Centre de Recherches Mathématiques
CRM Proceedings and Lecture Notes

$J-$invariants of arithmetic semistable elliptic surfaces and graphs

John McKay and Abdellah Sebbar

1. Introduction

This paper has three aspects: Firstly, the complete list of all the torsion-free genus zero congruence subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ was given in [**9**]. The interest in these groups first appeared in studying the properties of the Schwarz derivative acting on Hauptmoduls of some genus zero Fuchsian groups connected with Moonshine, see [**6, 7**]. The relationship between the subgroups and $\mathrm{PSL}_2(\mathbb{Z})$ can be well understood by exhibiting the relation between the Hauptmoduls of the subgroups and the Hauptmodul for $\mathrm{PSL}_2(\mathbb{Z})$, namely the elliptic modular function j. This includes all the information about the ramification, the cusps, and the cusp widths as well as the index.

Secondly, from a geometric point of view, each of these subgroups yields a semistable elliptic surface over $\mathbb{P}^1$, that is the singular fibres consist solely of cycles of irreducible curves. The singular fibres being above the cusps, the number of irreducible components is nothing but the cusp width. Moreover, the above relations between the Hauptmoduls which express the $j-$function as a rational function of the Hauptmoduls provide also the $J-$invariant of the elliptic surfaces (or of the generic fibre which is an elliptic curve over the function field of the base curve). Thus, we are describing the $J-$invariants of all the modular elliptic surfaces over $\mathbb{P}^1$ arising from a congruence subgroup (we call these surfaces arithmetic elliptic surfaces) which are semistable.

We mention that, among the above elliptic surfaces, there are some with four singular fibres (equivalently the associated group has index 12 in $\mathrm{PSL}_2(\mathbb{Z})$). These surfaces describe all the semistable elliptic surfaces over $\mathbb{P}^1$ with four singular fibres (without the modularity property), since according to Beauville [**1**], there are only six such surfaces. See also [**10**] for more properties of these surfaces and the associated groups.

Finally, to understand the situation at the level of cosets, we describe the permutation action of the generators x and y of the modular group which correspond respectively to the transformations T and TS where

$$T:\tau\mapsto -1/\tau \quad S:\tau\mapsto\tau+1 \quad (\Im(\tau)>0).$$

If the index of a subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ is μ, then x and y generate a transitive permutation subgroup on the μ cosets. If x and y act fixed-point freely, the subgroup is torsion-free (hence free). Moreover, the disjoint cycle decomposition of the permutation xy provides the genus as well as the cusp widths for each subgroup. There is a one-to-one between these data (up to a simultaneous conjugacy of x and y by an element of $\mathfrak{S}_\mu$), and nonisomorphic Schreier graphs with μ nodes which are connected (transitivity), have no loops (being torsion-free) and whose circuits lengths correspond to the permutaion decompositions of xy and to the cusp widths. We give the graphs of all the subgroups in question, from which the generating permutations on the cosets may be derived.

2. Torsion-free genus zero congruence subgroups

Let $\mathrm{PSL}_2(\mathbb{Z})$ be the modular group acting on the upper half-plane $\mathfrak{H} = \{\tau \in \mathbb{C},\ \Im(\tau) > 0\}$ in the usual way. Let Γ be a subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ of finite index μ. If a nonidentity element in Γ corresponds to a matrix A, then A has a fixed point in $\mathfrak{H}$ when $|\mathrm{tr}(A)|$ is 0 (resp. 1), in which case A is of order 2 (resp. 3); such a transformation is called elliptic and the fixed point is an elliptic fixed point. If $|\mathrm{tr}(A)| = 2$, A has a single fixed point on the real line which is rational or ∞, and A has infinite order; the transformation is parabolic and the fixed point is a cusp. The hyperbolic transformations correspond to $|\mathrm{tr}(A)| > 2$.

The quotient space $\Gamma\backslash\mathfrak{H}$ can be made into a compact Riemann surface $\Gamma\backslash\mathfrak{H}^*$ by adjoining the cusps. The genus Γ of this Riemann surface is called the genus of the group Γ. Let ν_k ($k = 2, 3$) be the number of inequivalent elliptic fixed points of order k (which are fixed by a transformation of order k), and let h be the number of the inequivalent cusps, then, the Riemann-Hurwitz formula yields

$$g = 1 + \frac{\mu}{12} - \frac{\nu_2}{4} - \frac{\nu_3}{3} - \frac{h}{2}. \tag{2.1}$$

Let m_i, $i = 1 \dots r$, be the orders of the $r = \nu_2 + \nu_3$ conjugacy classes of elliptic subgroups . The signature of Γ is given by $(g; m_1 \dots, m_r; h)$. Using this data, the group Γ is defined by generators and relations. We focus on the case where $g = r = 0$, and so Γ can be canonically generated by a set of parabolic elements $P_1, \dots, P_h$ with the relation $P_1 \dots P_h = 1$. In this case the formula (2.1) becomes

$$\mu = 6(h - 2). \tag{2.2}$$

It is shown in [**9**] that there are exactly 33 conjugacy classes of genus zero congruence subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ with no elliptic elements. They can be described as follows:

- The principal congruence groups $\Gamma(n)$ for $n = 2, 3, 4, 5$.
- The congruence subgroups $\Gamma_0(n)$ for $n = 4, 6, 8, 9, 12, 16, 18$.
- The congruence subgroups $\Gamma_1(n)$ for $n = 5, 7, 8, 9, 10, 12$.
- The intersections $\Gamma_0(4) \cap \Gamma(2)$, $\Gamma_0(3) \cap \Gamma(2)$, $\Gamma_0(8) \cap \Gamma(2)$, $\Gamma_0(2) \cap \Gamma(3)$, $\Gamma_1(8) \cap \Gamma(2)$, $\Gamma_0(16) \cap \Gamma_1(8)$ and $\Gamma_0(25) \cap \Gamma(5)$.
- The congruence subgroups

$$\Gamma(m; m/d, \varepsilon, \chi) = \left\{ \pm \begin{pmatrix} 1 + \frac{m}{\varepsilon\chi}\alpha & d\beta \\ \frac{m}{\chi}\gamma & 1 + \frac{m}{\varepsilon\chi}\delta \end{pmatrix},\quad \gamma \equiv \alpha \mod \chi \right\}. \tag{2.3}$$

where the quadruples $(m, d, \varepsilon, \chi)$ take the values $(8, 2, 1, 2)$, $(12, 2, 1, 2)$, $(16, 1, 2, 2)$, $(27, 1, 3, 3)$, $(8, 4, 1, 2)$,
$(12, 2, 1, 2)$, $(16, 2, 2, 2)$, $(24, 1, 2, 2)$, $(32, 1, 4, 2)$ These congruence group are studied by Larcher [**5**] whose notation we use.

The possible indices for these 33 groups in $\mathrm{PSL}_2(\mathbb{Z})$ are 6, 12, 24, 36, 48 and 60.

3. Torsion-free Hauptmoduls and relations with j

Let Γ be a genus zero torsion-free congruence subgroup of $\mathrm{PSL}_2(\mathbb{Z})$. The stabilizer of a cusp Γ is a subgroup of finite index of the cusp stabilizer in $\mathrm{PSL}_2(\mathbb{Z})$. This index is, by definition, the width of the cusp in Γ. The sum of the all cusp widths is the index of Γ in $\mathrm{PSL}_2(\mathbb{Z})$. Moreover, the level of Γ occurs as a cusp width, and for the groups listed in the previous section, it corresponds to the cusp at 0 and is the lcm of all cusp widths. The smallest cusp width is at ∞ and is the gcd of all cusp widths.

The cusp widths are useful in describing the relationship between the Hauptmoduls for the above groups and the j function. For such a group Γ, the Riemann surface $\Gamma\backslash\mathfrak{H}^*$ is an étale covering of $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathfrak{H}^*$ ramified above 0, 1 and ∞. The ramification indices above ∞ are given by the cusp widths. While the index above 0 (resp. 1) is simply given by the order of vanishing of j (resp. $j-1$) at i (resp. ω), namely 2 (resp. 3), where $\omega \neq 1$ is a cube root of unity. From these considerations it is possible to give an explicit expression for j in terms of a Hauptmodul for Γ. Many of the 33 groups occur in Monstrous Moonshine and have Hauptmoduls expressed in terms of the Dedekind η function; for these the expressions are straightforward [**2, 3**], while for the remaining cases, the relations are provided by desymmetrization of order 2 or 3, or by taking the square root, depending on the index of Γ in some Moonshine group. Using these methods we express the j function in terms of a Hauptmodul for each group. The Hauptmoduls are normalized to have to have a q–expansion at ∞ of the form $f(\tau) = \frac{1}{q} + 0 + \sum_{k\geq 1} a_n q^n$.

In the table below we list all the torsion-free genus congruence subgroups together with their indices μ and their cusp widths.

μ	*Group*	*Widths*		μ	*Group*	*Widths*
6	$\Gamma(2)$	2^3		60	$\Gamma(5)$	5^{12}
	$\Gamma_0(4)$	$4, 1^2$			$\Gamma_0(25)\cap\Gamma_1(5)$	$25^2, 1^{10}$
12	$\Gamma(3)$	3^4		36	$\Gamma_0(2)\cap\Gamma(3)$	$6^4, 3^4$
	$\Gamma_0(4)\cap\Gamma(2)$	$4^2, 2^2$			$\Gamma_1(9)$	$9^3, 3^2, 1^3$
	$\Gamma_1(5)$	$5^2, 1^2$			$\Gamma(9;3,1,3)$	$9^2, 3^6$
	$\Gamma_0(6)$	$6, 3, 2, 1$			$\Gamma_1(10)$	$10^2, 5^2, 2^2, 1^2$
	$\Gamma_0(8)$	$8, 2, 1^2$			$\Gamma_0(18)$	$18, 9, 2^3, 1^3$
	$\Gamma_0(9)$	$9, 1^3$			$\Gamma(27;27,3,3)$	$27, 3, 1^6$
24	$\Gamma(4)$	4^6		48	$\Gamma_1(8)\cap\Gamma(2)$	$8^4, 4^2, 2^4$
	$\Gamma_0(3)\cap\Gamma(2)$	$6^3, 2^3$			$\Gamma(8;2,1,2)$	$8^2, 4^8$
	$\Gamma_1(7)$	$7^3, 1^3$			$\Gamma_1(12)$	$12^2, 6, 4^2, 3^2, 2, 1^2$
	$\Gamma_1(8)$	$8^2, 4, 2, 1^2$			$\Gamma(12;6,1,2)$	$12, 6^4, 4, 2^4$
	$\Gamma_0(8)\cap\Gamma(2)$	$8^2, 2^4$			$\Gamma_0(16)\cap\Gamma_1(8)$	$16^2, 4^2, 2^2, 1^4$
	$\Gamma(8;4,1,2)$	$8, 4^3, 2^2$			$\Gamma(16;8,2,2)$	$16^2, 2^8$
	$\Gamma_0(12)$	$12, 4, 3^2, 1^2$			$\Gamma(24;24,2,2)$	$24, 8, 3^4, 1^4$
	$\Gamma_0(16)$	$16, 4, 1^4$			$\Gamma(32;32,4,2)$	$32, 8, 1^8$
	$\Gamma(16;16,2,2)$	$16, 2^3, 1^2$				

Cusp widths of torsion-free genus 0 congruence subgroups of $\mathrm{PSL}_2(\mathbb{Z})$

It can be seen from (2.3) that each subgroup Γ not of index 6 in the above list has a lifting to a subgroup $\bar{\Gamma}$ of $\mathrm{SL}_2(\mathbb{Z})$ which contains no element of element of trace equal to -2, therefore, following Kodaira [**4**], one can construct elliptic surfaces with base curve being the quotient $\Gamma\backslash\mathfrak{H}^*$ and having singular fibres over the cusps. The three properties of these groups, namely genus zero, being torsion-free, and the absence of elements of trace -2 imply that the singular fibres of these elliptic surfaces are all of type I_n following Kodaira's notation, where n is the cusp width at the base point of the singular fibre. The two cases of index 6 provide surfaces of type I_2^*, I_2, I_2 and I_4, I_1, I_1^*. The rational functions expressing the j-function in terms of the Hauptmoduls of the subgroups represent the $J-$invariant of the elliptic surfaces. It is also easy to deduce a Weierstrass equation for these surfaces from these $J-$invariants (note that the numerators of the expressions below are all cubes).

The following tables give the expressions of the $j-$function as a rational function of the Hauptmodul t for each group, which are also the $J-$invariants of the semistable arithmetic elliptic surfaces over $\mathbb{P}^1$.

The degree of each rational function is given by the index of the corresponding group in $\mathrm{PSL}_2(\mathbb{Z})$. The exponents in the denominators are the cusp widths (amitting that at ∞) for each group. When the expressions for the rational functions are very long, we express them by a composition of simple rational functions to reduce their length. The expressions are over $\mathbb{Q}$ except for two at index 36 which are over $\mathbb{Q}(\omega)$, $\omega = \exp(2\pi i/3)$.

$\Gamma(2)$	$\frac{(t^2+192)^3}{(t-8)^2(t+8)^2}$
$\Gamma_0(4)$	$\frac{(t^2+240t+2112)^3}{(t+8)(t-8)^4}$

Index 6

$\Gamma(3)$	$\frac{t^3(t+6)^3(t^2-6t+36)^3}{(t-3)^3(t^2+3t+9)^3}$
$\Gamma_0(4)\cap\Gamma(2)$	$\frac{(t^4+224t^2+256)^3}{t^2(t-4)^4(t+4)^4}$
$\Gamma_1(5)$	$\frac{(t^4+248t^3+4064t^2+22312t+40336)^3}{(t+5)(t^2-t-31)^5}$
$\Gamma_0(6)$	$\frac{(t+7)^3(t^3+237t^2+1443t+2287)^3}{(t+3)^2(t+4)^3(t-5)^6}$
$\Gamma_0(8)$	$\frac{(t^4+240t^3+2144t^2+3840t+256)^3}{t(t+4)^2(t-4)^8}$
$\Gamma_0(9)$	$\frac{(t+6)^3(t^3+234t^2+756t+2160)^3}{(t^2+3t+9)(t-3)^9}$

Index 12

$\Gamma(4)$	$\frac{(t^4-4t^3+8t^2+16t+16)^3(t^4+4t^3+8t^2-16t+16)^3}{t^4(t-2)^4(t+2)^4(t^2+4)^4}$
$\Gamma_0(3)\cap\Gamma(2)$	$\frac{(t^2+3)^3(t^6+225t^4-405t^2+243)^3}{t^6(t-1)^2(t+1)^2(t-3)^6(t+3)^6}$
$\Gamma_1(7)$	$\frac{(t^2+5t+7)^3(t^6+247t^5+3840t^4+22695t^3+63800t^2+85943t+44647)^3}{(t+3)(t+2)(t^3+t^2-16t-29)^7}$
$\Gamma(8;4,1,2)$	$\frac{(t^8-8t^7+252t^6-1848t^5+6150t^4-11960t^3+15356t^2-12680t+4993)^3}{(t-2)^2(t-1)^4(t^2+2t-7)^4(t-3)^8}$
$\Gamma_0(8)\cap\Gamma(2)$	$\frac{(t^8+240t^6+2144t^4+3840t^2+256)^3}{t^2(t^2+4)^2(t-2)^8(t+2)^8}$
$\Gamma_1(8)$	$\frac{(t^8+248t^7+4092t^6+27528t^5+98790t^4+205640t^3+248636t^2+161720t+43873)^3}{(t+2)(t+1)^2(t+3)^4(t^2-2t-7)^8}$
$\Gamma_0(12)$	$\frac{(t^2+6t-3)^3(t^6+234t^5+747t^4+540t^3-729t^2-486t-243)^3}{t^3(t-1)(t+3)^3(t+1)^4(t-3)^{12}}$
$\Gamma(16;16,2,2)$	$\frac{(t^8+232t^7+252t^6-6888t^5+13350t^4+12760t^3-60484t^2+59560t-18527)^3}{(t-2)(t-1)^2(t^2+2t-7)^2(t-3)^{16}}$
$\Gamma_0(16)$	$\frac{(t^8+240t^7+2160t^6+6720t^5+17504t^4+26880t^3+34560t^2+15360t+256)^3}{t(t^2+4)(t+2)^4(t-2)^{16}}$

Index 24

$\Gamma_0(2)\cap\Gamma(3)$	$\frac{(t^3+4)^3(t^3+6t^2+4)^3(t^6-6t^5+36t^4+8t^3-24t^2+16)^3}{t^6(t+1)^3(t^2-t+1)^3(t-2)^6(t^2+2t+4)^6}$
$\Gamma_1(9)$	$\frac{(t+6)^3(t^3+234t^2+756t+2160)^3}{(t^2+3t+9)(t-3)^9}\circ\left(t-\frac{1}{t+1}-\frac{1}{t+2}\right)$
$\Gamma(9;3,1,3)$	$\frac{t^3(t+6)^3(t^2-6t+36)^3}{(t-3)^3(t^2+3t+9)^3}\circ\left(t-\frac{\omega^2}{t+\omega}-\frac{\omega^2}{t+2\omega}\right)$
$\Gamma_1(10)$	$\frac{(t^6+236t^5+1440t^4+1920t^3+3840t^2+256t+256)^3}{t^2(t^3-7t^2+8t+16)^5}\circ\left(t+\frac{t}{t+1}\right)$
$\Gamma_0(18)$	$\frac{(t^3+6t^2+4)^3(t^9+234t^8+756t^7+2172t^6+1872t^5+3024t^4+48t^3+3744t^2+64)^3}{t^2(t^2-t+1)(t^2+2t+4)^2(t+1)^9(t-2)^{18}}$
$\Gamma(27;27,3,3)$	$\frac{(t+6)^3(t^3+234t^2+756t+2160)^3}{(t^2+3t+9)(t-3)^9}\circ\left(t-\frac{\omega^2}{t+\omega}-\frac{\omega^2}{t+2\omega}\right)$

Index 36

4. Coset graphs

For each torsion-free genus zero congruence subgroup Γ of the modular group we give a coset graph for a permutation representation of the modular group on the cosets of the stabilizer of a point, Γ. These graphs (actually multigraphs) display the permutation action of generators x and y with the relations $x^2 = y^3 = 1$ on each coset Γx_i of $\mathrm{PSL}_2(\mathbb{Z}) = \cup\Gamma x_i,\ i = 1, \cdots, \mu$, where μ is the index Γ in $\mathrm{PSL}_2(\mathbb{Z})$. The graphs have μ nodes where μ takes the values $6, 12, 24, 36, 48, 60$. The permutation shape of the parabolic element xy, which is the cusp shape in Γ, is determined by following the path of xy. The graphs are built from simple edges (x), and triangles (y) which are assumed to be positively oriented.

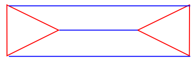
$\Gamma(8;2,1,2)$	$\frac{(t^8-8t^7+252t^6-1848t^5+6150t^4-11960t^3+15356t^2-12680t+4993)^3}{(t-2)^2(t-1)^4(t^2+2t-7)^4(t-3)^8} \circ (t^2+2)$
$\Gamma_1(8)\cap\Gamma(2)$	$\frac{(t^{16}+232t^{14}+732t^{12}-1192t^{10}+710t^8-1192t^6+732t^4+232t^2+1)^3}{t^2(t-1)^2(t+1)^2(t^2+1)^4(t^2-2t-1)^8(t^2+2t-1)^8}$
$\Gamma(12;6,1,2)$	$\frac{(t^2+3)^3(t^6+225t^4-405t^2+243)^3}{t^6(t-1)^2(t+1)^2(t-3)^6(t+3)^6} \circ \left(t-\frac{1}{t-1}\right)$
$\Gamma_1(12)$	$\frac{(t^2+6t-3)^3(t^6+234t^5+747t^4+540t^3-729t^2-486t-243)^3}{t^3(t-1)(t+3)^3(t+1)^4(t-3)^{12}} \circ \left(t-\frac{1}{t+1}\right)$
$\Gamma(16;8,2,2)$	$\frac{(t^8+232t^7+252t^6-6888t^5+13350t^4+12760t^3-60484t^2+59560t-18527)^3}{(t-2)(t-1)^2(t^2+2t-7)^2(t-3)^{16}} \circ (t^2+2)$
$\Gamma_0(16)\cap\Gamma_1(8)$	$\frac{(t^8+240t^7+2160t^6+6720t^5+17504t^4+26880t^3+34560t^2+15360t+256)^3}{t(t^2+4)(t+2)^4(t-2)^{16}} \circ \left(t-\frac{1}{t}\right)$
$\Gamma(24;24,2,2)$	$\frac{(t^2+6t-3)^3(t^6+234t^5+747t^4+540t^3-729t^2-486t-243)^3}{t^3(t-1)(t+3)^3(t+1)^4(t-3)^{12}} \circ \left(t-\frac{1}{t-1}\right)$
$\Gamma(32;32,4,2)$	$\frac{(t^8+240t^7+2160t^6+6720t^5+17504t^4+26880t^3+34560t^2+15360t+256)^3}{t(t^2+4)(t+2)^4(t-2)^{16}} \circ \left(t+\frac{1}{t}\right)$

Index 48

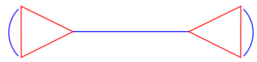
$\Gamma(5)$	$\frac{(t^2+3t+1)^3(t^4+t^3+11t^2-4t+16)^3(t^4-4t^3+11t^2-14t+31)^3}{(t-1)^5(t^4+t^3+6t^2+6t+11)^5} \circ \left(t-\frac{1}{t}\right)$
$\Gamma_0(25)\cap\Gamma_1(5)$	$\frac{(t^4+228t^3+494t^2-228t+1)^3}{t(t^2-11t-1)^5} \circ \frac{(t^5+3t^4+4t^3+2t^2+t)}{t^4-2t^3+4t^2-3t+1}$

Index 60

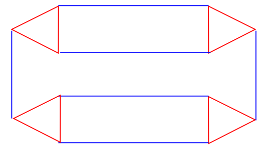
The Cayley graph of the group $\mathrm{PSL}_2(\mathbb{Z})$ consists of the (infinite) free trivalent tree with each node replaced by a positively oriented triangle. Our graphs are factor graphs of this one. The verification that each graph is on the coset of one of the groups above is carried out by finding a simultaneous conjugation of a set of matrix generators into recognizable form. (This corresponds to reordering the nodes of the graphs or to permuting the cosets).



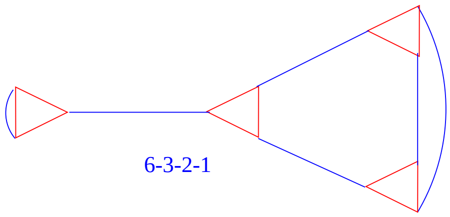
2-2-2



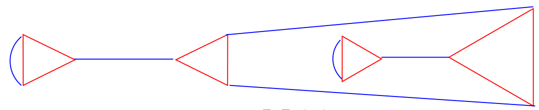
4-1-1



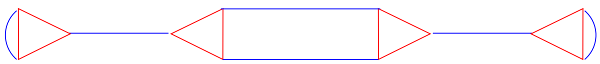
4-4-2-2



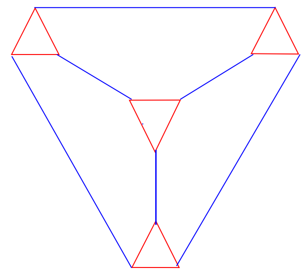
6-3-2-1



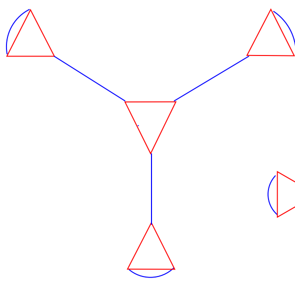
5-5-1-1



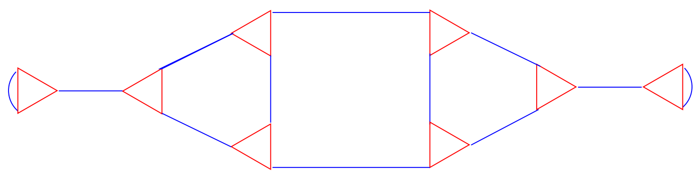
8-2-1-1



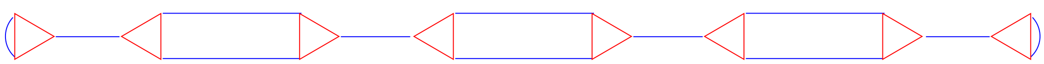
3-3-3-3



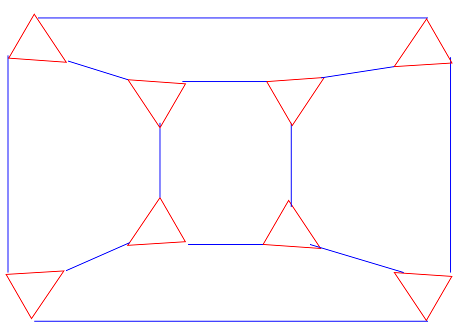
9-1-1-1



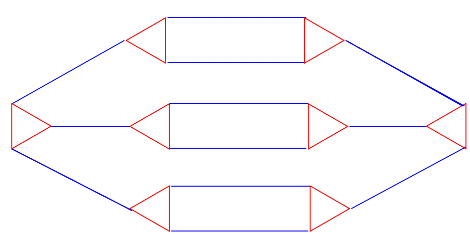
12-4-3-3-1-1



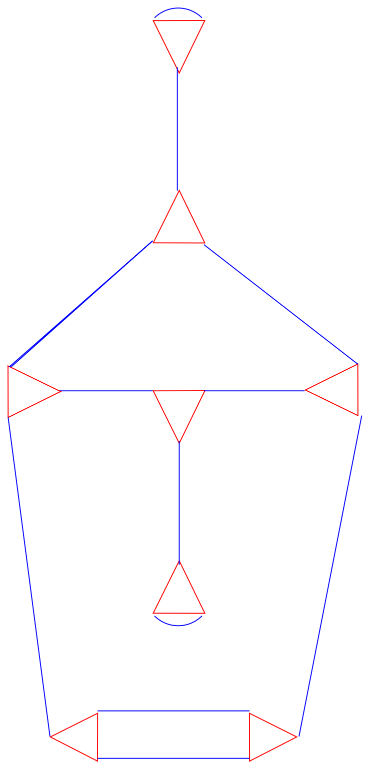
16-2-2-2-1-1



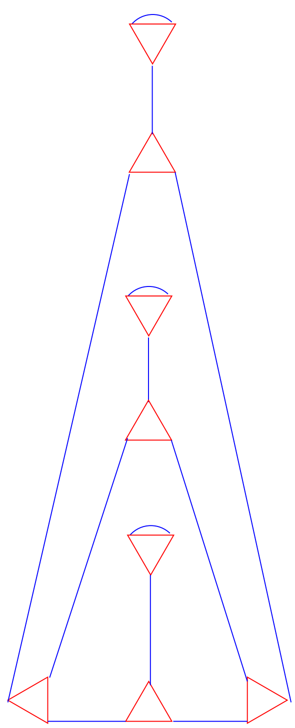
4-4-4-4-4-4



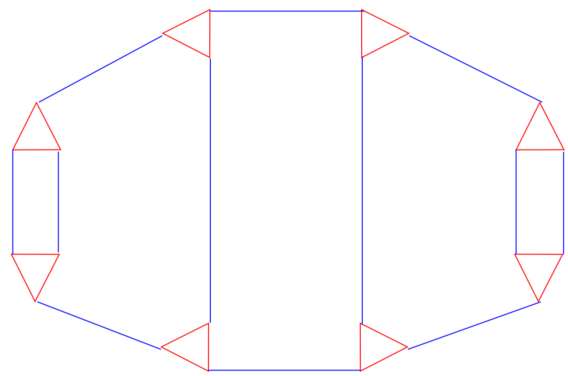
6-6-6-2-2-2



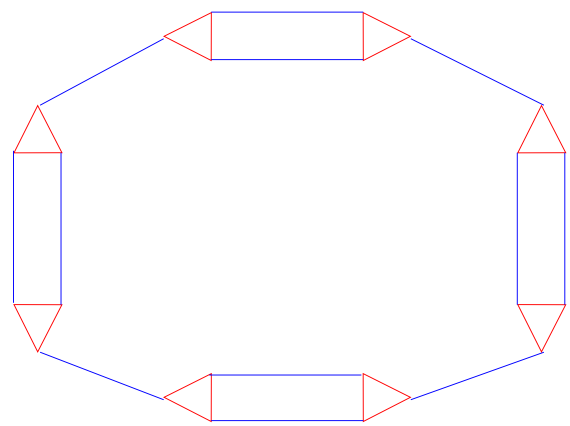
8-8-4-2-1-1



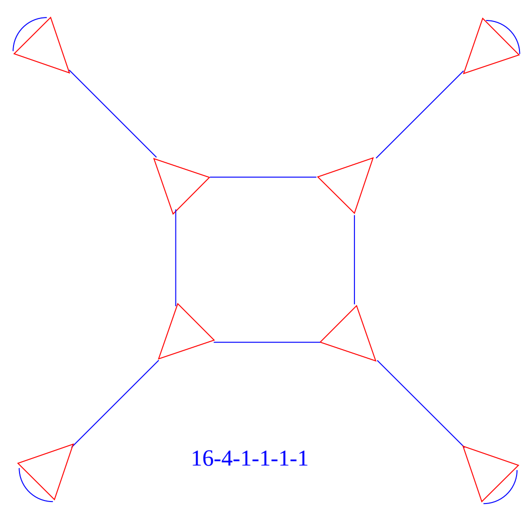
7-7-7-1-1-1



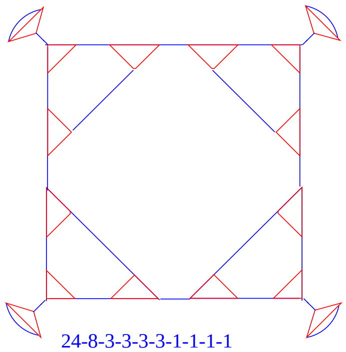
8-4-4-4-2-2



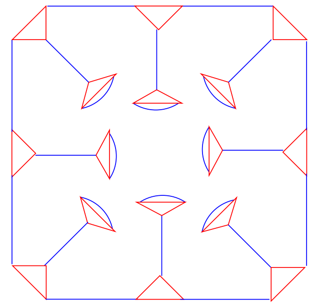
8-8-2-2-2-2



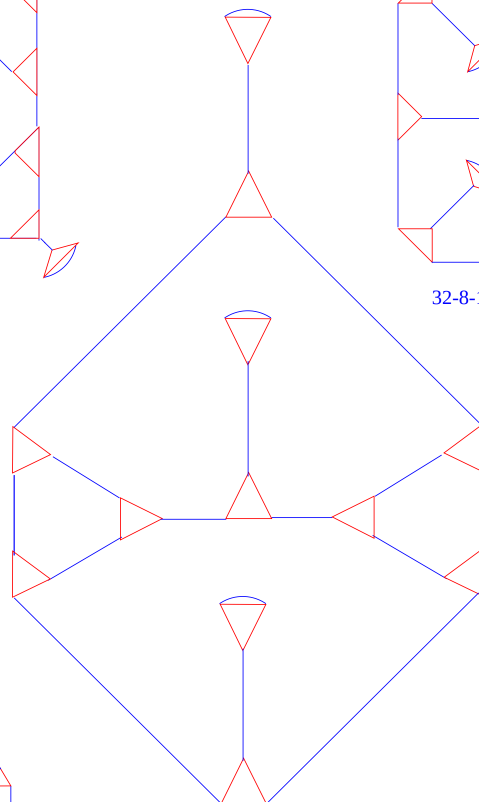
16-4-1-1-1-1



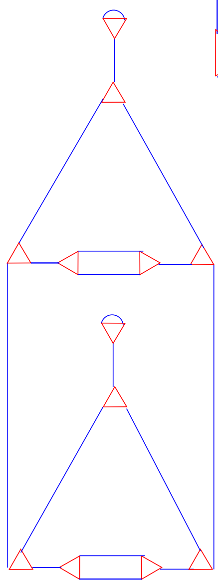
24-8-3-3-3-3-1-1-1-1



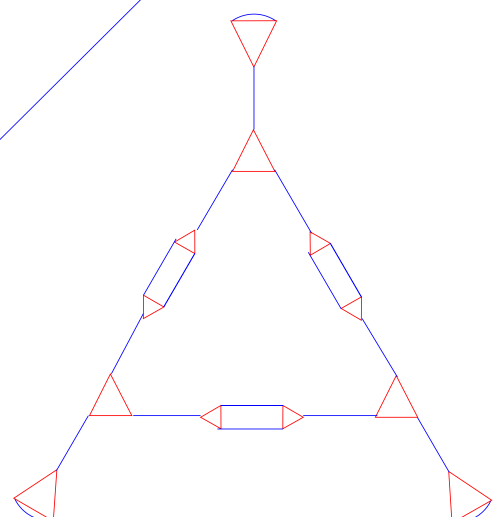
32-8-1-1-1-1-1-1-1-1



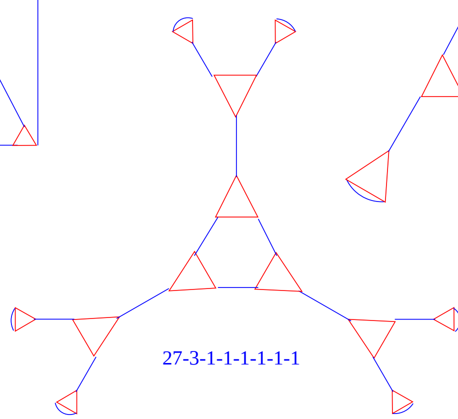
9-9-9-3-3-1-1-1



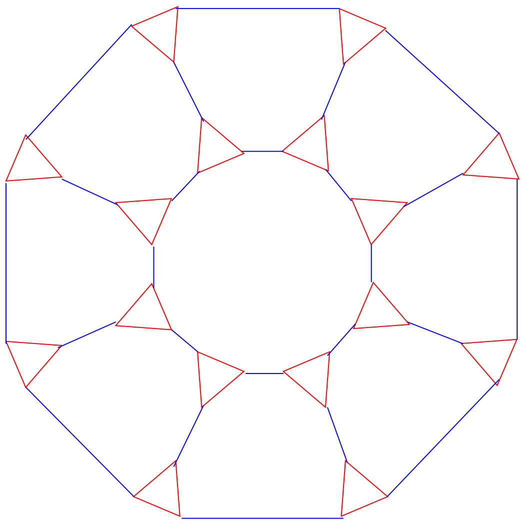
10-10-5-5-2-2-1-1



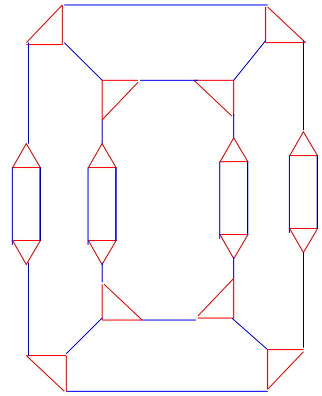
18-9-2-2-2-1-1-1



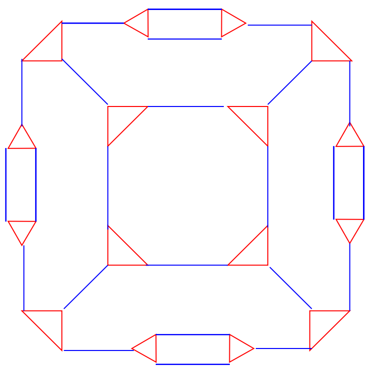
27-3-1-1-1-1-1-1



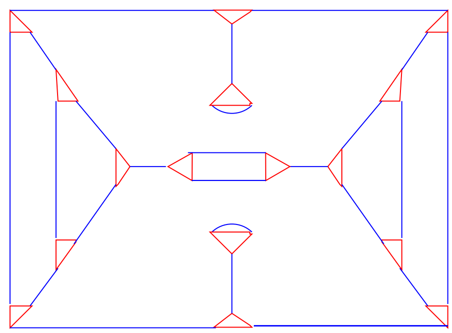
8-8-4-4-4-4-4-4-4-4



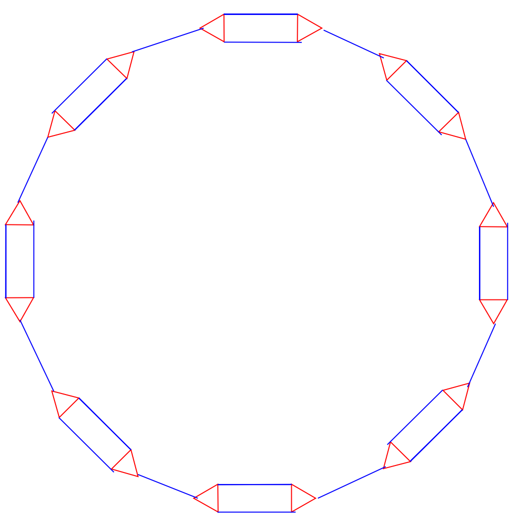
8-8-8-8-4-4-2-2-2-2



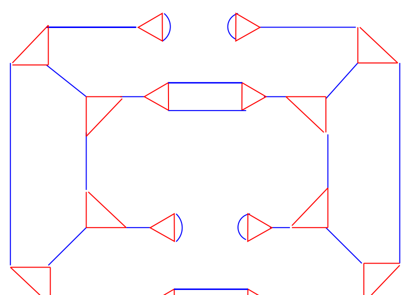
12-6-6-6-6-4-2-2-2-2



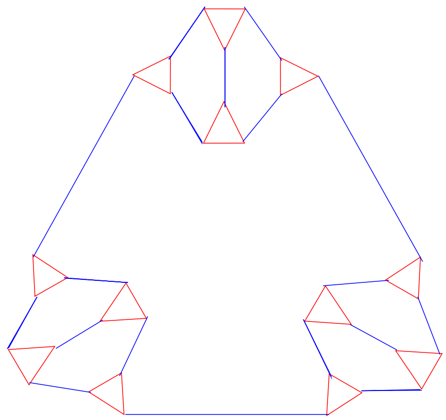
12-12-6-4-4-3-3-2-1-1



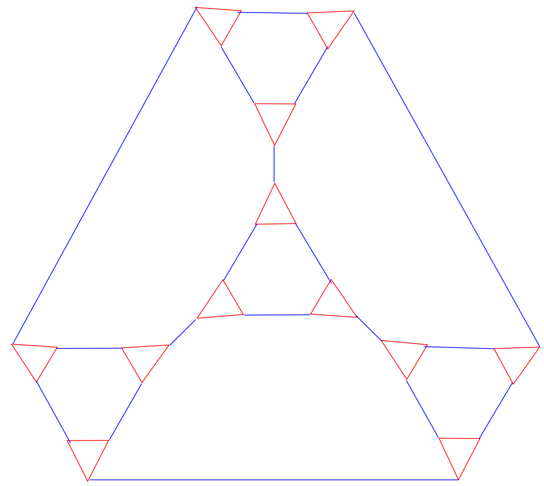
16-16-2-2-2-2-2-2-2-2



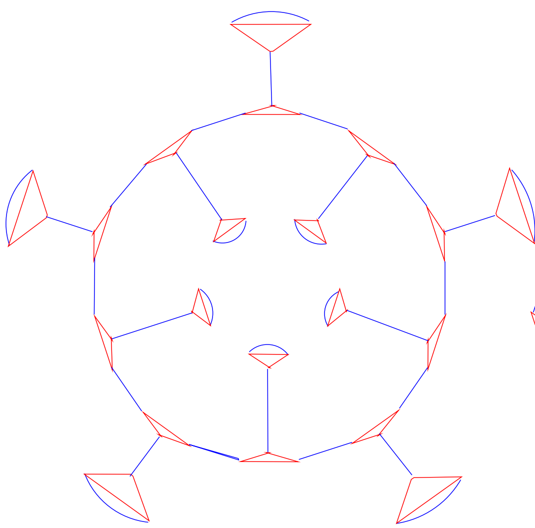
16-16-4-4-2-2-1-1-1-1



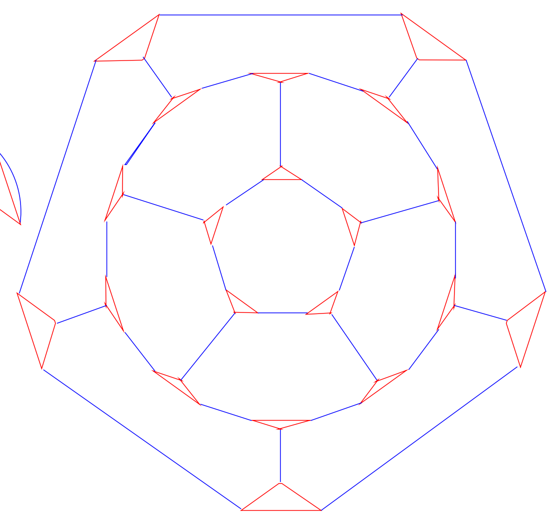
9-9-3-3-3-3-3-3



6-6-6-6-3-3-3-3



25-25-1-1-1-1-1-1-1-1-1-1



5-5-5-5-5-5-5-5-5-5-5-5

Acknowledgement. We acknowledge with thanks the help of Tim Hsu, Alexander Hulpke, and Simon Norton in the course of this work. Computer algebra systems, Gap, Magma, and Maple were used extensively.

References

[1] Beauville, A. Les familles stables de courbes elliptiques sur $\mathbb{P}_1$ admettant quatre fibres singulières, C. R. Acad. Sci. Paris Ser. 1 Math. 294 (1982), no. 19, 657-660.

[2] Conway, J. H. and Norton, S. P. Monstrous moonshine. Bull. London Math. Soc. 11 (1979), no. 3, 308–339.

[3] Ford, D., McKay, J. and Norton, S. More on replicable functions. Comm. Algebra 22 (1994), 5175–5193.

[4] Kodaira, K. On compact complex analytic surfaces II, Ann. Math. 77 (1963), 563-626.
[5] Larcher, H. The cusp amplitudes of the congruence subgroups of the classical modular group. II. Illinois J. Math. 28 (1984), 312–338.
[6] McKay, J., Sebbar, A. Fuchsian groups, Schwarzians, and theta functions, C. R. Acad. Sci. Paris, t.327, Serie I, p.343-348,1998.
[7] McKay, J., Sebbar, A. Fuchsian groups, automorphic functions and Schwarzians, *To appear.*
[8] Rankin, R. Modular forms and functions. Cambridge University Press, 1977.
[9] Sebbar, A. Classification of genus zero torsion-free congruence groups in $\mathrm{PSL}_2(\mathbb{R})$, *preprint.*
[10] Sebbar, A. Modular subgroups, forms, curves, and surfaces. *These proceedings*

John McKay, Department of Mathematics and CICMA. Concordia University,, 1455 de Maisonneuve Blvd. West, Montreal, Quebec H3G 1M8, Canada.

E-mail address: mckay@cs.concordia.ca

Abdellah Sebbar, Department of Mathematics and Statistics, University of Ottawa. Ottawa, ON K1N 6N5, Canada

E-mail address: sebbar@mathstat.uottawa.ca