Faculty of Science »
Department of mathematics and statistics

# MAT 1341A (Fall 2012) Final Exam

11 December, 2012.

Instructor - Barry Jessup

| | |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| 8 | |
| 9 | |
| 10 | |
| sub-total | |
| 11 [6pts] | |
| 12 [9pts] | |
| 13 [6pts] | |
| 14 [6pts] | |
| 15 [3pts] | |
| Total | |

Family Name:______________________

First Name:______________________

Student number:____________________

### Some Advice

Take a few minutes to read the entire paper before you begin to write, and read each question carefully. The multiple choice questions are only worth 1 point and the others are worth more. Make a note of the questions you feel confident you can do, and try those first: <u>you do not</u> have to do the questions in the order given.

### Instructions

1. You have 3 hours to complete this exam.
2. This is a closed book exam, and no notes of any kind are allowed. **The use of calculators**, cell phones, pagers or any text storage or communication device **is not permitted.**
3. Questions 1 to 10 are multiple choice. These questions are worth <u>1 point</u> each and no part marks will be given. Please record your answers in the spaces opposite.
4. Questions 11 – 15 require a complete solution, and their respective values are indicated above.
   **Spend your time accordingly.**
   Answer these questions in the space provided, and use the backs of pages if necessary.
5. **The correct answer requires justification written legibly and logically: you must convince me that you know why your solution is correct.**
6. Where it is possible to check your work, do so.

**Good luck! Bonne chance!**

**1.** Which of the following subsets of $\mathbf{F}(\mathbf{R})$ are subspaces of $\mathbf{F}(\mathbf{R})$?

$$\begin{aligned} S &= \{f \in \mathbf{F}(\mathbf{R}) \mid f(-x) + f(x) = 0\} \\ T &= \{f \in \mathbf{F}(\mathbf{R}) \mid f(-1)f(0) = 0\} \\ U &= \{f \in \mathbf{F}(\mathbf{R}) \mid f(0) \geq 0\} \\ V &= \{f \in \mathbf{F}(\mathbf{R}) \mid f(0) + f(1) = 0\} \end{aligned}$$

A. $V$ and $S$

B. $U$ and $S$

C. $S$ and $T$

E. $V$ and $T$

E. $U$ and $T$

F. $U$ and $V$.

**2.** If $a, b$ and $c$ are scalars and $u, v$ and $w$ are vectors in some vector space V, which of the following statements are always true?

I. The set $\{u, v, w\}$ of vectors is linearly independent if $au + bv + cw = 0$ when $a = b = c = 0$.

II. The set $\{u, v, w\}$ of vectors is linearly independent if $au + bv + cw = 0$ only if $a = b = c = 0$.

III. The set $\{u, v\}$ spans $V$ if $\{u, v\}$ is linearly independent.

IV. The set $\{u, v, w\}$ spans $V$ if every vector in $V$ is a linear combination of $u - 2v, u + v + w$ and $v - 3w$.

A. Only I & II are true.

B. Only II & IV are true.

C. Only II & III are true.

D. Only I & III & IV are true.

E. Only II & III & IV are true.

F. Only I is true.

**3.** Let $A = \begin{bmatrix} -1 & 2 & -1 \\ -5 & 7 & -3 \\ 3 & -4 & 2 \end{bmatrix}$. Which one of the following is true?

A. The second column of $A^{-1}$ is $\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$.

B. The third column of $A^{-1}$ is $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$.

C. The third row of $A^{-1}$ is $[\,-1 \quad 2 \quad 3\,]$.

D. The first row of $A^{-1}$ is $[\,-2 \quad 0 \quad 1\,]$.

E. The matrix $A$ is not invertible.

F. The second row of $A^{-1}$ is $[\,-1 \quad -1 \quad 2\,]$.

**4.** Suppose $n \geq 2$. In a linear system $Ax = b$, with $n$ equations and $n$ unknowns, the rank of $A$ is $n-1$ and the rank of the augmented matrix $[A \mid b]$ is also $n-1$. Which one of the following statements is true?

A. The system has no solution.

B. The system has a unique solution.

C. The system has infinitely many solutions.

D. The system has exactly $n-1$ solutions.

E. The determinant of $A$ is non-zero.

F. Such a system cannot exist.

**5.** Compute $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}^{2012}$

A. $\begin{bmatrix} 1 & 0 & 2012 & 0 \\ 0 & 1 & 0 & 0 \\ 2012 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 0 & 2012 & 2012 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ C. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2012 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ E. $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ F. $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

**6.** Let $\mathbf{M}_{22}$ denote, as usual, the vector space of real $2 \times 2$ matrices and let $K = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$, and define

$$C = \{A \in \mathbf{M}_{22} \mid KA = AK\}.$$

A. $C$ is a subspace of $\mathbf{M}_{22}$ of dimension 1

B. $C$ is a subspace of $\mathbf{M}_{22}$ of dimension 2

C. $C$ is a subspace of $\mathbf{M}_{22}$ of dimension 3

D. $C$ is a subspace of $\mathbf{M}_{22}$ of dimension 4

E. $C$ is a subspace of $\mathbf{M}_{22}$ of dimension 0

F. $C$ is not a subspace of $\mathbf{M}_{22}$.

**7.** Let $A$ be a $6 \times 4$ matrix. Answer the following questions:

(1) Can the system $Ax = 0$ have a non trivial solution?

(2) Can the columns of $A$ span $\mathbf{R}^6$?

(3) Can the columns of $A$ be linearly independent?

A. Yes, No, Yes.

B. Yes, Yes, Yes.

C. Yes, No, No.

D. No, No, Yes.

E. No, No, No.

F. No, Yes, Yes.

**8.** The set of vectors $\{(3,0,4),(0,1,0),(-4,0,3)\}$ is an orthogonal basis of $\mathbf{R}^3$. Find $c_3 \in \mathbf{R}$ such that

$$(0,1,1) = c_1(3,0,4) + c_2(0,1,0) + c_3(-4,0,3) \quad \text{for some scalars } c_1, c_2 \in \mathbf{R}.$$

That is, find the third Fourier coefficient of $(0,1,1)$ with respect to the ordered orthogonal basis of $\mathbf{R}^3$ above.

A. $\frac{3}{25}$

B. $\frac{3}{5}$

C. 3

D. 1

E. $\frac{1}{25}$

F. $-\frac{3}{25}$

**9.** Which two of the following statements are **false**?

(i) For all invertible $n \times n$ matrices $A$ and $B$, $\det(A^{-1}BA) = \det B$

(ii) For all invertible $n \times n$ matrices $A$ and $B$, $\det\left(A^{-1}B^{-1}AB\right) = 1$

(iii) For all $n \times n$ matrices $A$ and $B$, $(A^tB^t)^t = AB$

(iv) For all invertible $n \times n$ matrices $A$ and $B$, $(ABA^{-1})^{-1} = A^{-1}B^{-1}A$

(v) For all $n \times n$ matrices $A$ and $B$, $\det\left(A^tB\right) = \det\left(B^tA\right)$

A. (i) and (iii)

B. (ii) and (iii)

C. (iii) and (iv)

D. (ii) and (iv)

E. (ii) and (v)

F. (i) and (v)

**10.** Let $A$ be an $n \times n$ matrix. One of the the following statements in not equivalent to the statement:

**The number 0 is not an eigenvalue of A.**

Which one?

A. The homogeneous system $Ax = 0$ has a non-trivial solution.

B. $A$ is invertible.

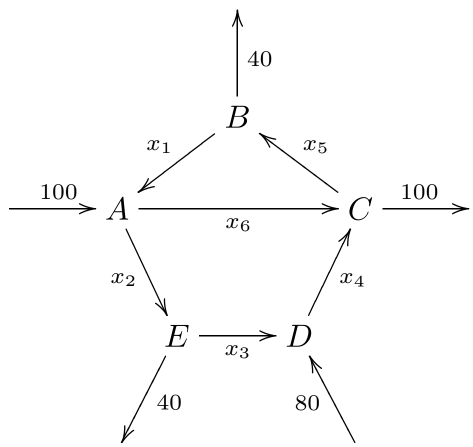
C. The determinant of $A$ is not zero.

D. The columns of $A$ span $\mathbf{R}^n$.

E. The rows of $A$ are linearly independent.

F. The rank of $A$ is $n$.

**11.** Consider the network of streets with intersections A, B, C, D and E below. The arrows indicate the direction of traffic flow along the **one-way streets**, and the numbers refer to the **exact** number of cars observed to enter or leave A, B, C, D and E during one minute. Each $x_i$ denotes the unknown number of cars which passed along the indicated streets during the same period.




a) Write down a system of linear equations which describes the the traffic flow, **together with all the constraints** on the variables $x_i$, $i = 1, \ldots, 6$.
(Do not perform any operations on your equations: this is done for you in (b). *Do not simply copy out the equations implicit in (b). You will not get any marks if you do this.*)

**11(b).** The reduced row-echelon form of the augmented matrix of the system in part (a) is

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & 0 & -40 \\ 0 & 1 & 0 & 0 & -1 & 1 & 60 \\ 0 & 0 & 1 & 0 & -1 & 1 & 20 \\ 0 & 0 & 0 & 1 & -1 & 1 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Give the general solution. (Ignore the constraints from (a) at this point.)

c) If $\overline{AC}$ were closed due to roadwork, find the minimum flow along $\overline{ED}$, **using your results from (b).**

*(You must justify all your answers.)*

**12.** Let $U = \text{span}\{(1,0,0,1),(0,1,0,0),(1,1,0,0),(1,4,0,-1)\}$

a) Find a basis of $U$ which is a subset of the given spanning set.

b) Find an orthogonal basis of $U$.

c) Find the best approximation to $(1,\ -1,\ 2,\ -1)$ by vectors in $U$.

d) Extend your basis in (b) to an **orthogonal** basis of $\mathbf{R}^4$.

**13.** $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$

a) Compute $\det(A - \lambda I_3)$ and hence show that the eigenvalues of $A$ are 0 and 2.

b) Find a basis of $E_0 = \{v \in \mathbf{R}^3 \mid Av = 0\}$.

c) Find a basis of $E_2 = \{v \in \mathbf{R}^3 \mid Av = 2v\}$.

d) Find an invertible matrix $P$ and a diagonal matrix $D$ such that $P^{-1}AP = D$. Explain why your choice of $P$ is invertible.

**14.** Let $u = (1, -1, 1)$ and define a linear transformation $S : \mathbf{R}^3 \to \mathbf{R}^3$ by

$$S(v) = u \times v, \qquad v \in \mathbf{R}^3,$$

where "$\times$" denotes the cross product. *(You do <u>not</u> have to prove that $S$ is linear.)*

a) If $(x, y, z) \in \mathbf{R}^3$, show that $S(x, y, z) = (-y - z, x - z, x + y)$.

b) Find the standard matrix of $S$.

c) Find a basis for $\ker S$ and describe it geometrically.

d) Find a basis for $\operatorname{im} S$ and describe it geometrically.

**15.** State whether the following are true or false.

**If you say the statement may be false, you must give an explicit example - with numbers! If you say the statement is true, you must give a clear explanation -e.g. by quoting a theorem from class.**

i) Every diagonalizable matrix is invertible.

ii) If a 2 by 2 matrix $A$ satisfies $A^2 = 0$, then $A = 0$.

iii) The rows of a $12 \times 11$ matrix are always linearly dependent.

(This page is intentionally (almost) blank).