RATIONAL EQUIVARIANT FORMS

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Abstract. We investigate the notion of equivariant forms as functions on the upper half-plane commuting with the action of a discrete group. We put an emphasis on the rational equivariant forms for a modular subgroup that are parameterized by generalized modular forms. Furthermore, we study this parametrization when the modular subgroup is of genus zero as well as their behavior under the effect of the Schwarz derivative.

1. Introduction

In this paper we introduce and study the notion of equivariant forms for a modular subgroup. These are meromorphic functions on $\mathbb{H}$, the upper-half of the complex plane $\mathbb{C}$, which commute with the action of a discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$, the group of 2 by 2 matrices with real entries and determinant 1. More precisely, a meromorphic function h on $\mathbb{H}$ is called an *equivariant form* for Γ if it satisfies

$$h(\alpha \cdot z) = \alpha \cdot h(z) \quad \text{for all } z \in \mathbb{H} \text{ and } \alpha \in \Gamma\,,$$

in addition to some precise conditions at the cusps of Γ, and where $\alpha \cdot z$ denotes the usual action:

$$\alpha \cdot z = \frac{az+b}{cz+d} \quad \text{if } \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\,.$$

These functions appeared first in the works by Brady [Br] and Heins [He] as quotients of pseudo-periods of the Weierstrass elliptic ζ functions. More recently, these type of functions also appeared in [AAS1] in connection with modular forms and where the terminology of *equivariant forms* was first coined. Moreover, a wide class of functions known as the *rational equivariant forms* were provided for the modular group $\mathrm{SL}_2(\mathbb{Z})$ that are parameterized by modular forms yielding various interesting applications to modular differential equations and the analysis of the critical points of modular forms for $\mathrm{SL}_2(\mathbb{Z})$. This study, however, was limited to the modular group alone and the proper definition of the equivariant form was not available for an arbitrary subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

2000 *Mathematics Subject Classification.* Primary 11F11.

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In this paper, we undertake the task of generalizing the theory of equivariant forms to all the finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$. Beside providing the rigorous definitions for an arbitrary subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, we construct infinitely many equivariant forms that are parameterized by the so-called generalized modular forms for Γ. These were introduced by Knopp and Mason in [K-M] as a generalization of classical modular forms and are defined as follows: A generalized modular form for Γ of weight k and character $\mu : \Gamma \longrightarrow \mathbb{C}^\times$ is a meromorphic function f on $\mathbb{H}$ satisfying

$$f(\alpha \cdot z) \;= \mu(\alpha)\,(cz+d)^k\, f(z),\ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\,,\ z \in \mathbb{H}\,,$$

in addition to specific growth conditions at the cusps. The character μ is not necessary unitary, and when it is trivial, we refer to f as a modular form. If f is such a generalized modular form, then the function

$$h(z) \;=\; z \,+\, k\frac{f(z)}{f'(z)} \quad (\star)$$

is an equivariant form for Γ.

One of the principal results in this paper is that an equivariant form h is attached to a generalized modular form f as in $(\star)$ if and only if the poles of $1/[h(z)-z]$ in $\mathbb{H}\cup\{\text{cusps}\}$ are simple with rational residues. We refer to such equivariant forms as *the rational equivariant forms.* It turns out that when Γ is of genus zero, and the rationality condition holds, then f is actually a modular form (with trivial character). Furthermore, we provide a wide class of equivariant forms that are not rational in the above sense.

These new and fascinating objects turn out to be very natural and very rich. They are closely related to modular forms and can also be associated to other algebraic and topological theories such that the intertwining operators, the equivariant K-theory as well as the theory of concomitants in projective differential geometry. What is more fascinating is that the equivariant forms can be viewed as geometric objects. More precisely, in a forthcoming work, [E-S], it is shown that the equivariant forms are sections of the canonical line bundle of the modular curve of the underlying modular subgroup. Moreover, the set of equivariant forms for a given modular subgroup is endowed with the structure of an infinite dimensional affine space.

The paper is organized as follows: In Section 2 we provide some necessary preliminaries about subgroups of the modular group as well as modular forms. In Section 3, we introduce the notion of equivariant forms for a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ together with a special slash operator that allows us to define the meromorphy at the cusps. In Section 4 we show that each generalized modular form yields an equivariant form as in $(\star)$ above that we refer to as the rational equivariant forms. In the meantime, we provide infinitely many examples of equivariant forms which do not arise in this way. The main result of Section 5 is a theorem stating the necessary and sufficient conditions for an equivariant form to be rational. In section 6, we restrict ourselves to the genus zero modular subgroups in which case the generalized modular forms are simply modular forms. We also put an emphasis on the equivariant forms without fixed points in $\mathbb{H}$ and prove a uniqueness theorem about them. In Section 7, a second connection with modular forms lies in the fact that the Schwarz derivative of an equivariant form is a weight 4 modular form. Finally, Section 8 contains examples of equivariant forms arising from classical modular forms.

2. Preliminaries

The main references for this section are [Sh] and [Ra]. Let $\mathrm{SL}_2(\mathbb{R})$ be the group of 2x2 matrices with real entries and determinant 1. It acts on the upper half of the complex plane

$$\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$

by linear fractional transformations

$$\alpha \cdot z = \frac{az+b}{cz+d}, \; z \in \mathbb{H}, \; \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

The Möbius group $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm I\}$ is the full automorphism group of $\mathbb{H}$. For α as above, $z \in \mathbb{H}$, set $j_\alpha(z) = cz + d$. Then for a meromorphic function f defined on $\mathbb{H}$ and a positive integer k, we define the slash operator on f by

$$f|_k[\alpha](z) = j_\alpha(z)^{-k} f(\alpha \cdot z). \tag{2.1}$$

The map $j : \mathrm{SL}_2(\mathbb{R}) \times \mathbb{H} \to \mathbb{C}^*$ defines what is called an automorphic factor and satisfies the cocycle relation

$$j_{\alpha\beta}(z) = j_\alpha(\beta \cdot z) j_\beta(z)$$

for all $\alpha, \; \beta \in \mathrm{SL}_2(\mathbb{R})$.

We notice that, since $-\alpha \cdot z = \alpha \cdot z$, $j_{-\alpha}(z)^k = -j_\alpha(z)^k$ when k is odd, so that $f|_k[-\alpha](z) = -f|_k[\alpha](z)$.

In this paper, we will mainly be concerned with the modular group $\mathrm{SL}_2(\mathbb{Z})$ of matrices in $\mathrm{SL}_2(\mathbb{R})$ with integer entries as well as its subgroups of finite index to which we refer as the modular subgroup.

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A modular subgroup Γ is called a congruence group of level N if it contains the principal congruence group $\Gamma(N)$ defined by

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) |\, a \equiv d \equiv 1 \mod (N),\, b \equiv c \equiv 0 \mod (N) \right\}$$

and N is the smallest positive integer such that $\Gamma(N) \subseteq \Gamma$.

Let k be a positive integer. A function f on $\mathbb{H}$ is called a meromorphic modular form or simply a modular form of weight k for a modular subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ if

(1) f is meromorphic on $\mathbb{H}$,
(2) for all $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathbb{H}$, we have $f|_k[\alpha](z) = f(z)$,
(3) f is meromorphic at the cusps.

The last condition means the following. Let $\mathfrak{s} \in \mathbb{Q} \cup \{\infty\}$ be a cusp of Γ. Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma \cdot \mathfrak{s} = \infty$ and set $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then, the function $f|_k[\gamma^{-1}](z)$ is invariant under $\gamma \Gamma_{\mathfrak{s}} \gamma^{-1} =< T^{l_{\mathfrak{s}}} >$, $l_{\mathfrak{s}}$ being the cusp width at $\mathfrak{s}$ and $\Gamma_{\mathfrak{s}}$ the isotropy group of $\mathfrak{s}$ inside Γ. Hence, it has a Fourier expansion in the local parameter at infinity $q_{\mathfrak{s}} := e^{2\pi i z / l_{\mathfrak{s}}}$ if k is even and $q_{\mathfrak{s}} = e^{\pi i z / l_{\mathfrak{s}}}$ if k is odd. The meromorphy condition means we have the Fourier series expansion

$$f|_k[\gamma^{-1}](z) = \sum_{n=n_{\mathfrak{s}}}^{\infty} a_n^{\mathfrak{s}} q_{\mathfrak{s}}^n$$

with the integer $n_{\mathfrak{s}}$ being finite. If $n_{\mathfrak{s}} \geq 0$ for every cusp $\mathfrak{s}$ and if f is holomorphic on $\mathbb{H}$ then f is called a holomorphic modular form. A holomorphic modular form is called a cusp form if it vanishes at all cusps, in other words $n_{\mathfrak{s}} > 0$ for all cusps $\mathfrak{s}$. When $k = 0$ the modular form is called a modular function.

There is also the notion of generalized modular forms due to M. Knopp and G. Mason, [K-M], and which will play a major role in this paper. A meromorphic function f on $\mathbb{H}$ is called a generalized modular form of weight k for a modular subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ if

1. f is meromorphic on $\mathbb{H}$,
2. for all $\alpha \in \Gamma$ and $z \in \mathbb{H}$, we have
$$f|_k[\alpha](z) = \mu(\alpha)\, f(z)\,,$$
where $\mu : \Gamma \longrightarrow \mathbb{C}^\times$ is a character.
3. f is meromorphic at the cusps.

The analysis at the cusps is similar to the above case of modular forms with a slight modification. The generalized modular forms look like classical modular forms with a multiplier system except for the fact that the character μ need not be unitary. It can easily be shown that for $\mathfrak{s}$, γ and $l_{\mathfrak{s}}$ as above

$$f|_k[\gamma^{-1}T^{l_{\mathfrak{s}}}] = e^{2\pi i\kappa_{\mathfrak{s}}} f|_k[\gamma^{-1}], \text{ for some } \kappa_{\mathfrak{s}} \in \mathbb{C}\,.$$

In other words, $\mu(T^{l_{\mathfrak{s}}}) = e^{2\pi i\kappa_{\mathfrak{s}}}$. Hence the function $\widetilde{f}_{\gamma^{-1}}(z) := e^{2\pi i\kappa_{\mathfrak{s}}} f|_k[\gamma^{-1}](z)$ is $l_{\mathfrak{s}}$ invariant, and therefore has a $q_{\mathfrak{s}}-$expansion. The meromorphy is then interpreted in a similar manner as for classical modular forms. If the character μ is trivial, then the generalized modular form becomes a modular form for Γ. For the motivation behind the generalized modular forms, their properties and applications, we refer to [K-M].

The examples of classical modular forms that will be used in this paper are the two Eisenstein series:

$$E_4(z) \;=\; 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n,$$

$$E_6(z) \;=\; 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n.$$

where $\sigma_k(n)$ is the sum of the k-th powers of the positive divisors of n. The Eisenstein series E_4 and E_6 are modular forms for $\mathrm{SL}_2(\mathbb{Z})$ of weight 4 and 6 respectively. We also introduce the Eisenstein series E_2 given by

$$E_2(z) \;=\; 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n,$$

which is not a modular form but it is rather referred to as a quasimodular form of weight 2 and depth 1. Moreover, E_2 satisfies

$$E_2(z) \;=\; \frac{1}{2\pi i}\frac{\Delta'(z)}{\Delta(z)}, \tag{2.2}$$

where Δ is the weight 12 cusp form for $\mathrm{SL}_2(\mathbb{Z})$ given by

$$\Delta(z) \;=\; q\prod_{n\geq 1}(1-q^n)^{24}.$$

The Eisenstein series satisfy the Ramanujan relations

$$\frac{6}{\pi i}\,E_2' \;=\; E_2^2 - E_4\,, \tag{2.3}$$

$$\frac{3}{2\pi i}\,E_4' \;=\; E_4E_2 - E_6\,, \tag{2.4}$$

$$\frac{1}{\pi i}\,E_6' \;=\; E_6E_2 - E_4^2\,. \tag{2.5}$$

The Jacobi theta functions are given by

$$\begin{aligned}\theta_2(z) &= \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}\ ,\\ \theta_3(z) &= \sum_{n=-\infty}^{\infty} q^{n^2}\ ,\\ \theta_4(z) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}\ ,\end{aligned}$$

These theta functions are modular forms of weight $1/2$ for the level 2 congruence subgroup $\Gamma(2)$.

There is a systematic way to produce modular forms from two given ones using the so-called Rankin-Cohen brackets. For a modular form f of weight k and a modular form g of weight l, the Rankin-Cohen bracket of order $n\geq 0$ is defined by

$$[f,g]_n \;=\; \sum_{r=0}^{n} (-1)^r \binom{k+n-1}{n-r}\binom{l+n-1}{r} f^{(r)}g^{(n-r)}\,.$$

It defines a modular form of weight $k+l+2n$. For instance

$$[f,g]_0 = fg\ ,\quad [f,g]_1 = kfg' - lf'g$$

and

$$[f,g]_2 = \frac{k(k+1)}{2}fg'' - (k+1)(l+1)f'g' + \frac{l(l+1)}{2}f''g\,.$$

A modular subgroup Γ is an example of a Fuchsian group of the first kind. Its element are divided into three classes:

(1) The elliptic elements, which have a trace equal to 0 or ± 1. Each elliptic element has a finite order and has a fixed element in $\mathbb{H}$ called an elliptic fixed point. The other fixed point is the complex conjugate of the first one.
(2) The parabolic elements which have a trace equal to ± 2. Each parabolic element has an infinite order and fixes a *cusp* that is either a rational number or infinity.
(3) The hyperbolic elements which have a trace greater than 2.

If we denote by $X(\Gamma)$ the quotient $\Gamma\backslash\mathbb{H}\cup\{cusps\}$. Then $X(\Gamma)$ is a compact Riemann surface whose genus g is also referred to as the genus of Γ. Let ν_∞ be the number of inequivalent cusps and r be the number of inequivalent elliptic points. Let $m_1, ..., m_r$ be the orders of the stabilizers of all conjugacy classes of elliptic points. Then we say that Γ has signature $(g;\ m_1, ...,\ m_r;\ \nu_\infty)$. The algebraic structure of the group can be determined by its signature. In fact, the group has the following presentation:

generators :

$$A_1,\ B_2, ...,\ A_g,\ B_g;\ E_1, ...,\ E_r;\ P_1, ...,\ P_{\nu_\infty}$$

relations :

$$E_1^{m_1} = ... = E_r^{m_r} = \prod_{i=1}^{\nu_\infty} P_i \prod_{i=1}^{r} E_i \prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1} = 1.$$

The generators P_i are parabolic, the E_i are elliptic, and the A_i and B_i are hyperbolic.

3. Equivariant forms

The systematic treatment of equivariant forms has been initiated in [AAS1] in connection with the study of certain Schwarz differential equation involving modular forms. They appeared as meromorphic functions on the upper half-plane $\mathbb{H}$ commuting with the action of a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$. This notion also appeared previously in the work of M. Heins in [He] and M. Brady in [Br] in connection with elliptic functions. More precisely, for a lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $z = \omega_2/\omega_1 \in \mathbb{H}$, the Weierstrass ζ-function is defined by $\zeta' = -\wp$ where $\wp$ is the Weierstrass elliptic $\wp$-function. If η_1 and η_2 are the pseudo-periods of ζ, then

$$h_0 = \omega_1 \eta_2$$

depends only on z and satisfies $h_0(\alpha \cdot z) = \alpha \cdot h_0(z)$, for $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. In this paper, we will not pursue this elliptic point of view but rather we will be interested in the modular point of view as was initiated in [AAS1].

We begin with the definition of an equivariant form for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ following [AAS1]. A meromorphic function h on $\mathbb{H}$ is called an equivariant form if it satisfies

(1) $h(\gamma \cdot z) = \gamma \cdot h(z)$ for all $z \in \mathbb{H}$ and all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,
(2) $h(z) - z$ is meromorphic at ∞.

Here $\gamma \cdot h(z)$ is obtained by extending the action of $\mathrm{SL}_2(\mathbb{Z})$ to all of $\mathbb{C}$. Furthermore, since $h(z) - z$ is 1-periodic, it has a Fourier expansion in $q = \exp(2\pi i z)$. To say that $h(z) - z$ is meromorphic means that this Fourier expansion has

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finitely many negative powers of q

$$h(z) - z = \sum_{n \geq n_0} a_n q^n .$$

A trivial example of an equivariant form is the identity $h_0(z) = z$. For the rest of this paper, if h is not the identity, we set

$$\widehat{h}(z) = \frac{1}{h(z) - z} ,$$

and it will be more convenient to consider the meromorphy of $\widehat{h}$ rather than that of $h(z) - z$ once $\mathrm{SL}_2(\mathbb{Z})$ is replaced by a modular subgroup Γ. To define the equivariant forms for such subgroups, one needs to make precise the meromorphy of $\widehat{h}$ at every cusp of Γ. We start by defining a certain "double-slash" operator. If f is a meromorphic function on $\mathbb{H}$ and $\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, let

$$f\|[\gamma](z) = j_\gamma(z)^{-2} f(\gamma \cdot z) - r j_\gamma(z)^{-1} , \tag{3.1}$$

where again $j_\gamma(z) = rz + s$. This defines an action of $\mathrm{SL}_2(\mathbb{R})$ on the space of meromorphic functions on $\mathbb{H}$. Indeed, for elements $\beta = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\gamma = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbb{R})$, we have, on one hand,

$$f\|[\beta\gamma](z) = j_{\beta\gamma}(z)^{-2} f(\beta\gamma \cdot z) - (cp + dr) j_{\beta\gamma}(z)^{-1} .$$

On the other hand,

$$\begin{aligned} (f\|[\beta])\,\|[\gamma](z) &= j_\gamma(z)^{-2} f\|[\beta](\gamma \cdot z) - r j_\gamma(z)^{-1} \\ &= j_\gamma(z)^{-2} \left(j_\beta(\gamma \cdot z)^{-2} f(\beta\gamma \cdot z) - c j_\beta(\gamma \cdot z)^{-1} \right) - r j_\gamma(z)^{-1} \\ &= j_{\beta\gamma}(z)^{-2} f(\beta\gamma \cdot z) - c j_\gamma(z)^{-2} j_\beta(\gamma \cdot z)^{-1} - r j_\gamma(z)^{-1} . \end{aligned}$$

One easily checks that

$$c j_\gamma(z)^{-2} j_\beta(\gamma \cdot z)^{-1} + r j_\gamma(z)^{-1} = (cp + dr) j_{\beta\gamma}(z)^{-1} ,$$

which yields

$$f\|[\beta\gamma](z) = (f\|[\beta])\,\|[\gamma](z),$$

Proposition 3.1. *Let h be a meromorphic function on $\mathbb{H}$ and let Γ be a modular subgroup. If $\gamma \in \Gamma$ and $z \in \mathbb{H}$, then*

$$h(\gamma \cdot z) = \gamma \cdot h(z) \ \text{ if and ony if } \ \widehat{h}\|[\gamma](z) = \widehat{h}(z) . \tag{3.2}$$

Proof. For $\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \Gamma$ we have

$$h(\gamma \cdot z) = \gamma \cdot h(z) \iff \widehat{h}(\gamma \cdot z) = j_\gamma(z) j_\gamma(h(z)) \widehat{h}(z) \iff$$

$$j_\gamma(z)^{-2} \widehat{h}(\gamma \cdot z) = \frac{j_\gamma(h(z))}{j_\gamma(z)} \widehat{h}(z) .$$

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However, $j_\gamma(h(z)) = r(h(z) - z) + j_\gamma(z)$ so that

$$\frac{j_\gamma(h(z))}{j_\gamma(z)}\widehat{h}(z) = \widehat{h}(z) + rj_\gamma(z)^{-1}\,.$$

The proposition follows. □

Notice that if Γ contains -1_2 and since $j_{-\gamma}(w) = -j_\gamma(w)$ we have

$$\widehat{h}\|[-\gamma](z) = \widehat{h}\|[\gamma](z).$$

If $-1_2 \notin \Gamma$, then we would have two types of cusps: the regular and the irregular cusps. When $\mathfrak{s}$ is regular, $\Gamma_\mathfrak{s}$ is conjugate to $\langle T^{l_\mathfrak{s}}\rangle$ so that the function $\widehat{h}\|[\gamma^{-1}]$ is $l_\mathfrak{s}$-periodic with $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma\cdot\mathfrak{s} = \infty$. In the second case where $\mathfrak{s}$ is irregular, the isotropy group $\Gamma_\mathfrak{s}$ is conjugate to $\langle -T^{l_\mathfrak{s}}\rangle$. An easy computation shows, however, that, as h is equivariant for Γ,

$$\widehat{h}\|[\gamma^{-1}]\|[T^{l_\mathfrak{s}}] = \widehat{h}\|[\alpha(-\gamma^{-1})] = \widehat{h}\|[-\gamma^{-1}]\ ,$$

for some $\alpha \in \Gamma$. It follows from the definition of the slash operator $\|$ and the fact that $j_{-\gamma^{-1}} = -j_{\gamma^{-1}}$ that

$$\widehat{h}\|[\gamma^{-1}]\|[T^{l_\mathfrak{s}}] = \widehat{h}\|[\gamma^{-1}]\,,$$

that is $\widehat{h}\|[\gamma^{-1}]$ is also $l_\mathfrak{s}$-periodic.

We now proceed to define the notion of an equivariant form for a modular subgroup Γ. Let $\mathfrak{s}$ be a cusp of Γ, that is $\mathfrak{s}$ is in $\mathbb{Q}\cup\{\infty\}$, and choose $\gamma = \begin{pmatrix} p & q \\ r & s\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma\cdot\mathfrak{s} = \infty$. Then the isotropy group of $\mathfrak{s}$, $\Gamma_\mathfrak{s} = \{\alpha \in \Gamma |\ \alpha\cdot\mathfrak{s} = \mathfrak{s}\}$, is conjugate by γ to the infinite cyclic group generated by $T^{l_\mathfrak{s}}$, with $T = \begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}$ and $l_\mathfrak{s}$ is a positive integer known as the cusp width of Γ at the cusp $\mathfrak{s}$.

If h is a meromorphic function on $\mathbb{H}$ which commutes with the action of Γ on $\mathbb{H}$, then $\widehat{h}\|[\gamma^{-1}](z)$ is invariant under $\gamma\Gamma_\mathfrak{s}\gamma^{-1} = \langle T^{l_\mathfrak{s}}\rangle$ and hence it is $l_\mathfrak{s}$-periodic. Therefore, it has a Fourier expansion in the local parameter $q_\mathfrak{s} = \exp(2\pi iz/l_\mathfrak{s})$ of the form

$$\widehat{h}\|[\gamma^{-1}](z) = \sum_{m\geq m_\mathfrak{s}} a_m q_\mathfrak{s}^m.$$

We say that h is meromorphic at $\mathfrak{s}$ if $\widehat{h}\|[\gamma^{-1}](z)$ is meromorphic at ∞ in the sense that the integer $m_\mathfrak{s}$ is finite. It is important to note that if this holds at a cusp $\mathfrak{s}$, then it also holds at any cusp that is Γ-equivalent to $\mathfrak{s}$.

Definition 3.1. *An equivariant form for Γ is a meromorphic function on $\mathbb{H}$ which commutes with the action of Γ and which is meromorphic at every cusp of Γ.*

The rest of this paper is devoted to study these objects which turn out to be very rich in structure.

Proposition 3.2. *Let $\Gamma_1,\ \Gamma_2$ be two modular subgroups. Suppose that Γ_1 and Γ_2 are conjugate, that is $\Gamma_1 = \alpha\Gamma_2\alpha^{-1}$ for some $\alpha \in SL_2(\mathbb{Z})$. If h_1 is an equivariant form for Γ_1, then*

$$h_2(z) = \alpha^{-1} \circ h_1 \circ \alpha(z)$$

is an equivariant form for Γ_2.

Proof. The commuting of h_2 with the action of Γ_2 follows easily from that of h_1 with the action of Γ_1. A straightforward computation shows that

$$\widehat{h}_2(z) \;=\; \widehat{h}_1\|[\alpha](z).$$

Therefore, h_2 is also meromorphic at the cusps. □

4. Rational equivariant forms

In this section, we construct a wide class of equivariant forms that arise from generalized modular forms. We also focus on the effect of the geometry of the modular subgroup on the analytic properties of the equivariant forms.

Theorem 4.1. *Let Γ be a modular subgroup and let f be a generalized modular form for Γ of weight k and character μ. Then the function*

$$h_f(z) \;=\; z \,+\, k\,\frac{f(z)}{f'(z)} \tag{4.1}$$

is an equivariant form for Γ.

Proof. Set $h(z) = h_f(z)$ where f be a generalized modular form for Γ Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. We have

$$\begin{aligned} h(\alpha\cdot z) \;&=\; \frac{az+b}{cz+d} + \frac{k(cz+d)^k\mu(\alpha)f(z)}{ck(cz+d)^{k+1}\mu(\alpha)f(z) + (cz+d)^{k+2}\mu(\alpha)f'(z)} \\ &=\; \frac{(az+b)(ckf(z)+(cz+d)f'(z)) + kf(z)}{(cz+d)(ckf(z)+(cz+d)f'(z))} \\ &=\; \frac{akf(z)+(az+b)f'(z)}{ckf(z)+(cz+d)f'(z)}\,. \end{aligned}$$

Meanwhile, we have

$$\alpha \cdot h(z) = \frac{ah(z)+b}{ch(z)+d}$$

$$= \frac{(az+b)f'(z)+akf(z)}{(cz+d)f'(z)+ckf}\,.$$

Therefore h commutes with the action of Γ. Furthermore, it is clear that $h(z)$ is meromorphic on $\mathbb{H}$. Let $\mathfrak{s}$ be a cusp of Γ and choose $\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma \cdot \mathfrak{s} = \infty$. We have

$$\widehat{h}\|[\gamma^{-1}](z) = j_{\gamma^{-1}}(z)^{-2}\widehat{h}(\gamma^{-1}\cdot z) + rj_{\gamma^{-1}}(z)^{-1}$$

$$= \frac{j_{\gamma^{-1}}(z)^{-2}f'(\gamma^{-1}\cdot z)}{kf(\gamma^{-1}\cdot z)} + rj_{\gamma^{-1}}(z)^{-1}$$

$$= \frac{j_{\gamma^{-1}}(z)^{-2}f'(\gamma^{-1}\cdot z) + rkj_{\gamma^{-1}}(z)^{-1}f(\gamma^{-1}\cdot z)}{kf(\gamma^{-1}\cdot z)}\,.$$

Recall that

$$f|_k[\gamma^{-1}](z) = j_{\gamma^{-1}}(z)^{-k}f(\gamma^{-1}\cdot z)\,,$$

so that

$$(f|_k[\gamma^{-1}](z))' = j_{\gamma^{-1}}(z)^{-k-2}f'(\gamma^{-1}\cdot z) + rkj_{\gamma^{-1}}(z)^{-k-1}f(\gamma^{-1}\cdot z)\,.$$

It follows that

$$\widehat{h}\|[\gamma^{-1}](z) = \frac{(f|_k[\gamma^{-1}](z))'}{k\,f|_k[\gamma^{-1}](z)}\,. \tag{4.2}$$

Since f is meromorphic at the cusp $\mathfrak{s}$ as a generalized modular form, it follows that $\widehat{h}\|[\gamma^{-1}](z)$ has a meromorphic $q_{\mathfrak{s}}$-expansion at ∞ where $q_{\mathfrak{s}} = e^{2\pi i z/l_s}$, $l_{\mathfrak{s}}$ being the cusp width at $\mathfrak{s}$. In other words, the meromorphy of h_f at the cusps as an equivariant is a consequence of the meromorphy of f as a generalized modular form. Therefore, h is an equivariant form. □

Remark 4.2. For the case of a modular form, the fact that $h_f(z)$ commutes with the action of Γ has already appeared in [Sm].

The above theorem makes explicit how to construct equivariant forms from generalized modular forms. We also have

Proposition 4.3. *Let f be a generalized modular form of weight k and character μ for a modular subgroup Γ. If c is a non-zero constant and n is a positive integer, then f, cf and f^n yield the same equivariant form, that is*

$$h_f \,=\, h_{cf} \,=\, h_{f^n}\,.$$

Proof. This is straightforward noting that the weight of f^n is nk. ☐

If f is a weight k generalized modular form, and $h = h_f$ is the corresponding equivariant form, then $z_0 \in \mathbb{H}$ is a pole of $\widehat{h}$ if and only if it a zero of f. In fact if n is the multiplicity of f at z_0, then z_0 is a simple pole of $\widehat{h}$ with residue n/k; a rational number. We will see in the next section that if $\widehat{h}$ has only simple poles with rational residues, in addition to some rationality conditions at the cusps, then the equivariant form h arises from a generalized modular form, that is, $h = h_f$ as in Theorem 4.1. Thus an equivariant form $h = h_f$ for some generalized modular form f is referred to as a *rational equivariant form.* Notice that the trivial equivariant form $h_0(z) = z$ is a also rational since it corresponds to the modular functions (of weight 0).

In view of Proposition 3.2, if h_f is a rational equivariant form for Γ associated with a generalized modular form f on Γ, then for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, $h_\gamma := \gamma^{-1} \circ h_f \circ \gamma$ is an equivariant form for $\gamma^{-1}\Gamma\gamma$. As we have

$$\widehat{h}_\gamma(z) = \widehat{h}\|[\gamma](z)$$

and hence

$$h_\gamma(z) = z + \frac{kf|_k[\gamma](z)}{(f|_k[\gamma](z))'}\,,$$

we conclude

Proposition 4.4. *Let h_f be a rational equivariant form corresponding to a weight k modular form f on Γ. Then for all $\gamma \in SL_2(\mathbb{Z})$, the equivariant form $\gamma^{-1}\circ h_f \circ \gamma$ is a rational equivariant form corresponding to the weight k modular form $f|_k[\gamma]$ on $\gamma^{-1}\Gamma\gamma$.*

It turns out that the rational equivariant forms account only for a small class of equivariant forms. Indeed, Theorem 4.1 can be generalized as follows.

Theorem 4.5. *Let Γ be a modular subgroup and let f and g be generalized modular forms of weights k and $k+2$ respectively and having the same character, then*

$$h(z) \,=\, z \,+\, k\,\frac{f(z)}{f'(z)+g(z)} \tag{4.3}$$

is an equivariant form for Γ.

Proof. The proof is similar to that of Theorem 4.1 □

The formula (4.3) can be used to produce non-rational equivariant forms. Indeed, Let $f = E_4$ and $g = E_6$, then the equivariant form

$$h(z) \;=\; z + 4\,\frac{E_4(z)}{E_4'(z) + E_6(z)}$$

is not rational. To see this, one ca easily see that the cubic root of unity ρ is a simple pole of $\widehat{h}$ with a residue equal to $1 + 2\pi i/3$.

A fundamental example of an equivariant form that will play a crucial role in the rest of this paper is obtained when we take $f = \Delta$, the weight 12 cusp form (the modular discriminant). We thus obtain the equivariant form for $\mathrm{SL}_2(\mathbb{Z})$ or any modular subgroup Γ:

$$h_1(z) \;=\; z \;+\; 12\,\frac{\Delta(z)}{\Delta'(z)} \;=\; z \;+\; \frac{6}{\pi i E_2(z)}\,. \tag{4.4}$$

It is worth mentioning that $h_1(z)$ can also be obtained via Theorem 4.5. Indeed, if f is a modular form of weight k, then one can show that

$$\delta_k f(z) = \frac{6}{i\pi} f'(z) - kE_2(z)f(z) \tag{4.5}$$

is modular form of weight $k+2$. Taking $g = -\frac{i\pi}{6}\delta_k f$ in (4.3) yields $h(z) = h_1(z)$.

We end this section with the behavior of equivariant forms at the elliptic fixed points.

Proposition 4.6. *Let h be an equivariant form for a modular subgroup Γ. If z_0 is an elliptic fixed point, then $h(z_0) = z_0$ or $h(z_0) = \overline{z_0}$.*

Proof. If γ fixes $z_0 \in \mathbb{H}$, then it also fixes $h(z_0)$. □

5. The criteria for rationality

We saw in the previous section that if $h = h_f$ is a rational equivariant form for a modular subgroup Γ, then $\widehat{h}$ has simple zeros in $\mathbb{H}$ with rational residues. Moreover, for a cusp $\mathfrak{s}$ of Γ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma \cdot \mathfrak{s} = \infty$, one can see from (4.2) that

$$\frac{1}{2i\pi} \lim_{z \to i\infty} \widehat{h}\|[\gamma^{-1}](z) \;=\; \frac{n}{kl_{\mathfrak{s}}} \;\in\; \mathbb{Q}\,,$$

where k is the weight of the generalized modular form f, n is the order of infinity in the $q_{\mathfrak{s}}$-expansion of $f|_k[\gamma^{-1}](z)$ and $l_{\mathfrak{s}}$ is the cusp width at $\mathfrak{s}$ of Γ.

The goal of this section is to show that these conditions are also sufficient for an equivariant form to be rational.

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Lemma 5.1. *Let Γ be a modular subgroup, and let h be an equivariant form for Γ. Then $\widehat{h}$ has only a finite number of poles in the closure of a fundamental domain of Γ.*

Proof. If $\mathcal{D}$ is the closure of a fundamental domain, then it has only a finite number of cusps. By definition, $\widehat{h}$ is meromorphic at these cusps and thus, if a cusp is pole of $\widehat{h}$ then it is an isolated pole. Therefore, every cusp that is a pole has a neighborhood in $\mathcal{D}$ made of the intersection of $\mathcal{D}$ with the interior of a horocycle, that is a circle in $\mathbb{H}$ tangent to the real line at the rational cusp or a horizontal line if the cusp is at infinity, and which does contain a pole other than the cusp itself. Excluding these neighborhoods yields a compact polygon inside $\mathbb{H}$ that must contains only a finite number of poles of $\widehat{h}$. The lemma follows. □

Lemma 5.2. *Suppose that h is equivariant for a modular subgroup Γ such that $\widehat{h}$ has only simple poles in $\mathbb{H}$, then the set of the residues at these poles is finite.*

Proof. Let z_0 be a simple pole of $\widehat{h}$, then in particular $h(z_0) = z_0$. Therefore

$$\begin{aligned} \mathrm{Res}_{z_0}\widehat{h}(z) &= \lim_{z\mapsto z_0} \frac{z - z_0}{h(z) - z} \\ &= \lim_{z\mapsto z_0} \frac{1}{\dfrac{h(z) - h(z_0)}{z - z_0} - 1} \\ &= \frac{1}{h'(z_0) - 1}\ . \end{aligned}$$

On the other hand, even though h does not take the same value at the Γ-orbit of z_0, the derivative h' does. Indeed, differentiating $h(\gamma\cdot z) = \gamma\cdot h(z)$ for $\gamma \in \Gamma$ yields

$$\frac{h'(\gamma\cdot z)}{j_\gamma(z)^2} = \frac{h'(z)}{j_\gamma(h(z))^2}\ .$$

Hence, if $z = z_0$ then $h'(\gamma\cdot z_0) = h'(z_0)$. It follows that the set of values of $h'(z_0)$ where z_0 is a simple pole of $\widehat{h}$ is the same as the set of values restricted to a single fundamental domain. Now, according to Lemma 5.1, this set is finite. □

We now state the criteria for an equivariant form to be rational.

Theorem 5.3. *Let Γ be a modular subgroup and let h be an equivariant form for Γ. Then h is rational if and only if*

1. *The poles of* $\widehat{h}$ *in* $\mathbb{H}$ *are all simple with rational residues.*
2. *For each cusp* $\mathfrak{s}$ *of* Γ *and* $\gamma \in SL_2(\mathbb{Z})$ *with* $\gamma \cdot \mathfrak{s} = \infty$*, we have*

$$\frac{1}{2i\pi} \lim_{z \to i\infty} \widehat{h}\|[\gamma^{-1}](z) \in \mathbb{Q}\,. \tag{5.1}$$

Proof. From the discussion at the beginning of this section, it is clear that conditions 1. and 2. are necessary. We will show that they are also sufficient. Let h be an equivariant form for a modular subgroup Γ as in the theorem. Conditions 1. and 2. provide us with a finite number of rational numbers, namely the values of the residues of $\widehat{h}$ which are in finite number according to Lemma 5.2, and the values of the limits (6.1) for the finite number of cusps that are inequivalent relative to Γ. Notice that these limits are the same at equivalent cusps according to the formula (4.2). Let k be a positive integer that is a multiple of the denominators of all these rational numbers. Fix $z_0 \in \mathbb{H}$ that is not a pole of $\widehat{h}(z)$ and define a function f by

$$f(z) = \exp\left(\int_{z_0}^{z} k\widehat{h}(u)du\right).$$

This function is well defined as the integral is independent of the path of integration. Indeed, let Σ_1, Σ_2 be two paths joining z_0 and z that do not contain any poles of $\widehat{h}(z)$. Then

$$\int_{\Sigma_1 \cup -\Sigma_2} k\widehat{h}(u)du \;=\; 2\pi k i \sum \mathrm{Res}(\mathrm{\widehat{h}(z), z}) \;\in\; 2\pi \mathrm{i}\mathbb{Z}\,,$$

where the sum is over the poles of $\widehat{h}(z)$ interior to $\Sigma_1 \cup -\Sigma_2$ which we orient positively. Thus,

$$\int_{\Sigma_1} k\widehat{h}(u)du = \int_{\Sigma_2} k\widehat{h}(u)du + 2\pi m i$$

for some $m \in \mathbb{Z}$, and so f is well-defined. We extend f to a meromorphic function on the set S of poles of $\widehat{h}(z)$ in the following way. Let r (an integer) be the residue of $k\widehat{h}(z)$ at a pole z_1. If $r > 0$ we define $f(z_1) = 0$ to make f holomorphic at z_1 and the order of f at z_1 is r. If $r < 0$ then z_1 is a pole of f of order $-r$. Thus f is a well-defined meromorphic function on $\mathbb{H}$ satisfying

$$h(z) \;=\; z \;+\; \frac{kf(z)}{f'(z)}\,.$$

We now proceed to study the modular properties of f. For $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we have

$$f(\alpha \cdot z) \;=\; g_\alpha(z) f(z)\,,$$

where

$$g_\alpha(z) \;=\; \exp\left(\int_z^{\alpha\cdot z} k\widehat{h}(u)du\right) .$$

Taking the logarithmic derivative of g_α yields

$$\frac{g'_\alpha(z)}{g_\alpha(z)} \;=\; \frac{d}{dz}(\alpha\cdot z)k\widehat{h}(\alpha\cdot z) - k\widehat{h}(z) .$$

Since h is equivariant and $\dfrac{d}{dz}\alpha\cdot z = (j_\alpha(z))^{-2}$, one shows that

$$\frac{g'_\alpha(z)}{g_\alpha(z)} \;=\; \frac{kc}{j_\alpha(z)} .$$

Therefore,

$$g_\alpha(z) = \mu(\alpha){j_\alpha}^k \quad \text{for some } \mu(\alpha)\in\mathbb{C}^\times .$$

In fact, this defines a group character $\mu:\ \Gamma\longrightarrow\ \mathbb{C}^\times$. Indeed, for $\alpha,\ \beta\in\Gamma$ we have

$$\begin{aligned} g_{\alpha\beta}(z) &= \mu(\alpha\beta)(j_{\alpha\beta}(z))^k \\ &= \exp\left(\int_z^{\alpha\beta\cdot z} k\widehat{h}(u)du\right) \\ &= \exp\left(\int_{\beta\cdot z}^{\alpha\cdot(\beta\cdot z)} k\widehat{h}(u)du\right)\exp\left(\int_z^{\beta\cdot z} k\widehat{h}(u)du\right) \\ &= g_\alpha(\beta\cdot z)g_\beta(z)\ , \end{aligned}$$

which implies that

$$\mu(\alpha\beta)j_{\alpha\beta}(z)^k = \mu(\alpha)j_\alpha(\beta\cdot z)^k\mu(\beta)j_\beta(z)^k .$$

In the meantime, $j_{\alpha\beta}(z) = j_\alpha(\beta\cdot z)j_\beta(z)$ as j is an automorphic factor. Therefore,

$$\mu(\alpha\beta) = \mu(\alpha)\mu(\beta) .$$

As for the meromorphy of f at the cusps, let $\mathfrak{s}\in\mathbb{Q}\cup\{\infty\}$ and $\gamma\in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma\cdot\mathfrak{s}=\infty$. An easy computation shows that $f|_k[\gamma^{-1}](z)$ is $l_\mathfrak{s}$ invariant, that is

$$f|_k[\gamma^{-1}](z) = \sum_{n=n_\mathfrak{s}}^{\infty} b_n q_\mathfrak{s}^n,\ q_\mathfrak{s} = e^{2\pi i z/l_\mathfrak{s}}\ ,$$

$l_{\mathfrak{s}}$ being the cusp width of Γ at $\mathfrak{s}$. To prove that $n_{\mathfrak{s}}$ is finite it suffices to show that $f|_k[\gamma^{-1}](z)$ is meromorphic at infinity. Indeed, we have

$$\begin{aligned}
f(z)|_k[\gamma^{-1}] &= j_{\gamma^{-1}}(z)^{-k} f(\gamma^{-1}\cdot z)\\
&= j_{\gamma^{-1}}(z)^{-k}\exp\left(\int_{z_0}^{\gamma^{-1}\cdot z} k\widehat{h}(u)du\right)\\
&= j_{\gamma^{-1}}(z)^{-k}\exp\left(\int_{\gamma\cdot z_0}^{z} j_{\gamma^{-1}}(w)^{-2}\widehat{h}(\gamma^{-1}\cdot w)kdw\right)\\
&= j_{\gamma^{-1}}(z)^{-k}\exp\left(\int_{\gamma\cdot z_0}^{z} \left(\widehat{h}||_2[\gamma^{-1}](w) - rj_{\gamma^{-1}}(z)^{-1}\right)kdw\right)\\
&= j_{\gamma^{-1}}(\gamma\cdot z_0)^{-k}\exp\left(\int_{\gamma\cdot z_0}^{z}\left(a_0+\sum_{n\geq 1} a_n\exp(2\pi iwn/l_s)\right)kdw\right)\\
&= j_{\gamma}(z_0)^{k}\exp(ka_0(z-\alpha\cdot z_0))\exp\left(\sum_{n\geq 1}\int_{\gamma\cdot z_0}^{z} a_n(\exp(2\pi inw/l_s)kdw\right)\\
&= \exp(ka_0 z)\cdot\ \text{holomorphic factor at infinity}\,,
\end{aligned}$$

where we have used the fact the series converges normally for $\Im z > y_0 > 0$ and the fact that since z_0 is not a fixed point of h then neither are the images $\gamma\cdot z_0$ for all $\gamma\in\Gamma$. Furthermore, since $ka_s = 2\pi ikm_s \in 2\pi i\mathbb{Z}$ using (6.1), we have

$$f(z)|_k[\gamma^{-1}] = q_{\mathfrak{s}}^{m_{\mathfrak{s}}}\cdot\ \text{holomorphic factor at infinity}\,.$$

Thus $f(z)|_k[\gamma^{-1}]$ is meromorphic at infinity. Therefore f is a generalized modular form for Γ of weight k and character μ. □

6. The genus zero condition

Theorem 5.3 gives the necessary and sufficient conditions for an equivariant form h to be rational, meaning that $h = h_f$ where f is a generalized modular form. The following theorem provides us with a sufficient condition under which f is actually a modular form (with trivial character).

Theorem 6.1. *Let Γ be a genus zero modular subgroup and let h be an equivariant form for Γ. Suppose that*

1. *the poles of $\widehat{h}$ in $\mathbb{H}$ are all simple with rational residues,*
2. *for each cusp $\mathfrak{s}$ of Γ and $\gamma\in SL_2(\mathbb{Z})$ with $\gamma\cdot\mathfrak{s}=\infty$, we have*

$$\frac{1}{2i\pi}\lim_{z\to i\infty}\widehat{h}||[\gamma^{-1}](z)\in\mathbb{Q}\,, \tag{6.1}$$

then $h = h_f$ is rational with f being a modular form for Γ.

Proof. According to Theorem 5.3, if conditions 1. and 2. hold, then $h = h_f$ with f being a generalized modular form of weight k and character μ. We may suppose that k is even, otherwise we replace f by f^2 which does not change h. We will analyze the effect of the character μ on various elements of Γ: If α is a parabolic element, and let $\mathfrak{s}$ be a cusp with $\alpha \cdot \mathfrak{s} = \mathfrak{s}$. Choose $\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ so that $\gamma \cdot \mathfrak{s} = \infty$. Since $\gamma\alpha\gamma^{-1}\infty = \infty$ we have $\gamma\alpha\gamma^{-1} = T^{m_{\mathfrak{s}}}$ for an integer $m_{\mathfrak{s}}$ divisible by the cusp width $l_{\mathfrak{s}}$ at $\mathfrak{s}$. Making the substitution $u = \gamma^{-1} \cdot w = \begin{pmatrix} s & -q \\ -r & p \end{pmatrix} \cdot w$ in the expression of $g_\alpha(z)$ for $z \in \mathbb{H}$ as in the proof of Theorem 5.3 yields

$$\begin{aligned} g_\alpha(z) &= \exp\left(\int_{\gamma\cdot z}^{\gamma\alpha\cdot z} j_{\gamma^{-1}}(w)^{-2}\widehat{h}(\gamma^{-1}\cdot w)kdw\right) \\ &= \exp\left(\int_{\gamma\cdot z}^{\gamma\alpha\cdot z} \left(-rj_{\gamma^{-1}}(w)^{-1} + \widehat{h}||_2[\gamma^{-1}](w)\right)kdw\right) \\ &= j_{\gamma^{-1}}(\gamma\alpha\cdot z)^k j_{\gamma^{-1}}(\gamma\cdot z)^{-k}\exp\left(\int_{\gamma\cdot z}^{\gamma\alpha\cdot z} \widehat{h}||_2[\gamma^{-1}](w)kdw\right) \\ &= j_\gamma(\alpha\cdot z)^{-k} j_\gamma(z)^k \exp\left(\int_{\gamma\cdot z}^{T^{m_{\mathfrak{s}}}\gamma\cdot z}\left(a_0 + \sum_{n\geq 1} a_n \exp(2\pi i n w/l_{\mathfrak{s}})\right)kdw\right) \\ &= j_\gamma(\alpha\cdot z)^{-k} j_\gamma(z)^k \exp\left(\int_{\gamma\cdot z}^{\gamma\cdot z + m_{\mathfrak{s}}}\left(a_0 + \sum_{n\geq 1} a_n \exp(2\pi i n w/l_{\mathfrak{s}})\right)kdw\right), \end{aligned}$$

since $j_{\gamma^{-1}}(\gamma\cdot z)j_\gamma(z) = 1$ and by assumption (6.1)

$$\widehat{h}||_2[\gamma^{-1}](z) = a_0 + \sum_{n\geq 1} a_n q_{\mathfrak{s}}^n,\ q_{\mathfrak{s}} = e^{2\pi i z/l_{\mathfrak{s}}}.$$

Hence,

$$g_\alpha(z) = j_\gamma(\alpha\cdot z)^{-k} j_\gamma(z)^k \exp(k a_0 m_{\mathfrak{s}}) = j_\gamma(\alpha\cdot z)^{-k} j_\gamma(z)^k ,$$

as $ka_0 \in 2\pi i\mathbb{Z}$ by assumption and that the function $z \mapsto \exp(2\pi i z/l_{\mathfrak{s}})$ is $l_{\mathfrak{s}}$-periodic with $l_{\mathfrak{s}}$ dividing $m_{\mathfrak{s}}$. Recall that

$$g_\alpha(z) = \mu(\alpha)\, j_\alpha(z)^k.$$

Hence

$$\mu(\alpha) = (j_\alpha(z) j_\gamma(\alpha\cdot z))^{-k} j_\gamma(z)^k .$$

Taking the limit as z tends to $\mathfrak{s}$ and since $\alpha\cdot\mathfrak{s} = \mathfrak{s}$, we get

$$\mu(\alpha) = (j_\alpha(\mathfrak{s}) j_\gamma(\alpha\cdot\mathfrak{s}))^{-k} j_\gamma(\mathfrak{s})^k = j_\alpha(\mathfrak{s}).$$

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If $\mathfrak{s} = \infty$ then $j_\alpha(\mathfrak{s}) = 1$, otherwise the fixed point of $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\mathfrak{s} = (a-d)/2c$ and therefore

$$j_\alpha(\mathfrak{s}) = \frac{a+d}{2} = \pm 1$$

as α is parabolic with a trace equal to ± 2. Since k has been chosen to be even, we have $\mu(\alpha) = 1$. Therefore, the character μ is trivial on all the parabolic elements of Γ.

Suppose now that $\alpha \in \Gamma$ is an elliptic element with order m_α, say. Since $\alpha^{m_\alpha} = 1$ we have $\mu(\alpha)^{m_\alpha} = 1$. If m_Γ is the lcm of the orders of the (finite number of) classes of elliptic elements, then for each elliptic element α, we have $\mu(\alpha)^{m_\Gamma} = 1$.

If Γ has genus 0, then it can be generated by parabolic and elliptic elements only and thus the character μ satisfies $\mu(\alpha)^{m_\Gamma} = 1$ for all $\alpha \in \Gamma$. Therefore, the function $F = f^{m_\Gamma}$ is a modular form of weight km_Γ for Γ. Meanwhile, using Proposition 4.3, we have

$$h(z) = z + \frac{kf(z)}{f'(z)} = z + \frac{km_\Gamma F(z)}{F'(z)}.$$

The meromorphy at cusps follows similarly as in Theorem 5.3. Therefore $h = h_F$ is a rational equivariant form with F being a modular form.

□

We have encountered the equivariant form h_1 given by

$$h_1(z) = z + 12\frac{\Delta}{\Delta'} = z + \frac{6}{\pi i E_2(z)},$$

for which $h_1(z) - z \neq 0$ for all $z \in \mathbb{H}$. Moreover, the limit of $h_1(z) - z$ as z tends to ∞ is $6/i\pi \neq 0$. In general, we say that an equivariant form h for a modular subgroup Γ does not have a fixed point if:

- $h(z) - z$ does not vanish in $\mathbb{H}$.
- h has no fixed points at the cusps of Γ in the sense that for every cusp $\mathfrak{s}$ and $\gamma \in \Gamma$ with $\gamma \cdot \infty = \mathfrak{s}$, we have $\lim_{z \mapsto i\infty} \widehat{h}||_2[\gamma^{-1}](z)$ is finite.

Proposition 6.2. *Let f be a holomorphic modular form for Γ and $h = h_f$ the corresponding rational equivariant form, then h does not have fixed points at the cusps if and only if f is a cusp form for Γ.*

Proof. This is an immediate consequence of (4.2) □

In the case the modular subgroup Γ is of genus 0, the following theorem indicates that the fundamental example h_1 is unique among the equivariant forms without fixed points with an integrality condition at the cusps.

Theorem 6.3. *Let h be an equivariant form without fixed points for a genus zero modular subgroup Γ. Suppose that for every cusp $\mathfrak{s}$ and $\gamma \in SL_2(\mathbb{Z})$ with $\gamma \cdot \mathfrak{s} = \infty$, we have*

$$\frac{6}{\pi i} \lim_{z \mapsto \infty} \widehat{h}||_2[\gamma^{-1}](z) \in \mathbb{Z}^+ ,$$

then

$$h(z) = h_1(z) = z + \frac{6}{\pi i E_2(z)} .$$

Proof. Fix z_0 in $\mathbb{H}$. It is clear from the conditions of the theorem that

$$f(z) = \exp\left(\int_{z_0}^{z} \frac{12dw}{h(w) - w}\right)$$

defines a non-vanishing holomorphic function of $\mathbb{H}$. We will prove that f is a weight 12 cusp form for Γ.

Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. We have $f(\alpha \cdot z) = g_\alpha(z) f(z)$, where

$$g_\alpha(z) = \exp\left(\int_{z}^{\alpha \cdot z} 12\widehat{h}(u)du\right) .$$

Taking the logarithmic derivative of g_α yields

$$\frac{g'_\alpha(z)}{g_\alpha(z)} = \frac{12c}{cz + d}$$

since h is equivariant. Hence, $g_\alpha(z) = \mu(\alpha) j_\alpha(z)^{12}$, for a character character μ of Γ. Similarly to the proof of Theorem 5.3, f is a non-vanishing weight 12 generalized modular form with a character μ that is trivial on the parabolic elements and has a finite order at the elliptic elements.

As for the holomorphy at the cusps, let $\mathfrak{s}$ be a cusp and let, as above, $\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma \cdot \mathfrak{s} = \infty$. We have

$$\begin{aligned} f(z)|_{12}[\gamma^{-1}] &= j_{\gamma^{-1}}(z)^{-12} f(\gamma^{-1} \cdot z) \\ &= j_{\gamma^{-1}}(z)^{-12} \exp\left(\int_{\gamma \cdot z_0}^{z} j_{\gamma^{-1}}(w)^{-2} \widehat{h}(\gamma^{-1} \cdot w) 12 dw \right) \\ &= j_{\gamma^{-1}}(z)^{-12} \exp\left(\int_{\gamma \cdot z_0}^{z} \left(\widehat{h}||_2[\gamma^{-1}](w) - r j_{\gamma^{-1}}(z)^{-1} \right) 12 dw \right) \\ &= j_{\gamma^{-1}}(\gamma \cdot z_0)^{-12} \exp\left(\int_{\gamma \cdot z_0}^{z} (a_0 + \sum_{n \geq 1} a_n \exp(2\pi i w n / l_{\mathfrak{s}})) 12 dw \right) \\ &= \exp(12 a_0 z) \text{ holomorphic factor at infinity }, \end{aligned}$$

Since $12 a_{\mathfrak{s}} = 2\pi i m,\ m \geq 1$, it is clear that $f(z)|_{12}[\gamma^{-1}]$ vanishes at ∞ with an order of vanishing that is a multiple of the cusp width.

Let m_Γ be the least common multiple of the orders of the elliptic generators of Γ, then it is clear that f^{m_Γ} is a cusp form of weight $12 m_\Gamma$ with an order of vanishing at every cusp being at least m_Γ. Therefore, keeping in mind that Δ vanishes at a cusp with order equal to the cusp width in Γ, $(f/\Delta)^{m_\Gamma}$ is a holomorphic modular function of weight 0. Thus, it is constant since Γ is of genus zero as $(f/\Delta)^{m_\Gamma}$ would define a holomorphic function on the genus 0 Riemann surface $X(\Gamma)$. It follows that $f^{m_\Gamma} = c \Delta^{m_\Gamma}$ for some constant c. This concludes the proof since

$$h = h_f = h_{f^{m_\Gamma}} = h_{c \Delta^{m_\Gamma}} = h_\Delta = h_1 \,.$$

□

7. Effect of the Schwarz derivative

In [AAS1], the main motivation behind the introduction of the equivariant forms was the so-called Schwarz derivative. It is defined for a meromorphic function on a domain of $\mathbb{C}$ by

$$\{f, z\} = 2 \left(\frac{f''}{f'} \right)' - \left(\frac{f''}{f'} \right)^2 = 2 \frac{f'''}{f'} - 3 \frac{f''^2}{f'^2} \,.$$

One can check that it satisfies the following rules:

- Chain rule: If w is a function of z then

$$\{f,z\} \;=\; (dw/dz)^2\{f,w\} + \{w,z\}\,.$$

- If f is a linear fractional transform of z, then $\{f,z\}=0$.
- Inversion formula: If $w'(z_0)\neq 0$ for some point z_0, then in a neighborhood of z_0,

$$\{z,w\} \;=\; -(dz/dw)^2\{w,z\}\,.$$

There is a close relationship between the Schwarz derivative and second order linear differential equations that can be illustrated as follows: Let $Q(z)$ be a meromorphic function on a domain of $\mathbb{C}$, and consider the differential equation

$$y''(z) \,+\, \frac{1}{4}Q(z)y(z) = 0\,. \tag{7.1}$$

If y_1 and y_2 are two linearly independent solutions, then the quotient $f=y_1/y_2$ is a solution of the Schwarz differential equation

$$\{f,z\} \;=\; Q(z)\,. \tag{7.2}$$

Conversely, if w is a locally univalent function such that $\{w,z\}=Q(z)$, then $y_1 = \dfrac{w}{\sqrt{w'}}$, $y_2 = \dfrac{1}{\sqrt{w'}}$ are two linearly independent solutions to (7.2). As a consequence, we have

- $\{f,z\}=0$ if and only if f is a linear fractional transform of z.
- $\{w_1,z\}=\{w_2,z\}$ if and only if each function is a linear fraction of the other.

The Schwarz derivative has been an extremely useful tool in complex analysis, differential equations, projective geometry, dynamical systems among other fields. In the work [McSe1], it was shown that the Schwarz derivative can have an important role in the theory of modular forms as well. For instance, we have the following

Proposition 7.1. [McSe1] *We have*

i. *If f is a modular function for a discrete group G, then $\{f,z\}$ is a weight 4 modular form for G.*
ii. *If G is of genus 0 and f is a Hauptmodul for G, then $\{f,z\}$ is weight 4 modular form for the normalizer of G inside $SL_2(\mathbb{R})$.*

The first statement is an immediate consequence of the chain rule and the following fact:

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Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$. Then

$$\{w, z\} \;=\; \frac{\det \alpha}{(cz+d)^4}\,\{w, \alpha \cdot z\}\,. \tag{7.3}$$

The second statement is, however, less trivial.

One of the questions that [AAS1] tried to answer is about the converse to the second statement that one can formulate as follows:
Given a meromorphic function f on $\mathbb{H}$ such that its Schwarz derivative $\{f, z\}$ is a weight 4 modular form for a discrete group Γ (or simply for a modular subgroup). Is f a modular function (of weight 0) for a subgroup of G? While the answer depends on many factors involving the geometry of G or the analytic properties of f, it turns out that there are examples of functions f which are not invariant under any nontrivial element of G but their Schwarz derivatives are weight 4 modular forms. Among these are the equivariant forms. Indeed, if h is an equivariant form for a modular subgroup Γ, then h is by definition not invariant under any non-identity element of Γ, however, we have:

Theorem 7.2. *If h is an equivariant form for a modular subgroup Γ, then $\{h, z\}$ is a weight 4 modular form for Γ.*

Proof. Set $H(z) = \{h, z\}$ and let $\gamma \in \Gamma$. Then we have

$$\begin{aligned} H(\gamma \cdot z) &= \{h(\gamma \cdot z), \gamma \cdot z\} \\ &= \{\gamma \cdot h(z), \gamma \cdot z\} \quad \text{since } h \text{ is equivariant} \\ &= \{h(z), \gamma \cdot z\} \quad \text{by the properties of the Schwarz derivative} \\ &= j_\gamma(z)^4\, H(z) \quad \text{using (7.3)}\,. \end{aligned}$$

The meromorphy of $\{h, z\}$ follows from that of $h(z)$ as an equivariant form. ☐

Remark 7.3. One should keep in mind the following fact: The meromorphy of $h(z) - z$ at the cusps suggest that $h(z) = z+$ a meromorphic part at the cusp with z representing a logarithmic singularity at the cusp. However, in the expression of $\{h, z\}$, only h', h'' and h''' intervenes so that it becomes meromorphic too without the logarithmic singularity.

In addition to the rational equivariant forms that arise from modular forms, this theorem provides a second connection between equivariant forms and modular forms. It would be interesting to connect the two points of view. Indeed, suppose we are given a modular form f of weight k for a modular subgroup Γ and let h_f be the associated equivariant form

$$h_f(z) \;=\; z \;+\; k\,\frac{f(z)}{f'(z)}\,.$$

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This yields a modular form of weight 4 for Γ given by the Schwarz derivative $\{h_f, z\}$. One can see from the expression of $\{h_f, z\}$ that it has a double pole at the critical points of h_f as well as at the poles of h_f that are at least of order 2, and it is holomorphic elsewhere, see also [McSe1]. In the meantime, we have

$$h'_f(z) = \frac{(k+1)f'^2 - kff''}{f'^2} = \frac{-[f,f]_2}{(k+1)f'^2},$$

where $[.,.]_2$ is the Rankin-Cohen bracket of order 2 introduced in Section 2. Therefore, we have

Proposition 7.4. *Let f be a modular form of weight k on Γ. Then $f'^2 h'_f$ is a weight $2k+4$ modular form on Γ. Moreover, the poles of $\{h_f, z\}$ are located at the zeros of the second Cohen-Rankin bracket $[f,f]_2$ of f.*

Proof. This follows from the expression of h'_f and the fact that a zero of f' does not yield a pole of h'_f. □

Define the following sequence of modular forms for Γ:

$$\begin{aligned} f_0(z) &= f(z) \\ f_{n+1}(z) &= \{h_{f_n}, z\}, \quad n \geq 0. \end{aligned}$$

For integers $n \geq 1$, the modular form f_n has weight 4. The behavior of this sequence is not easy to study given an initial modular form f_0, however, we have the following

Proposition 7.5. *Let $f_0 = E_4$, the weight 4 Eisenstein series, then for all $n \geq 1$ we have*

$$f_n(z) = 4\pi^2 E_4(z).$$

Proof. Since $E_4(z) = 1 + 240q + \mathrm{o}(q)$, $q = \exp(2i\pi z)$, it is clear that

$$[E_4, E_4]_2 = 5\,(4E_4E''_4 - 5E'^2_4) = \alpha\, q + \mathrm{o}(q),$$

α being a constant, is a weight 12 holomorphic modular form that vanishes at ∞. Therefore,

$$h'_{E_4}(z) = -\frac{\alpha\Delta(z)}{5E'^2_4(z)}.$$

Further, $h_{E_4}(z)$ has only simple poles in $\mathbb{H}$. Therefore, the Schwarz derivative $\{h_{E_4}, z\}$ is a weight 4 modular form that is holomorphic on $\mathbb{H}$. An analysis of its q-expansion at ∞ yields

$$\{h_{E_4}, z\} = 4\pi^2 + \mathrm{o}(q),$$

and in particular it is holomorphic at ∞. In the meantime, the space of weight 4 holomorphic modular forms is 1-dimensional generated by E_4. Thus, we have

$$f_1(z) \;=\; \{h_{E_4}, z\} \;=\; 4\pi^2\, E_4(z)\,.$$

We now proceed by induction. Suppose that $f_n = 4\pi^2 E_4$, then

$$h_{f_n} \;=\; h_{4\pi^2 E_4} = h_{E_4} \quad \text{(using Proposition 4.3)}\,.$$

Hence

$$f_{n+1} \;=\; \{h_{f_n}, z\} \;=\; \{h_{E_4}, z\} \;=\; f_1 \;=\; 4\pi^2 E_4(z)\,.$$

The proposition follows. □

8. Examples

In this section we provide examples of equivariant forms arising from classical modular forms and functions as well as the generalized ones.

We start with the Dedekind modular invariant j-function and the Klein modular elliptic λ-function. They are modular respectively for the modular group and the principal congruence subgroup $\Gamma(2)$, and can be defined by

$$j(z) \;=\; \frac{E_4^3(z)}{\Delta(z)}\,,$$

$$\lambda(z) \;=\; \frac{\vartheta_2^4(z)}{\vartheta_3^4(z)}\,.$$

It is known that these two modular functions are Hauptmoduln in the sense that they generate the fields of modular functions for their respective invariance groups. Their logarithmic derivatives are meromorphic modular forms of weight 2 and are given by

$$j_1 := \frac{j'}{j} \;=\; -\frac{E_6}{E_4}$$

$$\lambda_1 := \frac{\lambda'}{\lambda} \;=\; \pi i \vartheta_4^4\,.$$

The rational equivariant form corresponding to j_1 and λ_1 are therefore

$$h_{j_1}(z) = \quad z + \frac{2j(z)j'(z)}{j(z)j''(z) - j'^2(z)}\,, h_{\lambda_1}(z) = \quad z + \frac{2\lambda(z)\lambda'(z)}{\lambda(z)\lambda''(z) - \lambda'^2(z)}\,.$$

On the other hand using Ramanujan relations for E_4 and E_6 one can easily show that

$$h_{j_1}(z) = z + \frac{E_4(z)E_6(z)}{\frac{\pi i}{6}E_2(z)E_4(z)E_6(z) - \frac{2\pi i}{3}\left(\Delta(z) + E_4^3(z)\right)}\,,$$

from which it follows that h_{j_1} has i and $\rho = e^{2\pi i/3}$ (the zeros of E_6 and E_4, respectively) as fixed points in $\mathbb{H}$. As for h_{λ_1}, Proposition 4.3 implies that

$$h_{\lambda_1}(z) = h_{\vartheta_4}(z) = z + \frac{\vartheta_4(z)}{2\vartheta_4'(z)} \ .$$

This rational equivariant form has however no fixed points in $\mathbb{H}$ because the Jacobi theta function are holomorphic and non-vanishing in $\mathbb{H}$ since it has an expression in terms of *eta*-products given by [Ra]:

$$\vartheta_4(z) = \frac{\eta(z)^2}{\eta(2z)} \ ,$$

where the Dedekind eta-function is given by

$$\eta(z) \ = \ \Delta(z)^{\frac{1}{24}} \ = \ q^{\frac{1}{24}} \prod_{n\geq 1} (1 - q^n) \, .$$

Moreover, h_{λ_1} has no fixed points at the cusps. Indeed, using the fact that $2\pi i E_2$ is the logarithmic derivative of the modular discriminant $\Delta(z)$, we have

$$h_{\lambda_1}(z) = z + \frac{1}{4\pi i(E_2(z) - E_2(2z))} \ .$$

Hence

$$\widehat{h}_{\lambda_1} = 4\pi i(E_2(z) - E_2(2z)) \ ,$$

from which it follows the holomorphy at cusps of h_{λ_1}, and so making it a rational equivariant form without fixed points. A similar situation holds for the Jacobi theta functions ϑ_2 and ϑ_3 whose *eta*-products are given by [Ra]:

$$\vartheta_2(z) = 2\frac{\eta(4z)^2}{\eta(2z)},$$

$$\vartheta_3(z) = \frac{\eta(2z)^5}{\eta(z)^2\eta(4z)^2} \ .$$

Their corresponding equivariant forms are also rational equivariant forms without fixed points. This similarly follows from the expression of these functions in terms of the Eisenstein series E_2 given by

$$\begin{aligned} h_{\vartheta_2}(z) &= z + \frac{1}{8\pi i(4E_2(4z) - E_2(2z))} \ , \\ h_{\vartheta_3}(z) &= z + \frac{24}{2\pi i(5E_2(2z) - E_2(z) - 4E_2(4z))} \ . \end{aligned}$$

Now, using the expressions in terms of E_2 of these equivariant forms and the Ramanujan's differential system, we see that $\{h_{\vartheta_l}, z\}$, for $l = 2,\ 3,\ 4$, is holomorphic on $\mathbb{H}^*$. Further, as the space of weight 4 modular forms on $\Gamma_0(2)$, $(ST)^{-1}\Gamma_0(2)(ST)$ and $\Gamma^0(2) = (ST)^{-2}\Gamma_0(2)(ST)^2$ is two dimensional with bases $\{\vartheta_2^8,\ \vartheta_3^4\vartheta_4^4\}$, $\{\vartheta_3^8,\ \vartheta_2^4\vartheta_4^4\}$ and $\{\vartheta_3^8,\ \vartheta_2^4\vartheta_3^4\}$ respectively, [Ra], and comparing the first few coefficient, we have

Proposition 8.1. *The Schwarz derivatives of* $h_{\vartheta_l},\ l = 2,\ 3,\ 4,$ *are respectively weight 4 holomorphic modular forms on* $\Gamma_0(2)$, $(ST)^{-1}\Gamma_0(2)(ST)$ *and* $\Gamma^0(2) = (ST)^{-2}\Gamma_0(2)(ST)^2$ *and satisfy*

$$\{h_{\vartheta_l}, z\} = \pi^2\vartheta_l(z)^8\ , \quad l = 2,\ 3,\ 4\,.$$

We note that since the Eisenstein series E_4 and E_6 vanish on $\mathbb{H}$, their corresponding equivariant forms

$$\begin{aligned} h_{E_4}(z) =& z + \frac{12E_4(z)}{2\pi i(E_2(z)E_4(z) - E_6(z))} \\ h_{E_6}(z) =& z + \frac{6E_6(z)}{\pi i(E_2(z)E_6(z) - E_4^2(z))} \end{aligned}$$

do have fixed points on $\mathbb{H}$. Proposition 7.5 shows that

$$\{h_{E_4}, z\} = 4\pi E_4(z)\,.$$

Also, one can show that

$$[E_6, E_6]_2 = 7(6E_6E_6'' - 7E_6'^2) = -49\pi^2E_4\Delta\,,$$

and that

$$\{h_{E_6}, z\} = 4\pi^2\frac{E_{12} + 288\Delta}{E_4^2}\ ,$$

where E_{12} is the weight 12 Eisenstein series.

We have already given an example of an equivariant form that is not rational. Namely, the equivariant form

$$h(z)\ =\ z + 4\,\frac{E_4(z)}{E_4'(z) + E_6(z)}\ .$$

We end this section by giving another example of a non-rational equivariant form having fixed points at cusps only. Consider the equivariant form

$$h(z)\ =\ z\ +\ \frac{12\Delta}{\Delta' + E_4^2E_6}\ =\ z + 12q + \mathrm{o}(q)\ \text{as}\ \ z \to i\infty\ .$$

Then h does not have fixed points in $\mathbb{H}$ as Δ is non-vanishing. Moreover, h does not satisfy the second condition of neither Theorem 5.3 nor Theorem 6.3 as $\widehat{h}$ has a pole at $i\infty$.

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References

[Br] Brady, Michael *Meromorphic solutions of a system of functional equations involving the modular group.* Proc. Amer. Math. Soc. 30 1971 271–277.

[E-S] Elbasraoui, Abdelkrim; Sebbar, Abdellah *Structure and geometry of equivariant forms*, In preparation

[He] Heins, Maurice *On the pseudo-periods of the Weierstrass zeta functions.* II. Nagoya Math. J. 30 1967 113–119.

[K-M] Knopp, Marvin; Mason, Geoffrey *Generalized modular forms*, Journal of Number Theory 99 (2003) 1-28.

[McSe1] McKay, John; Sebbar, Abdellah *Fuchsian groups, automorphic functions and Schwarzians*. Math. Ann. 318 (2000), no. 2, 255–275.

[Ra] Rankin, Robert *Modular forms and functions.* Cambridge University Press, 1977.

[AAS1] Sebbar Abdellah; Sebbar Ahmed *Schwarz, Equivariant functions and integrals of elliptic functions.*

[AAS2] Sebbar Abdellah; Sebbar Ahmed *Eisenstein Series and Modular Differential Equations.* To appear In Can. Math. Bull.

[Sh] G. Shimura, Goro *Introduction to the Arithmethic Theory of Automorphic Functions*, Princeton University Press, Princeton, New Jersey, 1971.

[Sm] Smart, John Roderick *On meromorphic functions commuting with elements of a function group.* Proc. Amer. Math. Soc. 33 (1972), 343–348.

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