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ZEROS OF THE EISENSTEIN SERIES E_2

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Abstract. In this paper we investigate the zeros of the Eisenstein series E_2. In particular, we prove that E_2 has infinitely many $\mathrm{SL}_2(\mathbb{Z})$-inequivalent zeros in the upper half-plane $\mathfrak{H}$, yet none in the standard fundamental $\mathfrak{F}$. Furthermore, we go on to investigate other fundamental regions in the upper half-plane $\mathfrak{H}$ for which there do or do not exist zeros of E_2. We establish infinitely many such regions containing a zero of E_2 and infinitely many which do not.

1. Introduction

Let $\mathfrak{H} = \{\tau \in \mathbb{C},\ \mathrm{Im}(\tau) > 0\}$ be the upper half-plane. The Eisenstein series are defined for every even integer $k \geq 2$ and $\tau \in \mathfrak{H}$ by

$$
\begin{aligned}
(1) \qquad E_k(\tau) &= 1 - \tfrac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \\
&= 1 - \tfrac{2k}{B_k} \sum_{n=1}^{\infty} \frac{n^{k-1} q^n}{1-q^n}, \qquad q = e^{2\pi i \tau}.
\end{aligned}
$$

Here B_k is the k-th Bernoulli number and $\sigma_k(n) = \sum_{d|n} d^k$.

These series play an important role in the theory of modular forms and quasi-modular forms. They have been the topic of extensive investigation for a long time from various points of view. For instance, from the analytic point of view, the study of the zeros of $E_k(z)$, $k \geq 4$, has been carried out by several authors. In 1963, K. Wohlfahrt proved in [6] that the zeros of E_k, $4 \leq k \leq 26$, are simple and lie in the arc of the unit circle $\left\{z = e^{i\theta} :\ \pi/2 \leq \theta \leq 3\pi/2\right\}$ in the fundamental domain $\mathfrak{F} = \{\tau \in \mathfrak{H},\ |\tau| \geq 1 \text{ and } |\mathrm{Re}(\tau)| \leq 1/2\}$ of the modular group $\mathrm{SL}_2(\mathbb{Z})$. He also conjectured that this holds for all $k \geq 4$. In 1970, F.K.C. Rankin and H.P.F. Swinnerton-Dyer [5] proved Wohlfahrt's conjecture. In 1982, R.A. Rankin [4] generalized their result to a certain class of Poincaré series. However, nothing has been proven for the Eisenstein series E_2, which is important in many fields. In fact, even whether it has a finite or an infinite number of zeros has not been known.

In this paper, we prove that there are infinitely many non-equivalent zeros of E_2 in $\mathfrak{H}$. In fact, since E_2 is not exactly a modular form but rather a quasi-modular form, two zeros τ_0 and τ_1 of E_2 are $\mathrm{SL}_2(\mathbb{Z})$-equivalent, that is $\tau_1 = \gamma \cdot \tau_0$ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ if and only if $\tau_1 = \tau_0 + n$ for an integer n. Thus, we restrict our investigation to the half-strip $\mathfrak{S} = \{\tau \in \mathfrak{H},\ -\frac{1}{2} < \mathrm{Re}(\tau) \leq \frac{1}{2}\}$, in which we prove

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that there are infinitely many zeros for E_2. Moreover, these zeros present a strange distribution in $\mathfrak{S}$. More precisely, the fundamental domain $\mathfrak{F}$ and infinitely many of its conjugates within $\mathfrak{S}$ contain no zero of E_2, while there are infinitely many conjugates of $\mathfrak{F}$ which contain zeros of E_2.

2. Eisenstein series: Some properties

The most familiar Eisenstein series are

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n, \tag{2}$$

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \tag{3}$$

$$E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n. \tag{4}$$

The series E_4 and E_6 are, respectively, modular forms of weight 4 and 6. However, the Eisenstein series E_2 is not a modular form. In fact, it transforms under the action of the modular group as follows (see [3]).

Proposition 2.1. *For* $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, *we have*

$$E_2(\alpha \cdot \tau) = (c\tau + d)^2 E_2(\tau) + \frac{6c}{\pi i}(c\tau + d), \tag{5}$$

where

$$\alpha \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

This proposition can be proved using the fact that E_2 is the logarithmic derivative of the modular discriminant $\Delta = \frac{1}{1728}(E_4^3 - E_6^2)$ of weight 12, the derivation being $\frac{1}{2\pi i}\frac{d}{d\tau}$.

These three functions were especially studied by Ramanujan [2], who proved that they satisfy the following differential equations:

$$\frac{1}{2\pi i}\frac{dE_2}{d\tau} = \frac{1}{12}(E_2^2 - E_4), \tag{6}$$

$$\frac{1}{2\pi i}\frac{dE_4}{d\tau} = \frac{1}{3}(E_2E_4 - E_6), \tag{7}$$

$$\frac{1}{2\pi i}\frac{dE_6}{d\tau} = \frac{1}{2}(E_2E_6 - E_4^2). \tag{8}$$

Thus the graded ring $\mathbb{C}[E_2,\ E_4,\ E_6]$ is closed under the differential operator $\frac{d}{d\tau}$. It is known that the space of all modular forms is exactly the graded ring $\mathbb{C}[E_4,\ E_6]$. We shall at this stage give some special values of E_2 at i and at the cubic root of unity $\rho = \frac{-1+i\sqrt{3}}{2}$:

$$E_2(i) = \frac{3}{\pi}, \tag{9}$$

$$E_2(\rho) = \frac{2\sqrt{3}}{\pi}. \tag{10}$$

This follows from the transformation formula for E_2 together with the appropriate transformations that fix i and ρ.

3. Zeros of the Eisenstein series E_2

In this section we prove that the series E_2 has infinitely many zeros, a fact that has not been known before. Set $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $S_n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$ for positive integers n.

Proposition 3.1. *The Eisenstein series E_2 has a zero τ_0, on the imaginary axis and a zero τ_1 on the axis $Re(z) = \frac{1}{2}$.*

Proof. It is clear that for $\tau = iy$, the series $E_2(\tau)$ is real and increasing on $(0,\ \infty)$. Meanwhile, $\lim_{y\to 0} E_2(iy) = -\infty$ and $\lim_{y\to\infty} E_2(iy) = 1$. It follows that E_2 has a unique zero, say τ_0, on the purely imaginary axis.

Similarly, $E_2(\tau)$ is real for $\tau = 1/2\ +\ iy$, $y > 0$. Furthermore, we have

$$\lim_{y\mapsto 0} E_2\left(\frac{1}{2} + iy\right) = -\infty.$$

Indeed, for $\alpha = S_2^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$ we have

$$E_2\left(\frac{1}{2} + iy\right) = -\frac{1}{y^2}\left(\frac{1}{4}E_2\left(-\frac{1}{2} + \frac{i}{4y}\right) - \frac{6y}{\pi}\right).$$

This gives the desired limit since $E_2\left(-\frac{1}{2} + \frac{i}{4y}\right)$ tends to 1 as y tends to 0. Combining this with the fact that $E_2(\rho) = E_2(\rho + 1) = \frac{2\sqrt{3}}{\pi}$ yields the existence of a zero τ_1 of real part $1/2$ and whose imaginary part is less than $\sqrt{3}/2$. Here again we used the transformation formula in Proposition 2.1 with $\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. $\square$

As for the location of these two zeros, and taking into account the special value of E_2 at i and ρ given respectively by (9) and (10), we have

Proposition 3.2. *The zeros τ_0 and τ_1 are contained respectively in the fundamental domains $S\mathfrak{F}$, $S_2\mathfrak{F}$.*

It is worth mentioning that numerical values of these two zeros appear in [1], where they are studied as equilibrium points of Green's functions.

Unlike the case of modular forms, the set of zeros of E_2 is not invariant under every conjugation by elements of $\mathrm{SL}_2(\mathbb{Z})$. In fact we have

Proposition 3.3. *Two zeros of E_2 are equivalent if and only if one is a translate of the other by an integer.*

Proof. Suppose that $z_1,\ z_2$ are any two zeros of E_2 in the half-plane $\mathfrak{H}$ that are equivalent modulo $\mathrm{SL}_2(\mathbb{Z})$. Say, $z_1 = \alpha \cdot z_2$, $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then, by the transformation formula for E_2 in Proposition 2.1, we have

$$E_2(z_1) = 0 = E_2(\alpha \cdot z_2) = (cz_2 + d)^2 E_2(z_2) + \frac{6c}{\pi i}(cz_2 + d) = \frac{6c}{\pi i}(cz_2 + d),$$

which is possible only when $c = 0$, and in this case we have $a = d = \pm 1$; that is, α is a translation. The converse follows from the invariance of E_2 under translation. $\square$

As a consequence we have

Corollary 3.4. *No two distinct zeros of E_2 in the half-strip $\mathfrak{S}$ are equivalent modulo the modular group $SL_2(\mathbb{Z})$.*

We now state the main results of this section.

Theorem 3.5. *The Eisenstein series E_2 has infinitely many zeros in the half-strip $\mathfrak{S} = \{\tau \in \mathfrak{H},\ -\frac{1}{2} < Re(\tau) \leq \frac{1}{2}\}$.*

Proof. Let τ_0 be the unique zero of E_2 on the imaginary axis. Let $\alpha = \begin{pmatrix} t & u \\ v & w \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, where $tv \neq 0$. Then, by Equation (5), we have

$$E_2(\tau_0) = 0 = E_2(\alpha^{-1}\alpha \cdot \tau_0) = (-v(\alpha \cdot \tau_0) + t)^2 E_2(\alpha \cdot \tau_0) - \frac{6v}{\pi i}(-v(\alpha \cdot \tau_0) + t).$$

It follows that

$$(-v(\alpha \cdot \tau_0) + t)E_2(\alpha \cdot \tau_0) = \frac{6v}{\pi i},$$

which is equivalent to saying that

$$\frac{E_2(\alpha \cdot \tau_0)}{(\alpha \cdot \tau_0)E_2(\alpha \cdot \tau_0) + \frac{6}{\pi i}} = \frac{v}{t}.$$

This means that the map $f(z)$ defined by

$$f(z) = \frac{E_2(z)}{\left(zE_2(z) + \frac{6}{\pi i}\right)}$$

carries $\alpha \cdot \tau_0$ onto $r_0 = v/t$, and thus it maps any open neighborhood D_0 of $\alpha \cdot \tau_0$, which we choose in the interior of the fundamental domain $\alpha S\mathfrak{F}$ and on which it is holomorphic, onto an open neighborhood U_0 of r_0. Let $r_1 = a_1/b_1$ be a reduced fraction in $\mathbb{Q} \cap U_0 \setminus \{r_0\}$. Then there exists $z_1 \in D_0 \setminus \{\alpha \cdot \tau_0\}$ such that $f(z_1) = a_1/b_1$. Therefore,

$$(-a_1 z_1 + b_1)E_2(z_1) = \frac{6a_1}{\pi i}. \tag{11}$$

Choose $c_1,\ d_1 \in \mathbb{Z}$ such that $b_1 d_1 - a_1 c_1 = 1$. Then

$$\gamma_1 := \begin{pmatrix} d_1 & -c_1 \\ -a_1 & b_1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

If we set $\tau_1 = \gamma_1 \cdot z_1$, then, using (5) and (11), we have $E_2(\tau_1) = 0$. Moreover, τ_1 is not equivalent to τ_0 modulo $\mathrm{SL}_2(\mathbb{Z})$; otherwise we would have, according to Proposition 3.3, that $\tau_0 := T^n \gamma_1 \cdot z_1$ for some $n \in \mathbb{Z}$ with $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Since $z_1 \in \alpha S\mathfrak{F}$, write $z_1 = \alpha \cdot z_1'$ for some $z_1' \in S\mathfrak{F}$. We have $\tau_0 = T^n \gamma_1 \alpha \cdot z_1'$ with τ_0 and z_1' being in the fundamental domain $S\mathfrak{F}$. Therefore, $T^n \gamma_1 \alpha = 1$, and hence $\tau_0 = z_1'$ and $\alpha \cdot \tau_0 = z_1$, a contradiction since we have chosen $z_1 \in D_0 \setminus \{\alpha\tau_0\}$. Thus τ_1 is a zero of E_2 that is not equivalent to τ_0.

It remains to show that two distinct rational numbers lead to two distinct zeros of E_2. Let $r_2 = a_2/b_2$ be a rational number in $U_0 \setminus \{r_0,\ r_1\}$. In the same way we construct a zero of E_2, $\tau_2 = \gamma_2 \cdot z_2$, that is not equivalent to τ_0 modulo $\mathrm{SL}_2(\mathbb{Z})$, with $z_2 \in \alpha S\mathfrak{F}$. Then τ_2 is not equivalent to τ_1 modulo $\mathrm{SL}_2(\mathbb{Z})$. Indeed if $\tau_1 = T^m \cdot \tau_2$ for some $m \in \mathbb{Z}$, then $\gamma_1 \alpha \cdot z_1' = T^m \gamma_2 \alpha \cdot z_2'$ with z_1' and z_2' being in the same fundamental domain $S\mathfrak{F}$. It follows that $\gamma_1 \alpha = T^m \gamma_2 \alpha$, and consequently $r_1 = r_2$.

This contradicts our choice of r_2. Hence, τ_2 is another zero of E_2 that is not equivalent to either τ_0 or τ_1. Finally, since the open set U_0 contains infinitely many rational numbers, we deduce that E_2 has infinitely many zeros in the half-strip $\mathfrak{S}$. □

Since E_2 is the logarithmic derivative of the discriminant Δ, from the above theorem we deduce

Corollary 3.6. *The discriminant Δ has infinitely many critical points.*

We now look at the multiplicity of the zeros of E_2.

Theorem 3.7. *The zeros of the Eisenstein series E_2 are all simple.*

Proof. Let z_0 be a zero of E_2. By (6), we have

$$\frac{1}{2\pi i}\frac{dE_2(z_0)}{d\tau} = \frac{1}{12}(E_2(z_0)^2 - E_4(z_0)) = \frac{-1}{12}E_4(z_0).$$

Therefore, to prove that this zero is simple, it suffices to show that $E_4(z_0) \neq 0$. It is known that E_4 has all its zeros at $\rho = \frac{-1+i\sqrt{3}}{2}$ and its conjugates modulo $\mathrm{SL}_2(\mathbb{Z})$ (see for instance [3]). Thus, it is enough to show that $E_2(\alpha\cdot\rho) \neq 0$ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. Using (5) and (10), we have for $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$:

$$E_2(\alpha\cdot\rho) = (c\rho + d)^2\frac{2\sqrt{3}}{\pi} + \frac{6c}{\pi i}(c\rho + d) = \frac{2\sqrt{3}}{\pi}(c^2 - cd + d^2)\ ,$$

which does not vanish unless $c = d = 0$, which is not the case since $ad - bc = 1$. This shows that E_2 does not vanish on the orbit of ρ and that consequently E_4 and E_2 have no common zeros. □

4. Distribution of the zeros of E_2

In this section, we will show that there are infinitely many fundamental regions within the half-strip $\mathfrak{S}$ that contain zeros of E_2, and we will also show that there are infinitely many such regions that do not contain any zero of E_2.

Theorem 4.1. *There is a positive integer c_0 such that for all integers $c \geq c_0$, there is a fundamental domain with a vertex at $1/c$ containing a zero of E_2.*

Proof. Let τ_0 again denote the unique zero of E_2 on the imaginary axis, and let $\alpha = \left(\begin{smallmatrix} t & u \\ v & w \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, so that $tv \neq 0$. As in the proof of Theorem 3.5 the map

$$f(z) = \frac{E_2(z)}{\left(zE_2(z) + \frac{6}{\pi i}\right)}$$

maps any neighborhood of $\alpha\cdot\tau_0$ onto a neighborhood of v/t. In particular, f maps a neighborhood D_0 of $S_1\tau_0$, chosen to be in the interior of $S_1S\mathfrak{F}$, onto a neighborhood U_0 of 1 (recall that $S_1 = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$). There exists a positive integer c_0 such that for all $c \geq c_0$, $1 + 1/c \in U_0$. For each $c \geq c_0$, let $z_c \in D_0$ be such that $f(z_c) = 1 + 1/c$. Therefore, if $\gamma_c = \left(\begin{smallmatrix} -1 & 1 \\ -1-c & c \end{smallmatrix}\right)$, then, as in the proof of Theorem 3.5, ${\gamma_c}^{-1}\cdot z_c$ is a zero E_2 belonging to ${\gamma_c}^{-1}S_1S\mathfrak{F}$. If we set $S_c = {\gamma_c}^{-1}S_1S = \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$ for $c \geq c_0$, then we have constructed a zero of E_2 in the fundamental domain $S_c\mathfrak{F}$ which has a vertex at the cusp $1/c$. □

Remark 4.1.

- Thanks to Proposition 3.3, the above theorem can be extended to include the cusps 0 and 1/2.
- An immediate consequence of this theorem is again the infiniteness of the number of zeros of the Eisenstein series E_2. Furthermore, it follows from Corollary 3.4 that all these zeros are inequivalent modulo $\mathrm{SL}_2(\mathbb{Z})$, as all these fundamental domains are contained in the half-strip $\mathfrak{S}$.

We now focus on the fundamental domains that contain no zeros of E_2.

Proposition 4.2. *The Eisenstein series E_2 has no zeros in the fundamental domain $\mathfrak{F}$ of $SL_2(\mathbb{Z})$.*

Proof. Let $\tau_0 = iy_0$ be the unique zero of E_2 on the imaginary axis. Using the transformation formula for E_2, we have

$$0 \,<\, E_2(-1/\tau_0) \,=\, \frac{6}{\pi} y_0 \,<\, 1.$$

This follows from the fact that $\mathrm{Im}(\tau_0) < 1$ (since $\tau_0 \in S\mathfrak{F}$) and thus $\mathrm{Im}(-1/\tau_0) > \mathrm{Im}(\tau_0)$, and the fact that E_2 is strictly increasing on the imaginary axis with the value 0 at τ_0 and the value 1 at $i\infty$. Therefore

$$y_0 \,<\, \frac{\pi}{6}. \tag{12}$$

If $\tau = x + iy \in \mathfrak{F}$ is a zero of E_2, then $\mathrm{Im}(\tau) > \sqrt{3}/2 > \pi/6 > y_0$ and therefore

$$\frac{1}{24}\,|1 - E_2(\tau)| = \left|\sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi i n \tau}\right| \le \sum_{n=1}^{\infty} \sigma_1(n) e^{-2\pi n y} < \sum_{n=1}^{\infty} \sigma_1(n) e^{-2\pi n y_0}.$$

The latter sum is simply $1/24(1 - E_2(\tau_0)) = 1/24$. Therefore

$$\frac{1}{24}|1 - E_2(\tau)| \,<\, \frac{1}{24}.$$

Hence $E_2(\tau)$ cannot be 0 if $\tau \in \mathfrak{F}$. $\square$

In the above proof we have used the inequality $\sqrt{3}/2 > \pi/6$, which is obvious numerically but is a consequence of a simpler inequality such as $\pi < 4$. In what follows we will rely on another inequality which is also numerically obvious:

$$e^{-\pi\sqrt{3}} \,<\, \frac{1}{200}. \tag{13}$$

It simply says that $0.00433 < 0.005$.

We will now investigate more fundamental domains that do not contain any zeros of E_2. For a fixed integer $c \ge 2$ we again set $S_c = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ and $S_{b,d}(c) = \begin{pmatrix} 1 & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $b,\ d \in \mathbb{Z}$, and $\delta_b = \begin{pmatrix} 0 & -1 \\ 1 & b \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $b \in \mathbb{Z}$. The fundamental domain $S_{b,d}(c)\mathfrak{F}$ has a vertex at the cusp $1/c$, as does $S_c\mathfrak{F}$. Also $\delta_b\mathfrak{F}$ has a vertex at the cusp 0, as does $S\mathfrak{F}$.

Let us examine more closely the fundamental domain $S_c\mathfrak{F}$. Its vertices are

$$\frac{1}{c}\ , \quad S_c \cdot \rho = \frac{c - \frac{1}{2} + i\frac{\sqrt{3}}{2}}{c^2 - c + 1}\ , \quad S_c \cdot (\rho + 1) = \frac{c + \frac{1}{2} + i\frac{\sqrt{3}}{2}}{c^2 + c + 1}.$$

It is clear that $\mathrm{Im}(S_c \cdot \rho) > \mathrm{Im}\, S_c \cdot (\rho+1)$ and $\mathrm{Re}\, S_c \cdot \rho > 1/c > \mathrm{Re}\, S_c \cdot (\rho+1)$. Thus

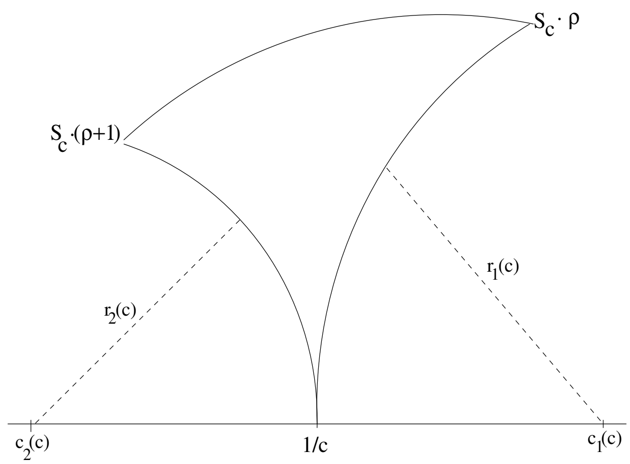

FIGURE 1

we have the following situation for the fundamental region $S_c\mathfrak{F}$ (see Figure 1).

The edge joining $1/c$ and $S_c \cdot \rho$ is an arc of the circle $\mathcal{C}_1(c)$ centered at $c_1(c) = (c-1)/c(c-2)$ and having radius $r_1(c) = 1/c(c-2)$, while the edge joining $1/c$ and $S_c \cdot (\rho+1)$ is an arc of the circle $\mathcal{C}_2(c)$ centered at $c_2(c) = (c+1)/c(c+2)$ with radius $r_2(c) = 1/c(c+2)$. In particular, any other fundamental domain having the cusp $1/c$ as a vertex is either within the circle $\mathcal{C}_1(c)$ or within the circle $\mathcal{C}_2(c)$.

The case $c = 2$ needs to be clarified, as the radius $r_1(2)$ is infinite and in this case the arc joining $1/2$ and $S_2 \cdot \rho$ is the vertical segment $\left[1/2,\ 1/2 + i\sqrt{3}/6\right]$ (see Figure 2). Moreover, as we are restricting the study to the half-strip $\mathfrak{S}$, we only consider those fundamental domains with vertex at the cusp $1/2$ that lie under the arc of the circle $\mathcal{C}_2(2)$. It has center at $c_2(2) = 3/10$ and radius $r_2(2) = 1/10$.

Lemma 4.3. *If we set*

$$M = \frac{1}{24}\left(1 - E_2\left(\frac{i\sqrt{3}}{2}\right)\right),$$

then we have

$$24^2\left(M^2 + \frac{M}{\pi}\right) < 1. \tag{14}$$

Proof. Set $q = exp(-\pi\sqrt{3})$. We have

$$0 < M = \sum_{n\geq 1} \sigma_1(n)\, q^n = \sum_{n\geq 1} \frac{nq^n}{1-q^n} \leq \frac{1}{1-q} \sum_{n\geq 1} nq^n = \frac{q}{(1-q)^3}.$$

Hence, using (13), we have

$$M \leq \frac{40000}{7880599}.$$

Therefore,

$$24^2\left(M^2 + \frac{M}{\pi}\right) < 24^2\left(M^2 + \frac{M}{3}\right) \leq \frac{61444600320000}{62103840598801} < 1.$$

$\square$

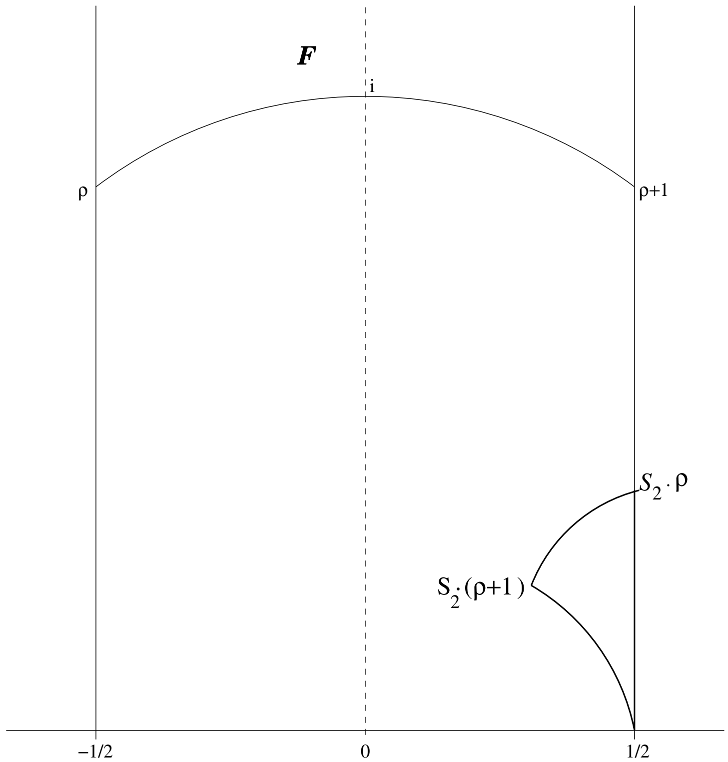

FIGURE 2

In the following, we will prove that the only fundamental domains having a vertex at the cusp $1/c$ that might contain a zero of E_2 are the $\beta_c\mathfrak{F}$, and the only fundamental domain having a vertex at the cusp 0 that might contain a zero is $S\mathfrak{F}$.

Theorem 4.4. *If $b \neq 0$, then E_2 has no zeros in $S_{b,d}(c)\mathfrak{F}$ or in $\delta_b\mathfrak{F}$.*

Proof. Suppose first that $c \geq 3$, and suppose there is a zero z_0 of E_2 in the fundamental domain $S_{b,d}(c)\mathfrak{F}$ where $S_{b,d}(c) = \begin{pmatrix} 1 & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. If $b \neq 0$, then, according to the discussion preceding the above lemma, the fundamental domain $S_{b,d}(c)\mathfrak{F}$ is either within the circle $\mathcal{C}_1(c)$ or $\mathcal{C}_2(c)$. We will show that in fact z_0 is outside the circles $\mathcal{C}_1(c)$ and $\mathcal{C}_2(c)$, which is a contradiction.

We have

$$E_2(S_{b,d}(c)^{-1} \cdot z_0) = \frac{-6c}{\pi i}(-cz_0 + 1),$$

so that

$$\begin{aligned} \sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi n i S_{b,d}(c)^{-1} \cdot z_0} &= \frac{1}{24} + \frac{c}{4\pi i}(-cz_0 + 1) \\ &= -\frac{c^2}{4\pi i}\left(z_0 - \left(\frac{1}{c} + \frac{\pi i}{6c^2}\right)\right). \end{aligned} \tag{15}$$

Since $S_{b,d}(c)^{-1} \cdot z_0 \in \mathfrak{F}$, we have

$$\operatorname{Im}\left(S_{b,d}(c)^{-1} \cdot z_0\right) \geq \frac{\sqrt{3}}{2}. \tag{16}$$

Hence

$$\left|\sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi n i S_{b,d}^{-1} \cdot z_0}\right| \leq \sum_{n=1}^{\infty} \sigma_1(n) e^{-n\pi\sqrt{3}} = M. \tag{17}$$

Therefore

$$\left|z_0 - \left(\frac{1}{c} + \frac{\pi i}{6c^2}\right)\right| \leq M\frac{4\pi}{c^2}; \tag{18}$$

that is, z_0 belongs to the disk $\mathcal{D}_0(c)$ of center $c_0(c) = \frac{1}{c} + \frac{\pi i}{6c^2}$ and radius $r_0(c) = M\frac{4\pi}{c^2}$. We will now show that the disk $\mathcal{D}_0(c)$ lies outside the circles $\mathcal{C}_1(c)$ and $\mathcal{C}_2(c)$ by showing respectively that $|c_0(c) - c_1(c)| > r_1(c) + r_0(c)$ and that $|c_0(c) - c_2(c)| > r_2(c) + r_0(c)$. Because the cusp $1/c$ and $c_0(c)$ are on the same vertical axis, we have

$$|c_1(c) - c_0(c)|^2 = r_1(c)^2 + \left(\frac{\pi}{6c^2}\right)^2.$$

Thus in order to prove that $|c_0(c) - c_1(c)| > r_0(c) + r_1(c)$ we only need to prove that

$$r_0(c)^2 + 2r_0(c)r_1(c) < \left(\frac{\pi}{6c^2}\right)^2.$$

In other words,

$$2\pi M^2 + \frac{Mc}{c-2} < \frac{\pi}{288}.$$

In the meantime, for $c \geq 4$, we have $c/(c-2) = 1 + 2/(c-2) \leq 2$. Thus it is enough to prove that $2\pi M^2 + 2M < \pi/288$, which is a consequence of Lemma 4.3.

Similarly, we prove that $|c_2 - c_0| > r_2 + r_0$. Indeed, as above, it is enough to show that

$$2\pi M^2 + \frac{Mc}{c+2} < \frac{\pi}{288},$$

which is a consequence of Lemma 4.3 since $c/(c+2) < 1$. Notice that $|c_2 - c_0| > r_2 + r_0$ is also valid for the cases $c = 2$ and $c = 3$. This proves the theorem for $c \geq 4$ and also for $c = 2$ since the circle $\mathcal{C}_1(c)$ is the vertical line $\operatorname{Re} z = 1/2$, and thus we only need to estimate the distance $|c_2 - c_0|$.

The case $c = 3$ involves different estimates since we cannot apply Lemma 4.3 for the above choice of M. As we noticed above z_0 is outside the circle $\mathcal{C}_2(3)$, and we only need to show that it is outside $\mathcal{C}_1(3)$. On the other hand, the fundamental domain $S_{-1,-2}(3)\mathfrak{F}$ is adjacent (on the right) to $S_3\mathfrak{F}$ (see Figure 3), and the disc $\mathcal{D}_0(3)$ is outside the circle $\mathcal{C}_3$ which joins the vertices $1/3$ and $S_{-1,-2}(3) \cdot \rho$. Indeed, this circle is centered at $8/21$ and has radius $1/21$. Moreover

$$|c_0(3) - 8/21| = \frac{\sqrt{324 + 49\pi^2}}{378} \approx 0.07518,$$

and

$$r_0(3) + \frac{1}{21} = \frac{4\pi M}{9} + \frac{1}{21} < \frac{4\pi}{9 \cdot 200} + \frac{1}{21} \approx 0.0546.$$

It follows that the only possible values of (b,d) for which $S_{b,d}\mathfrak{F}$ might contain a zero are $(b,d) = (-1,-2)$ leading to $S_{-1,-2}(3)\mathfrak{F}$ and $(b,d) = (0,1)$ leading to $S(3)\mathfrak{F}$. We now show that $z_0 \notin S_{-1,-2}(3)\mathfrak{F}$ by exhibiting a smaller disc $\mathcal{D}(3)$ containing z_0

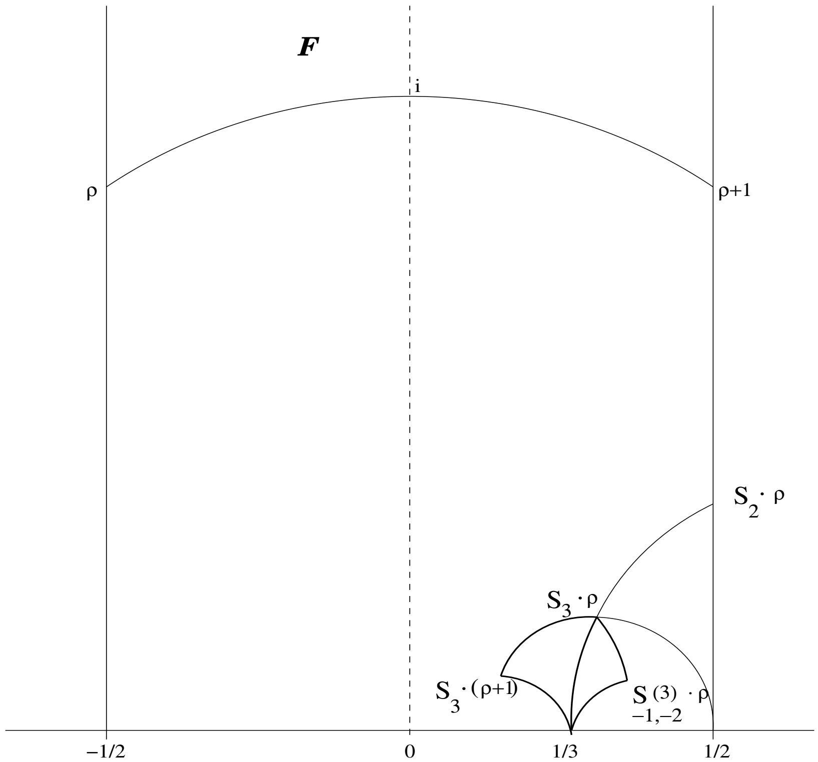

FIGURE 3

and lying outside the circle $\mathcal{C}_1(3)$ as the disc $\mathcal{D}_0(3)$ does not necessarily meet this condition. The transformation $S_{-1,-2}$ maps $\mathcal{D}_0(3)$ onto a disc $\mathcal{D}_0'(3)$ centered at

$$c_0'(3) = S_{-1,-2}(3)^{-1} \cdot c_0(3) \;=\; \frac{6i}{\pi} + \frac{2}{3}$$

and with radius $r_0'(3)$ that can easily be shown to satisfy $r_0'(3) < 0.26$. Therefore, we obtain a more precise lower bound to $\operatorname{Im} S_{-1,-2}(3) \cdot z_0$ as compared to (16):

$$\operatorname{Im} S_{-1,-2}(3) \cdot z_0 > \frac{6}{\pi} - 0.26.$$

We now replace M in Lemma 4.3 by

$$M' \;=\; \frac{1}{24}\,\left(1 - E_2\left(i(6/\pi - 0.26)\right)\right)$$

and obtain

$$2\pi M'^2 + 3M' < \pi/288.$$

Hence, as in the general case, we conclude that

$$\left| z_0 - \left(\frac{1}{3} + \frac{i\pi}{54}\right)\right| \;\leq\; M'\frac{4\pi}{9},$$

and therefore, the disc $D(3) = \mathcal{D}(1/3 + i\pi/54,\ 4\pi M'/9)$ is outside the circle $\mathcal{C}_1(3)$. It follows that there is no zero of E_2 in $S_{-1,-2}(3)\mathfrak{F}$ and thus in any $S_{b,d}(3)\mathfrak{F}$ for $b \neq 0$.

Finally, for the case of the cusp at 0, if z_0 is a zero of E_2 in $\delta_b\mathfrak{F}$, then z_0 is contained inside the circle centered at $\frac{\pi i}{6}$ and having radius $4M\pi$ which is clearly contained in $S\mathfrak{F}$. Therefore $b = 0$, since, otherwise, $\delta_b\mathfrak{F}$ and $S\mathfrak{F}$ are disjoint. $\square$

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