{1}{2}, -\frac{1}{3}, \frac{1}{6})$

C. $(-\frac{1}{3}, -\frac{1}{2}, -\frac{1}{6})$

D. $(\frac{1}{9}, \frac{1}{4}, \frac{1}{36})$

E. $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$

F. $(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}})$

9. Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & x \end{bmatrix}$. For which value(s) of x is A invertible?

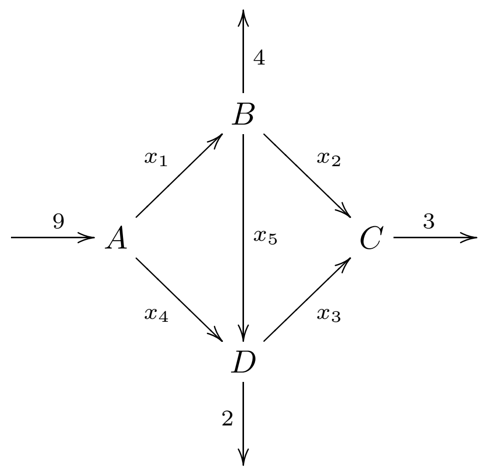
A. $x \neq -1$
B. $x \neq 1$
C. $x \neq 0$
D. $x = -1$
E. $x = 1$
F. $x \neq \pm 1$

10. Which of the following sets are linearly independent in $\mathbf{F}(\mathbf{R}) = \{f \mid f : \mathbf{R} \to \mathbf{R}\}$?

$S = \{x, x^2\}$
$T = \{1, x, x^2, (1-x)^2\}$
$U = \{1, 2\cos^2 x, 3\sin^2 x\}$

A. S and T.
B. S and U.
C. T and U.
D. S only.
E. T only.
F. S, T and U.

11. Consider the network of streets and intersections below. The arrows indicate the direction of traffic flow along the one-way streets, and the numbers refer to the exact number of cars observed to enter or leave the intersections during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



a) Write down a system of linear equations which describes the the traffic flow, **together with all the constraints** on the variables x_i, $i = 1, \ldots, 5$. (Do not perform any operations on your equations: this is done for you in (b), and *do not simply copy out the equations implicit in (b). You will not get any marks if you do this.*)

11 b). The reduced row-echelon form of the augmented matrix from part (a) is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 9 \\ 0 & 1 & 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Give the general solution. (Ignore the constraints at this point.)

c) If $\overline{BD}$ were closed due to roadwork, find the maximum and minimum flow along $\overline{AB}$,**using your results from (b).**

12. Let $U = \{(x, y, z, w) \in \mathbf{R}^4 \mid x + y + z - w = 0\}$.

a) Find a basis of U and give the dimension of U.

b) Find an orthogonal basis of U.

c) Find the best approximation to $(0, 1, 1, 1)$ by vectors in U.

13. $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}$.

a) Compute $\det(A - \lambda I_3)$ and hence show that the eigenvalues of A are 7 and 1.

b) Find a basis of $E_1 = \{x \in \mathbf{R}^3 \mid Ax = x\}$.

c) Find a basis of $E_7 = \{x \in \mathbf{R}^3 \mid Ax = 7x\}$.

d) Find an invertible matrix P such that $P^{-1}AP = D$ is diagonal, and give this diagonal matrix D. Explain why your choice of P is invertible.

e) Find an invertible matrix $Q \neq P$ such that $Q^{-1}AQ = \tilde{D}$ is also diagonal, and give this diagonal matrix $\tilde{D}$.

14. Let

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 2 & 0 & 6 \\ 1 & 2 & 4 & 2 \end{bmatrix}$$

a) Find a basis for the column space $\mathrm{col}\,(A)$ of A.

b) Give a complete geometric description of $\mathrm{col}\,(A)$.

c) Find a basis for the kernel, $\ker T$, of the linear transformation $T : \mathbf{R}^4 \to \mathbf{R}^3$ defined by

$$T(x) = Ax, \quad x \in \mathbf{R}^4.$$

d) Compute $\dim(\ker T) + \dim(\mathrm{im}\ T)$.

15. a) Let A be a real $n \times n$ matrix. Give 3 statements (in total) equivalent to

$$\text{“}\det A = 0\text{”},$$

one each in terms of:

(I) the rows of A

(II) the rank of A

(III) non-homogeneous systems $Ax = b$, for all $b \in \mathbf{R}^n$

15b) State whether the following are true or false. If true, explain why, if false, give a numerical example to illustrate.

i) $\begin{bmatrix} 6 & 0 \\ 1 & 5 \end{bmatrix}$ is diagonalizable.

ii) If 0 is an eigenvalue of 4×4 matrix A, then A is invertible.

iii) The columns of a 12×30 matrix are always linearly dependent.

16. (Four bonus marks) Make sure you finish and check the rest of the paper before trying this. As you know, bonus marks are much harder to earn.

In what follows, A denotes an $n \times n$ matrix.

a) Prove that if v and w are eigenvectors of A corresponding to distinct eigenvalues, then $\{v, w\}$ is linearly independent.

b) Prove that if all the eigenvalues of A are non-zero, then A is invertible.

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