Faculty of Science »
Department of mathematics and statistics

MAT 1341A Fall 2013 Final Exam

17 December, 2013.

Instructor - Barry Jessup

Family Name:______________________

First Name:________________________ Seat number: _________

Student number:____________________

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Some Advice

Take a few minutes to read the entire paper before you begin to write, and read each question carefully. The multiple choice questions are only worth 1 point and questions 11-15 are worth 6 points each. Make a note of the questions you feel confident you can do, and try those first: you do not have to do the questions in the order they are presented.

Instructions

1. You have 3 hours to complete this exam.
2. This is a closed book exam, and no notes of any kind are allowed. **The use of calculators**, cell phones or any text storage or communication device **is not permitted.**
3. Questions 1 to 10 are multiple choice. These questions have just one correct answer, are worth 1 point each and no part marks will be given. Please record your answers in the spaces opposite.
4. Questions 11 – 15 require a complete solution, and are worth 6 points each. Question 16 is a bonus question and should only be attempted after all other questions have been completed and checked.

   Spend your time accordingly.

   Answer questions 11 – 16 in the space provided, and use the backs of pages if necessary.
5. **The correct answer requires justification written legibly and logically: you must convince me that you know why your solution is correct.**
6. Where it is possible to check your work, do so.

Good luck! Bonne chance!

1. The subset $\{(x, y, z) \in \mathbf{R}^3 \mid 5x - 6y = -1\}$ is

A. a line in $\mathbf{R}^2$ with direction vector $(5,\ -6)$.
B. a line in $\mathbf{R}^2$ with direction vector $(-6,\ 5)$.
C. a plane in $\mathbf{R}^3$ through $(1,\ 1,\ 1)$ with normal vector $(5,\ -6,\ 0)$.
D. a plane in $\mathbf{R}^3$ through $(1,\ 1,\ 1)$ with normal vector $(-6,\ 5,\ 0)$.
E. a line in $\mathbf{R}^3$ with direction vector $(5,\ -6)$.
F. a line in $\mathbf{R}^3$ with direction vector $(-6,\ 5)$.

2. Which two of the following are subspaces of $\mathbf{F}(\mathbf{R}) = \{f \mid f : \mathbf{R} \to \mathbf{R}\}$?

$S = \{f \in \mathbf{F}(\mathbf{R}) \mid f(-1)f(1) = 0\}$
$T = \{f \in \mathbf{F}(\mathbf{R}) \mid f(-1) = f(1)\}$
$U = \{f \in \mathbf{F}(\mathbf{R}) \mid f(1) = 1\}$
$V = \{f \in \mathbf{F}(\mathbf{R}) \mid f(0) = 0\}$

A. S and T.
B. S and U.
C. S and V.
D. T and U.
E. T and V.
F. U and V.

3. Suppose $\{u, v, w\}$ is a set of vectors in a vector space V. Which of the following statements is equivalent to

"$\{u, v, w\}$ is linearly independent."

I. None of the vectors u, v or w is a multiple of any other single vector in $\{u, v, w\}$.

II. None of the vectors u, v or w is a linear combination of the other vectors in $\{u, v, w\}$.

III. If $a = b = c = 0$, then $au + bv + cw = 0$.

IV. If a, b, c are scalars, then $au + bv + cw = 0$ implies $a = b = c = 0$.

V. Both $\{u, v\}$ and $\{v, w\}$ are linearly independent.

A. I & II
B. I & III
C. II & III
D. II & V
E. II & IV
F. III & IV

4. The dimension of $\{A \in \mathbf{M}_{22}(\mathbf{R}) \mid A = A^t\}$ is:

A. 0
B. 1
C. 2
D. 3
E. 4
F. 5

5. For a non-homogeneous system of 2013 equations in 1012 variables, answer the following three questions:

∘ Can the system be inconsistent?
∘ Can the system have infinitely many solutions?
∘ Can the system have exactly one solution?

A. No, Yes, No.
B. Yes, Yes, Yes.
C. Yes, Yes, No.
D. No, No, No.
E. Yes, No, Yes.
F. No, No, Yes.

6. If the coefficient matrix A in a homogeneous system of 14 equations in 18 variables is known to have rank 8, how many parameters are there in the general solution?

A. 4
B. 6
C. 8
D. 10
E. 18
F. 0

7. Let A be a fixed 4×4 matrix and let $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Then, BA can always be obtained from A by :

A. Adding twice the second row of A to the fourth row of A
B. Adding twice the fourth row of A to the second row of A
C. Adding the second row of A to the fourth row of A
D. Adding the fourth row of A to the second row of A
E. Adding twice the second column of A to the fourth column of A
F. Adding twice the fourth column of A to the second column of A

8. If $M = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$, which one of the following statements is true ?

A. M is not invertible.

B. The 3^{rd} row vector of M^{-1} is $(1, -1, 1)$.

C. The 1^{st} row vector of M^{-1} is $(1, 2, 1)$.

D. The 2^{nd} row vector of M^{-1} is $(1, 0, -1)$.

E. The 2^{nd} column vector of M^{-1} is $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

F. M^{-1} is invertible, but none of B, C, D, or E is true.

9. Let A be an $n \times n$ matrix. Among the following statements, one is <u>not</u> equivalent to the other five. Which one is it?

A. A is invertible.

B. For any vector $b \in \mathbf{R}^n$, the system $Ax = b$ has a unique solution $x \in \mathbf{R}^n$.

C. The rows of A are linearly dependent.

D. A can be row-reduced to the identity matrix I_n.

E. The rank of A is n.

F. The columns of A span $\mathbf{R}^n$.

10. If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3$, find $\begin{vmatrix} a & 4g & d-2a \\ b & 4h & e-2b \\ c & 4i & f-2c \end{vmatrix}$.

A. 24

B. -24

C. 12

D. -12

E. 6

F. -6

11. a) Let $k \in \mathbf{R}$ and consider the linear system in the unknowns x, y and z:

$$\begin{array}{ccccccc} x & + & 2y & + & z & = & 0 \\ x & + & y & + & 2z & = & 0 \\ 2x & + & 3y & + & kz & = & 0 \end{array}$$

Find all values of k for which this system has

(i) a unique solution,
(ii) infinitely many solutions, and
(iii) no solutions.

11. b) Find all solutions of

$$\begin{array}{rcrcrcr} 2x & + & y & + & z & = & 14 \\ 3x & + & 3y & & & = & 15 \\ 5x & + & 4y & + & z & = & 29 \end{array}$$

such that x, y **and** z **are integers with** $x \geq 3, y \geq 0$ **and** $z \geq 2$.

12. $W = \text{span}\{(1,0,1,0),(0,1,0,1),(0,0,0,1),(1,1,1,0)\} \subset \mathbf{R}^4$.

a) Find a basis of W which is a subset of the given spanning set.

b) Find an orthogonal basis of W.

c) Find the best approximation to $(0,1,-1,1)$ in W.

d) Extend your basis of W in (b) to a basis of $\mathbf{R}^4$.

13. $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}.$

a) Show that the eigenvalues of A are 5 and -1.

b) Find a basis of $E_5 = \{v \in \mathbf{R}^3 \mid Av = 5v\}$.

c) Find a basis of $E_{-1} = \{v \in \mathbf{R}^3 \mid Av = -v\}$.

d) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. Explain why your choice of P is invertible.

14. Let $u = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and define a linear transformation $S : \mathbf{R}^3 \to \mathbf{R}^3$ by

$$S(v) = (u \cdot v)\, u, \qquad v \in \mathbf{R}^3,$$

where "$u \cdot v$" denotes dot product of u and v. *(You do* <u>*not*</u> *have to prove that S is linear.)*

a) If $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbf{R}^3$, show that $S(v) = \begin{bmatrix} x - y + z \\ -x + y - z \\ x - y + z \end{bmatrix}$.

b) Find a 3×3 matrix A such that $S(v) = Av$, where Av denotes the matrix product of A and v.

c) Find a basis for $\ker S = \{v \mid S(v) = 0\}$, and give a complete geometric description of $\ker S$.

d) Find a basis for $\operatorname{im} S = \{S(v) \mid v \in \mathbf{R}^3\}$, and give a complete geometric description of $\operatorname{im} S$.

15. State whether each of the following is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, you **must give an explicit example - with numbers!**
- If you say the statement is true, you must give a clear explanation - by quoting a theorem presented in class, or by giving a *proof valid for every case.*

a) If U and V are subspaces of $\mathbf{R}^2$, then their union

$$U \cup V = \{w \in \mathbf{R}^3 \mid w \in U \text{ or } w \in V\}$$

is also a subspace of $\mathbf{R}^2$.

ANSWER

b) $\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$ is diagonalizable.

ANSWER

15 (cont.)

c) If 0 is an eigenvalue of 7×7 matrix A, then A is invertible.

ANSWER

d) If A is an $m \times n$ matrix with $n > 1$ and $m > 1$, and if the reduced row echelon form of A has a row of zeros, then $\operatorname{rank} A < n$.

ANSWER

16. (4 bonus marks) Make sure you finish and check the rest of the paper before trying this. Bonus marks are much harder to earn.

Prove carefully that if A is an 11×11 anti-symmetric matrix (i.e. $A^t = -A$) , then 0 is an eigenvalue of A. (Your proof must be valid for all possible 11×11 anti-symmetric matrices.)