Faculty of Science »
Department of mathematics and statistics

MAT 1341A Final Exam, 2011

16-December, 2011.

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Family Name:______________________

First Name:______________________

Student number:______________________

Some Advice

Take 5 minutes to read the entire paper before you begin to write, and read each question carefully. The multiple choice questions are only worth 1 point and the others are worth more.
Make a note of the questions you feel confident you can do, and try those first: <u>you do not</u> have to try the questions in the order given.

Instructions

1. You have 3 hours to complete this exam.
2. This is a closed book exam, and no notes of any kind are allowed. **The use of calculators**, cell phones, pagers or any text storage or communication device **is not permitted.**
3. Questions 1 to 10 are multiple choice. These questions are worth <u>1 point</u> each and no part marks will be given. Please record your answers in the spaces opposite.
4. Questions 11 – 15 require a complete solution, and are worth <u>6 points</u> each, so spend your time accordingly. Answer these questions in the space provided, and use the backs of pages if necessary.
   Question 16 is a bonus question and should only be attempted after all other questions have been completed and checked.
5. **The correct answer requires justification written legibly and logically: you must convince me that you know why your solution is correct.**
6. Where it is possible to check your work, do so.

Good luck! Bonne chance!

1. Let $W = \{(x, y, z) \in \mathbf{R}^3 \mid x \geq 0,\ y \geq 0 \text{ and } z \geq 0\}$. Then,

A. W is a subspace of $\mathbf{R}^3$.

B. $(0, 0, 0) \notin W$ and W is not closed under multiplication by scalars.

C. W is closed under addition but W is not closed under multiplication by scalars.

D. W is closed under addition and W is closed under multiplication by scalars.

E. W is not closed under addition but W is closed under multiplication by scalars.

F. None of the other statements is true.

2. If the augmented matrix $[A|b]$ of a system $Ax = b$ is row-equivalent to

$$\begin{bmatrix} 1 & 0 & 0 & | & 5 \\ 0 & 1 & 1 & | & -2 \\ 0 & 0 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix},$$

which of the following statements is true?

A. The system is inconsistent

B. $X = (5,\ -2 - s,\ 1)$ is the solution for any value of s

C. $X = (5,\ -2,\ 1)$ is the unique solution of the system

D. $X = (5s,\ -2s,\ s)$ is a solution for any value of s

E. $X = (5t,\ -2 - s,\ s)$ is the solution for any value of s and t

F. $X = (5,\ -3,\ 1)$ is the unique solution to the system

3. For a non-homogeneous system of 12 equations in 15 unknowns, answer the following three questions:

∘ Can the system be inconsistent?
∘ Can the system have infinitely many solutions?
∘ Can the system have exactly one solution?

A. No, Yes, No.
B. Yes, Yes, Yes.
C. Yes, Yes, No.
D. No, No, No.
E. Yes, No, Yes.
F. No, No, Yes.

4. Suppose $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 4 \end{bmatrix}$. Which one of the following statements is true ?

A. A^{-1} does not exist.
B. The third row of A^{-1} is $[-1\ -1\,1]$.
C. The second row of A^{-1} is $[1\,2\ -1]$.
D. The first row of A^{-1} is $[2\ 0\ -1]$.
E. The second column of A^{-1} is $[0\ 2\ -1]^t$.
F. All of B, C, D, E are true.

5. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & 0 & x \end{bmatrix}$. For which value(s) of x is A invertible?

A. $x \neq -1$
B. $x \neq 1$
C. $x \neq 0$
D. $x = -1$
E. $x = 1$
F. $x \neq \pm 1$

6. Find the value of t for which $(4,\ 6,\ 3,\ t)$ belongs to $\text{span}\{(1, 3, -4, 1), (2, 8, -5, -1), (-1, -5, 0, 2)\}$.

A. 0
B. 4
C. 7
D. 11
E. 13
F. 15

7. The dimension of $K = \{A \in \mathbf{M}_{33}(\mathbf{R}) \mid A = -A^t\}$ is:

A. 0
B. 2
C. 3
D. 4
E. 6
F. 9

8. Which two of the following are subspaces of $\mathbf{F}(\mathbf{R}) = \{f \mid f : \mathbf{R} \to \mathbf{R}\}$?

$S = \{f \in \mathbf{F}(\mathbf{R}) \mid f(1)f(2) = 0\}$
$T = \{f \in \mathbf{F}(\mathbf{R}) \mid f(-x) = 2f(x), \forall x \in \mathbf{R}\}$
$U = \{f \in \mathbf{F}(\mathbf{R}) \mid f(1) > 1\}$
$V = \{f \in \mathbf{F}(\mathbf{R}) \mid f(6) = 0\}$

A. T and U.
B. T and V.
C. S and T.
D. S and V.
E. S and U.
F. U and V.

9. Let $A = \begin{bmatrix} 0 & 1 & 0 & -3 \\ 1 & 1 & 3 & 0 \\ 2 & 1 & 3 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix}$. The dimension of the row space of A is:

A. 4
B. 3
C. 2
D. 1
E. 0
F. Infinite

10. If $C = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and D is a $3 \times m$ matrix then the second row of the matrix CD is

A. not defined unless $m = 2$.
B. the same as the first row of D.
C. the same as the second row of D.
D. the sum of the first and the third row of D.
E. the sum of twice the second row of D and the third row of D.
F. twice the first row of D.

11. A dog named Lanso is advised by a nutritionist to take 5 units of vitamin A, 13 units of vitamin C and 23 units of vitamin D each day. Lanso can choose from the three brands I, II and III, and the amount of each vitamin in each capsule of the various brands is given below:

	I	II	III
vitamin A	1	1	0
vitamin C	2	1	1
vitamin D	4	3	1

Lanso is not capable of taking fractions of a capsule.

a) After **defining your variables**, write down a system of equations in these variables, **together with all constraints**, that determine the possible combinations of the numbers of capsules of each brand that will provide exactly the required amounts of vitamins for Lanso. (Do not perform any operations on your equations: this is done for you in (b). *Do not simply copy out the equations implicit in (b). You will not get any marks if you do this.*)

b) The reduced row-echelon form of the augmented matrix of the system in part (a) is:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 8 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Give the general solution. (Ignore the constraints from (a) at this point.)

11 c) Find all possible combinations of the numbers of capsules of each brand that will provide exactly the required amounts of vitamins for Lanso.

11 d) Lanso has a tight budget. If the respective costs (in cents) per capsule of brands I, II and II are 4, 2 and 3, determine the choice which will minimize Lanso's total cost each day, and give this minimum cost per day.

12. Let $W = \operatorname{span}\{(1,0,0,1), (0,-1,-1,0), (0,0,0,1), (0,1,1,1)\} \subset \mathbf{R}^4$.

a) Find a basis of W which is a subset of the given spanning set.

b) Find an orthogonal basis of W.

c) Find the best approximation to $(0,1,-1,1)$ in W.

d) Extend your basis of W in (b) to a basis of $\mathbf{R}^4$.

13. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

a) Compute $\det(A - \lambda I_3)$ and hence show that the eigenvalues of A are 2 and -1.

b) Find a basis of $E_2 = \{x \in \mathbf{R}^3 \mid Ax = 2x\}$.

c) Find a basis of $E_{-1} = \{x \in \mathbf{R}^3 \mid Ax = -x\}$.

d) Find an invertible matrix P such that $P^{-1}AP = D$ is diagonal, and give this diagonal matrix D. Explain why your choice of P is invertible.

e) Find an invertible matrix $Q \neq P$ such that $Q^{-1}AQ = \tilde{D}$ is also diagonal, and give this diagonal matrix $\tilde{D}$.

14. Let

$$A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 0 & 2 & 2 \\ 1 & -1 & 4 & 3 \end{bmatrix}$$

a) Find a basis for the column space $\operatorname{col}(A)$ of A.

b) Give a complete geometric description of $\operatorname{col}(A)$.

c) Find a basis for the kernel, $\ker T$, of the linear transformation $T : \mathbf{R}^4 \to \mathbf{R}^3$ defined by

$$T(x) = Ax, \quad x \in \mathbf{R}^4.$$

d) Compute $\dim(\ker T) + \dim(\operatorname{im} T)$.

15. a) Let A be a real $n \times n$ matrix. Give 3 statements (in total) equivalent to

$$\text{“}\det A \neq 0\text{”},$$

one each in terms of:

(I) the columns of A

(II) the reduced row-echelon form of A

(III) the homogeneous system $Ax = 0$.

15b) State whether the following are true or false. If true, explain why, if false, give a numerical example to illustrate.

i) If A and B are 2 by 2 matrices, then $\det(A+B)$ is always equal to $\det A + \det B$.

ii) If a 13×13 matrix A satisfies then $A^2 = 0$, then A is not invertible.

iii) The columns of a 3×4 matrix are always linearly dependent.

16. (Four bonus marks) Make sure you finish and check the rest of the paper before trying this. As you know, bonus marks are much harder to earn.

Suppose A is a 3 by 3 matrix such that $A = -A^t$ (i.e., A is antisymmetric). Prove that if $A \neq 0$, then the rank of A is exactly 2.

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