EQUIVARIANT FUNCTIONS AND VECTOR-VALUED MODULAR FORMS

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Abstract. For any discrete group Γ and any 2-dimensional complex representation ρ of Γ, we introduce the notion of $\rho-$equivariant functions, and we show that they are parameterized by vector-valued modular forms. We also provide examples arising from the monodromy of differential equations.

1. Introduction

Throughout this paper, by a discrete group, we mean a finitely generated Fuchsian group of the first kind, acting on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Let Γ be such a group. Let $\rho : \Gamma \longrightarrow \mathrm{GL}_2(\mathbb{C})$ be a 2-dimensional complex representation of Γ. A meromorphic function h on $\mathbb{H}$ is called a $\rho-$*equivariant function* with respect to Γ if

$$h(\gamma \cdot z) \;=\; \rho(\gamma) \cdot h(z) \quad \text{for all } \; z \in \mathbb{H}\,,\; \gamma \in \Gamma\,, \tag{1.1}$$

where the action on both sides is by linear fractional transformations. The set of $\rho-$equivariant functions for Γ will be denoted by $E_\rho(\Gamma)$.

In the case ρ is the defining representation of Γ, that is $\rho(\gamma) = \gamma$ for all $\gamma \in \Gamma$, then elements of $E_\rho(\Gamma)$ are simply called equivariant functions. These were studied extensively in [1, 2, 3] and have various connections to modular forms, quasi-modular forms, elliptic functions and to sections of the canonical line bundle of $X(\Gamma) = \overline{\Gamma\backslash\mathbb{H}}$. In particular, one shows that the set of equivariant functions for a discrete group Γ without the trivial one $h_0(z) = z$ has a vector space structure isomorphic to the space of weight 2 automorphic forms for Γ.

In this paper, we will treat the general case where ρ is an arbitrary 2-dimensional complex representation of Γ. The main result of this paper states that every $\rho-$equivariant function is parameterized by a 2-dimensional vector-valued modular form for ρ. More precisely, if

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$F = (f_1, f_2)^t$ is a vector-valued modular form for Γ and ρ (see Section 2), then $h_F = f_1/f_2$ is a $\rho-$equivariant function for Γ. We will show that, in fact, every $\rho-$equivariant function arises in this way. To achieve this parametrization, we use the fact that the Schwarz derivative of a $\rho-$equivariant function is a weight 4 automorphic form for Γ, in addition to the knowledge of the existence of global solutions to a certain second degree differential equation.

Finally, in the last section, we construct examples of $\rho-$equivariant functions when ρ is the monodromy representation of second degree ordinary differential equations.

2. Vector-valued modular forms

The theory of vector-valued modular forms was introduced long ago as a higher dimensional generalization of the classical (scalar) modular forms. Let Γ be a discrete group and let $\rho : \Gamma \longrightarrow \mathrm{GL}_n(\mathbb{C})$ be an $n-$dimensional complex representation of Γ. A vector-valued modular form of integral weight k for Γ and representation ρ is an $n-$tuple $F(z) = (f_1(z), \ldots, f_n(z))^t$ of meromorphic functions on the complex upper half-plane $\mathbb{H}$ satisfying

$$F(z)|_k\gamma\,(z) \;=\; \rho(\gamma)\,F(z) \;\text{ for all } \gamma \in \Gamma\,,\;\; z \in \mathbb{H}\,, \tag{2.1}$$

and some growth conditions at the cusps that are similar to those for classical automorphic forms. The slash operator in (2.1) is defined as usual by $F|_k\gamma(z) = (cz+d)^{-k}F(\gamma\cdot z)$ where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The set of these vector-valued modular forms will be denoted by $V_\rho(\Gamma)_k$, and we will refer to them as $\rho-$VMF of weight k.

We omit the multiplier system since it will not have any effect in this paper. If $F(z)$ is holomorphic, then it will be called a holomorphic vector-valued modular form.

The theory of vector-valued modular forms is fairly well understood when Γ is the modular group. See for instance [4] and the references therein. However, except for the genus 0 subgroups of the modular group, it is not even clear that nonzero vector-valued modular forms exist. In connection with equivariant functions, we have the following straightforward proposition.

Proposition 2.1. *Let Γ be a discrete group, and ρ an arbitrary 2-dimensional complex representation of Γ. If $F(z) = (f_1(z), f_2(z))^t$ is a vector-valued modular form for ρ of a certain weight, then $f_1(z)/f_2(z)$ is a $\rho-$equivariant function for Γ.*

We will prove below that every $\rho-$equivariant function arises in this way.

3. Differential equations

Let D be a domain in $\mathbb{C}$ and let f be a meromorphic function on D. Its Schwarz derivative, $S(f)$, is defined by

$$S(f) \;=\; \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 .$$

This is an important tool in projective geometry and differential equations. The main properties that will be useful to us are summarized as follows (see [5] for more details):

Proposition 3.1. *We have*

1. *If y_1 and y_2 are two linearly independent solutions to a differential equation $y''+Qy=0$ where Q is a meromorphic function on D, then $S(y_1/y_2)=2Q$.*
2. *If f and g are two meromorphic functions on D, then $S(f)=S(g)$ if and only if $f=\dfrac{ag+b}{cg+d}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C})$.*
3. *$S(f\circ\gamma)(z)=(cz+d)^4 S(f)$ provided $\gamma\cdot z\in D$, where $\gamma=\begin{pmatrix} * & * \\ c & d \end{pmatrix}$.*

In particular, we have

Proposition 3.2. *If f is a ρ-equivariant for a discrete group Γ, then $S(f)$ is an automorphic form of weight 4 for Γ.*

Now, consider the second order ordinary differential equation (ODE)

$$x''+Px'+Qx=0,$$

where P and Q are holomorphic functions in D. This ODE has two linearly independent holomorphic solutions in D if D is simply connected. For a fixed $z_0\in D$, set

$$y(z) \;=\; x(z)\exp\left(\int_{z_0}^{z}\frac{1}{2}P(w)dw\right).$$

The above ODE reduces to an ODE in normal form

$$y'' \;+\; gy \;=\; 0\,, \tag{3.1}$$

with

$$g=Q-\frac{1}{2}P'-\frac{1}{4}P^2.$$

When the domain D is not simply connected, we may not expect to find global solutions to (3.1) on D. However, under some conditions on g, global solutions do exist as it is illustrated in the following theorem which will be crucial for the rest of this paper.

Theorem 3.3. *Let D be a domain in $\mathbb{C}$. Suppose h is a nonconstant meromorphic function on D such that $S(h)$ is holomorphic in D, and let $g = \frac{1}{2}S(h)$. Then the differential equation*

$$y'' + gy = 0$$

has two linearly independent holomorphic solutions in D.

Proof. Let $\{U_i, i \in I\}$ be a covering of D by open discs with $\dim V(U_i) = 2$ for all $i \in I$ where $V(U_i)$ denotes the space of holomorphic solutions to $y''+gy = 0$ on U_i. Choose L_i and K_i to form a basis for $V(U_i)$. Using property (1) of Proposition 3.1, we have $S(K_i/L_i) = 2g = S(h)$ on U_i. Now, using property (2) of Proposition 3.1, we have $K_i/L_i = \alpha_i \cdot h$ for $\alpha_i \in \mathrm{GL}_2(\mathbb{C})$. In the meantime, on each connected component W of $U_i \cap U_j$, we have

$$(K_i, L_i)^t = \alpha_W (K_j, L_j)^t \ , \quad \alpha_W \in \mathrm{GL}_2(\mathbb{C})\,,$$

since each of (K_i, L_i) and (K_j, L_j) is a basis of $V(W)$. Hence, on W we have

$$\frac{K_i}{L_i} = \alpha_W \cdot \frac{K_j}{L_j}\,,$$

and therefore

$$\alpha_i h = \alpha_W \alpha_j h\,.$$

It follows that

$$\alpha_i \alpha_j^{-1} = \alpha_W$$

as h is meromorphic and nonconstant and thus it takes more than three distinct values on the domain D. Therefore, α_W does not depend on W. Moreover, on $U_i \cap U_j$ we have

$$\alpha_i^{-1}(K_i, L_i)^t = \alpha_j^{-1}(K_j, L_j)^t\,. \tag{3.2}$$

If we define f_1 and f_2 on U_i by

$$(f_1, f_2)^t = \alpha_i^{-1}(K_i, L_i)^t\,,$$

then using (3.2), we see that f_1 and f_2 are well defined all over D and they are two linearly independent solutions to $y'' + gy = 0$ on all of D as they are linearly independent over U_i. ☐

4. The correspondence

In this section, we will prove that every $\rho-$equivariant function arises from a vector-valued modular form as in Proposition 2.1. We start with the following property of the slash operator known as Bol's identity.

Proposition 4.1. *Let r be a nonnegative integer, $F(z)$ a complex function and $\gamma \in SL_2(\mathbb{C})$, then*

$$(F|_{-r}\gamma)^{(r+1)}(z) \;=\; F^{(r+1)}|_{r+2}\gamma(z)\,.$$

As a consequence, we have

Corollary 4.2. *Let r be a nonnegative integer, g a $\Gamma-$automorphic form of weight $2(r+1)$ and D a domain in $\mathbb{H}$ that is stable under the action of Γ. Denote by $V_r(D)$ the solution space on D to the differential equation*

$$f^{(r+1)} + gf = 0\,.$$

Then for all $\gamma \in \Gamma$,

$$f \in V_r(D) \quad \text{if and only if } f|_{-r}\gamma \in V_r(D)\,.$$

Corollary 4.3. *The operator $|_{-r}$ provides a representation ρ_r of Γ in $GL(V_r)$. Moreover, if f_1, f_2,. . . ,f_{r+1} form a basis of V_r (if the basis exists), then*

$$F = (f_1, f_2, \ldots, f_{r+1})^t$$

behaves as a ρ_r-VMF of weight $-r$ for Γ.

We know state the main result of this paper. Recall from Proposition 2.1 that if $F = (f_1, f_2)^t$ is a $\rho-$VMF, then $h_F = f_1/f_2$ is a $\rho-$equivariant function.

Theorem 4.4. *The map*

$$V_\rho(\Gamma)_{-1} \to E_\rho(\Gamma)$$

$$F \mapsto h_F$$

is surjective.

Proof. Suppose that h is a $\rho-$equivariant function for Γ. According Proposition 3.2, its Schwarz derivative $S(h)$ is an automorphic form of weight 4 for Γ. Let $g = \frac{1}{2}S(h)$ and D the complement in $\mathbb{H}$ of the set of poles of g. Then D is a domain that is stable under Γ since g is an automorphic form for Γ.

Using the same notation as in the previous section, we have, for $r = 1$, $S(f_1/f_2) = S(h)$ where $\{f_1, f_2\}$ are two linearly independent solutions in $V(D)$ provided by Theorem 3.3. Hence, by Proposition 3.1

$$\frac{f_1}{f_2} = \alpha \cdot h\ , \quad \alpha \in \mathrm{GL}_2(\mathbb{C})\,.$$

Also, using Corollary 4.2 with $r = 1$, we deduce that $F_1 = (f_1, f_2)^t$ is a ρ_1-VMF of weight -1 for Γ. Therefore,

$$\alpha^{-1}\rho_1\alpha = \rho\,.$$

Hence $F = \alpha^{-1}F_1$ is a $\rho-$VMF of weight -1 for Γ with $h_F = h$ on D. Since g has only double poles, then by looking at the form of the solutions near a singular point, and using the fact that f_1 and f_2 are holomorphic and thus single-valued, we see that f_1 and f_2 can be extended to meromorphic functions on $\mathbb{H}$. $\square$

Remark 4.5. *If f is an automorphic form of weight $k+1$ for Γ, then $(f_1, f_2) \longrightarrow (ff_1, ff_2)$ yields an isomorphism between $V_\rho(\Gamma)_{-1}$ and $V_\rho(\Gamma)_k$, and therefore, the above surjection in the theorem extends to $V_\rho(\Gamma)_k$ whenever it is nontrivial.*

5. Examples

In this section, we shall construct examples of $\rho-$VMF's and of $\rho-$equivariant functions when ρ is the monodromy representation of a differential equation.

Let U be a domain in $\mathbb{C}$ such that $\mathbb{C}\setminus U$ contains at least two points. The universal covering of U is then $\mathbb{H}$ as it cannot be $\mathbb{P}_1(\mathbb{C})$ because U is noncompact and it cannot be $\mathbb{C}$ because of Picard's theorem.

Let $\pi : \mathbb{H} \longrightarrow U$ be the covering map. We consider the differential equation on U

$$y'' + Py' + Qy = 0 \tag{5.1}$$

where P and Q are two holomorphic functions on U. This differential equation has a lift to $\mathbb{H}$

$$y'' + \pi^*Py' + \pi^*Qy = 0\,. \tag{5.2}$$

Let V be the solution space to (5.2) which is a 2-dimensional vector space since $\mathbb{H}$ is simply connected. Let γ be a covering transformation in $\mathrm{Deck}(\mathbb{H}/U)$ which is isomorphic to the fundamental group $\pi_1(U)$ and let $f \in V$. Then $\gamma^*f = f\circ\gamma^{-1}$ is also a solution in V. This defines the monodromy representation of $\pi_1(U)$:

$$\rho : \pi_1(U) \longrightarrow \mathrm{GL}(V)\,.$$

If f_1 and f_2 are two linearly independent solutions in V, we set $F = (f_1, f_2)^t$. Then we have

$$F \circ \gamma \,=\, \rho(\gamma) F \,.$$

Therefore, the quotient f_1/f_2 is a $\rho-$equivariant function on $\mathbb{H}$ for the group $\pi_1(U)$ which is a torsion-free discrete group.

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