Math. Ann. 318, 255–275 (2000) **Mathematische Annalen**

Digital Object Identifier (DOI) 10.1007/s002080000116

Fuchsian groups, automorphic functions and Schwarzians

John McKay · Abdellah Sebbar

Received April 8, 1999 / Revised January 10, 2000 /
Published online July 20, 2000 –

Abstract. In this paper, a close connection is established between the geometry of certain genus zero Fuchsian groups and the analytic properties of the automorphic forms obtained by applying a certain differential operator to the Hauptmoduls of the groups.

1. Introduction

Let G be a Fuchsian group of the first kind acting on the upper half-plane $\mathfrak{H}$ such that the compactification X of the open Riemann surface $G\backslash\mathfrak{H}$ has genus zero; we then say that G is of genus zero. If a function f defined on X generates the function field over $\mathbb{C}$ of X, then f is called a Hauptmodul for the genus zero group G, and is defined up to linear fractional transformations. Each Hauptmodul can be extended to a meromorphic function defined on $\mathfrak{H}$ to become an automorphic function with respect to G. A prototype is the elliptic modular function j which is a Hauptmodul for the modular group $\mathrm{PSL}_2(\mathbb{Z})$. When G contains the transformation $\tau \to \tau + 1$ which generates its translation subgroup (the stabilizer of ∞), then each Hauptmodul has a Fourier expansion in $q = \exp(2\pi i\tau)$, and one of them has the form

$$f(\tau) = \frac{1}{q} + \sum_{n\geq 1} a_n\, q^n, \qquad a_n \in \mathbb{C}. \tag{1.1}$$

To such an f and for each positive integer n, there exists a unique monic polynomial of degree n, $P_n = P_{n,f}$ whose coefficients depend on the coefficients $\{a_k\}$ of f. It is characterized by the property that $P_n(f) - 1/q^n$ is a power series in

J. McKay
Department of Mathematics and CICMA, Concordia University, 1455 de Maisonneuve Blvd. West, Montreal, Quebec H3G 1M8, Canada (e-mail: mckay@cs.concordia.ca)

A. Sebbar
Department of Mathematics and Statistics, University of Ottawa, Ottawa, ON K1N 6N5, Canada (e-mail: sebbar@mathstat.uottawa.ca)

q with no constant term, see Sect. 4. For example, if f is the elliptic modular function j normalized to have the form (1.1), then

$$P_n(j) = nT_n(j),$$

where T_n is the classical n-th Hecke operator.

For a meromorphic function f defined on some complex domain, there is a differential operator known as the Schwarzian (or the Schwarz derivative) defined by

$$\{f, z\} = 2\left(\frac{f''}{f'}\right)' - \left(\frac{f''}{f'}\right)^2.$$

which is invariant under linear fractional transformations of f. For f given formally by (1.1), the Schwarzian of f is completely described in terms of the q-coefficients of the Faber polynomial $P_n(f)$, see Proposition 4.1..

For an automorphic function f and for a Fuchsian group G, we find $\{f, \tau\}$ to be an automorphic form of weight 4 for G. When f is a Hauptmodul, then $\{f, \tau\}$ is generically invariant under a larger group, namely the normalizer of G in $\mathrm{PSL}_2(\mathbb{R})$, and for the inverse function $\tau(f)$, $\{\tau, f\}$ is a rational function of f. The analytic behaviour of $\{f, \tau\}$ is that it is holomorphic in $\mathfrak{H}$ except at elliptic fixed points where it has poles of order 2, and it is holomorphic at the cusps, see Proposition 6.2.

This leads us to restrict our attention to genus zero Fuchsian groups with no elliptic elements. For finiteness reasons [11] we consider only those groups which contain some $\Gamma_0(n)$ with finite index and such that the stabilizer of ∞ is generated by $\tau \to \tau + 1$. In other words, these are torsion-free genus zero groups with the cusp at ∞ having width 1. We determine all the n such that $\Gamma_0(n)$, or a conjugate, satisfies these conditions. There are 14 such groups which have Hauptmoduls given by eta-products, and there are only 3 more groups which are not $\Gamma_0(n)$ or a conjugate with the same property.

The Schwarzian of a Hauptmodul for such a group, being holomorphic on $\mathfrak{H}$ and at the cusps, is completely determined in terms of a canonical weight 4 automorphic form. For 14 groups, these forms are theta functions of variously normed rank 8 lattices (Sect. 7), and for the 3 remaining cases they are simple linear combinations of Eisenstein series and known cusp forms (Sect. 8). The theta functions arise only when the groups are, up to conjugacy, $\Gamma_0(n)$. The significance of the lattices of the theta functions involved is as yet unknown.

2. The Schwarzian

Let f be a meromorphic function over some region of the complex plane. We define the Schwarzian of f to be:

$$\{f, z\} = 2\left(\frac{f''}{f'}\right)' - \left(\frac{f''}{f'}\right)^2 = \frac{1}{f'^2}(2f'f''' - 3f''^2). \tag{2.1}$$

This function is the main subject of this note. It is an essential ingredient for solving the problem of mapping a circular disc or a half-plane onto a hyperbolic polygon, and was studied by Schwarz [8] in connection with differential equations and quadratic differentials.

By direct computation we have:

- If f and g are two meromorphic functions such that each function is a linear fractional transform of the other, then $\{f, z\} = \{g, z\}$.
- if w is a function of z, then

$$\{f, z\} = \{f, w\}(dw/dz)^2 + \{w, z\}. \tag{2.2}$$

- If f is a linear fractional transform of z, then$\{f, z\} = 0$.

It follows that if $w'(z_0) \neq 0$ for some z_0, then in a neighbourhood of this point, the inverse function $z(w)$ satisfies:

$$\{z, w\} = -\{w, z\}(dz/dw)^2. \tag{2.3}$$

If also $w = \frac{az+b}{cz+d}$ for some constants a, b, c, and d, then

$$\{f, z\} = \{f, w\}\,\frac{(ad-bc)^2}{(cz+d)^4}. \tag{2.4}$$

There is an important connection with second order linear differential equations. Let y_1 and y_2 be two linearly independent solutions to

$$y'' + \frac{1}{4}R(z)y = 0, \tag{2.5}$$

where $R(z)$ is a meromorphic function on a domain. If we set $f = y_1/y_2$, then f is a solution to $\{f, z\} = R(z)$, and conversely, if f is a locally univalent function which satisfies $\{f, z\} = R(z)$, then $y_1 = f/\sqrt{f'}$ and $y_2 = 1/\sqrt{f'}$ are two linearly independent solutions to (2.5). As an immediate consequence, $\{f, z\} = 0$ if and only if f is a linear fractional transform of z, and $\{f, z\} = \{g, z\}$ if and only if each function is a linear fractional transform of the other.

3. Genus zero Fuchsian groups

Henceforth we use z for a variable in any domain of the complex plane, and τ if this domain is the upper half-plane $\mathfrak{H}$, and we set $q = \exp(2\pi i\tau)$.

Let G be a Fuchsian group for the upper half-plane $\mathfrak{H}$ and f an automorphic form for G of weight k ($k \geq 0$), that is, a meromorphic function on $\mathfrak{H}$ satisfying

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau), \quad \tau \in \mathfrak{H}\,, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G,$$

with some growth condition at the cusps. We say that the elements of G leave f invariant even if $k \neq 0$.

In general the derivative of f is not an automorphic form for any weight. If $k = 0$, then f' is automorphic of weight 2. We can, however, show that the expression $kff'' - (k+1)f'^2$ is an automorphic form of weight $2k+4$. In particular, if f is an automorphic function (of weight 0), then f' has weight 2 and therefore $2f'f''' - 3f''^2$ has weight 8. Dividing by f'^2, we obtain $\{f, \tau\}$. And dividing by f'^4 we obtain $-\{\tau, f\}$ according to (2.3). Hence we have

Proposition 3.1. *If $f(\tau)$ is an automorphic function for a Fuchsian group G, then $\{\tau, f\}$ is an automorphic function and $\{f, \tau\}$ is an automorphic form of weight 4 for G.* □

Assume that G is of genus zero, in the sense that the compactification of the Riemann surface $G\backslash\mathfrak{H}$ is of genus zero. If f is a complex analytic embedding from this surface into the extended complex plane, then f induces an automorphic function for G defined on $\mathfrak{H}$. In the language of Fricke and Klein, f is a Hauptmodul; it generates the function field of the Riemann surface. The function f is determined up to a linear fractional transform. According to (2.3), $\{\tau, f\}$ is an automorphic function for G, hence it must be a rational function of f, say $R(f)$. Properties of $R(f)$ will be discussed in a later section. Now we give a more precise statement about the invariance group of $\{f, \tau\}$.

Proposition 3.2. *Let G be a genus zero Fuchsian group and f a Hauptmodul for G. Then $\{f, \tau\}$ is a weight 4 automorphic form for the normalizer of G in $PSL_2(\mathbb{R})$. Conversely, any element of $PSL_2(\mathbb{R})$ which leaves $\{f, \tau\}$ invariant normalizes G.*

Proof. Let g be an element of $\mathrm{PSL}_2(\mathbb{R})$ which normalizes G. The function $f(g\cdot\tau)$ defines an automorphic function for G, where

$$g \cdot \tau = \frac{a\tau+b}{c\tau+d} \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Now $f(g \cdot \tau)$ takes its values only once on the Riemann surface $G\backslash\mathfrak{H}$, therefore

$$f(g\cdot\tau) = \frac{\alpha f(\tau)+\beta}{\gamma f(\tau)+\delta}\,, \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{C})\,, \quad \tau \in \mathfrak{H}. \tag{3.1}$$

It follows that

$$\begin{aligned}\{f(g\cdot\tau), g\cdot\tau\} =& \{f(\tau), g\cdot\tau\}\\ =& (c\tau+d)^4\{f,\tau\},\end{aligned}$$

using (2.4), and so $\{f,\tau\}$ is an automorphic form of weight 4 for the normalizer of G in $\mathrm{PSL}_2(\mathbb{R})$. For the converse, let g be an element of $\mathrm{PSL}_2(\mathbb{R})$ which leaves $\{f,\tau\}$ invariant. It is clear that $\{f(g\cdot\tau),\tau\} = \{f,\tau\}$, hence $f(g\cdot\tau) = \gamma_g\cdot f(\tau)$ where γ_g is an element of $\mathrm{PGL}_2(\mathbb{C})$ and the map $g\to\gamma_g$ is a group homomorphism. For $x\in G$

$$f(gxg^{-1}\cdot\tau) = \gamma_g\cdot f(xg^{-1}\cdot\tau) = \gamma_g\cdot f(g^{-1}\cdot\tau) = \gamma_g\gamma_{g^{-1}}\cdot f(\tau) = f(\tau),$$

which implies that gxg^{-1} leaves f invariant. We need to show that any element g of $\mathrm{PSL}_2(\mathbb{R})$ which leaves the Hauptmodul f invariant is actually in G. Let τ_0 be any interior point in a fundamental region $\mathcal{D}$, then τ_0 and $g\cdot\tau_0$ are necessarily in the same G-orbit, otherwise we can bring $g\cdot\tau_0$ to D by applying an element of G, and this new point in D is not τ_0, but both have the same image by f which contradicts f being a Hauptmodul, hence we may assume that $xg\cdot\tau_0 = \tau_0$ for some $x\in G$, and can choose a neighbourhood $\mathcal{U}$ of τ_0 such that xg maps $\mathcal{U}$ inside $\mathcal{D}$. Let $\tau\neq\tau_0$ be an element of $\mathcal{U}$, then necessarily $xg\cdot\tau=\tau$ since otherwise we would have two distinct points inside a fundamental domain having the same image by f. Now xg has two distinct fixed points in $\mathfrak{H}$, which is possible only if $xg=1$. This implies that $g\in G$. □

Remark 3.1. The invariance group for the function $\{\tau, f\}$ is, in general, not larger than G.

4. Expansion at ∞

In this section we restrict ourselves to the class of functions of the form

$$f(q) = \frac{1}{q} + \sum_{n\geq 1} a_n q^n\ , \quad a_n\in\mathbb{C}. \tag{4.1}$$

We assume that these functions are meromorphic in the unit disc with a simple pole at 0. We will also consider them as functions of τ with $q=\exp(2\pi i\tau)$ and $\tau\in\mathfrak{H}$. Any genus zero Fuchsian group in which the translations are generated by $\tau\to\tau+1$ has a Hauptmodul of the form (4.1).

For $w\in\mathbb{C}$ and p in the unit disc, set

$$F_w(p) = \frac{f'(p)}{w-f(p)} - \frac{1}{p}\ , \tag{4.2}$$

then $F_w(p)$ is analytic in a neighbourhood of zero and has a Taylor expansion

$$F_w(p) = \sum_{n\geq 1} P_n(w)p^{n-1}. \tag{4.3}$$

Substituting (4.1) and (4.3) into (4.2) we obtain

$$\left(\sum_{n\geq 1} P_n(w)p^n\right)\left(\frac{-1}{p} + w - \sum_{n\geq 1} a_n p^n\right) = -w + \sum_{n\geq 1}(n+1)a_n p^n.$$

Identifying the coefficients of powers of p, we get $P_1(w) = w$ and for $n \geq 2$:

$$P_{n+1}(w) = wP_n(w) - \sum_{k=1}^{n-1} a_{n-k}P_k(w) - (n+1)a_n. \tag{4.4}$$

It follows that P_n is a monic polynomial of degree n. For example:

$$P_1(w) = w\ ,\quad P_2(w) = w^2 - 2a_1\ ,\quad P_3(w) = w^3 - 3a_1 w - 3a_2\ \ldots$$

We call P_n the n-th Faber polynomial associated with f. It follows from (4.4) that

$$P_n(f(q)) = \frac{1}{q^n} + g_n(q), \tag{4.5}$$

with $g_n(q)$ a power series in q and $g_n(0) = 0$. For each positive integer n, $P_n(w)$ is the unique polynomial such that $P_n(f(q))$ has the form (4.5). As an example, we may take the elliptic modular function J normalized to the following form

$$J = \frac{1}{q} + 0 + 196884q + \cdots .$$

For each $n \geq 1$, the action of the n-th Hecke operator T_n on J is given by

$$T_n(J)(\tau) = \frac{1}{n}\sum_{\substack{ad=n\\ 0\leq b<d}} J\left(\frac{a\tau + b}{d}\right).$$

This expression is invariant under the action of the modular group $\mathrm{SL}_2(\mathbb{Z})$ since multiplication by an element of this group permutes the matrices in the sum. It follows that $T_n(J)$ is a rational function of J, moreover $T_n(J)$ has no poles on $\mathfrak{H}$, and so is a polynomial in J. Writing its q-expansion yields

$$T_n(J) = \frac{1}{n}P_n(J),$$

which says that, for all $n \geq 1$, the n-th Hecke operator applied to J yields the unique polynomial satisfying (4.5). For Hauptmoduls for proper subgroups of the modular group this is no longer true for all $n \geq 1$.

Let us write

$$g_n(q) = n \sum_{m\geq 1} h_{m,n}\, q^m. \tag{4.6}$$

Since $P_1(w) = w$, we have $h_{m,1} = a_m$ for all $m \geq 1$. Also from (4.2) and (4.3) we have

$$\begin{aligned}\frac{f'(p)}{f(q)-f(p)} - \frac{1}{p} &= \sum_{n\geq 1}\left(\frac{1}{q^n} + n\sum_{m\geq 1} h_{m,n}\, q^m\right) p^{n-1}\\ &= \frac{1}{q-p} + \sum_{m,n\geq 1} n h_{m,n}\, q^m p^{n-1}.\end{aligned}$$

Differentiating this identity with respect to q, we get

$$-\frac{f'(p)f'(q)}{(f(p)-f(q))^2} + \frac{1}{(q-p)^2} = \sum_{m,n\geq 1} mnh_{m,n}\, q^{n-1} p^{m-1}. \tag{4.7}$$

The left side is symmetric in p and q. It follows that

$$h_{m,n} = h_{n,m} \quad \text{for } m, n \geq 1.$$

A function f with rational coefficient a_n is called replicable if the corresponding coefficients satisfy $h_{m,n} = h_{r,s}$ whenever $\gcd(m,n) = \gcd(r,s)$ and $\operatorname{lcm}(m,n) = \operatorname{lcm}(r,s)$.

Proposition 4.1. *For f given by* (4.1) *and $\{h_{m,n}\}$ given by* (4.5) *and* (4.6), *we have*

$$\frac{1}{4\pi^2}\{f,\tau\} = 1 + 12\sum_{m,n\geq 1} mn\, h_{m,n}\, q^{m+n}. \tag{4.8}$$

Proof. As $p \to q$, let us write

$$\begin{aligned} f(p) = f(q) + (p-q)f'(q) + \frac{1}{2}(p-q)^2 f''(q)\\ +\frac{1}{6}(p-q)^3 f'''(q) + o\left((p-q)^3\right),\end{aligned}$$

and

$$f'(p) = f'(q) + (p-q)f''(q) + \frac{1}{2}(p-q)^2 f'''(q) + o\left((p-q)^2\right).$$

It follows that

$$\frac{f'(p)f'(q)}{(f(p)-f(q))^2} - \frac{1}{(q-p)^2} = \frac{1}{12}\{f,q\} + O(p-q).$$

Substituting into (4.7), we obtain

$$\frac{1}{12}\{f,q\} = -\sum_{m,n\geq 1} mnh_{m,n}\, q^{n-1}p^{m-1} + O(p-q).$$

Since $\{q,\tau\} = (2\pi)^2$, we deduce by using (2.2) that

$$\{f,\tau\} = -(2\pi)^2 q^2\{f,q\} + (2\pi)^2.$$

The proposition follows. □

Remark 4.1. The identity (4.8) establishes a close connection between the Schwarzian and the replicability of f and it may also have an interpretation in terms of vertex algebras. In later sections we will see some illustrations of this identity.

5. Triangle groups and theta functions

Let τ be the mapping function which sends $\mathfrak{H}$ onto a hyperbolic triangle in $\mathfrak{H}$ with prescribed angles $\alpha_1\pi$ at $\tau(0)$, $\alpha_2\pi$ at $\tau(1)$ and $\alpha_3\pi$ at $\tau(\infty)$. Let f be the inverse function of τ. If $\alpha_1 \neq 0$, we may apply a linear fractional transform to τ so that the sides of the triangle which meet at $\tau(0)$ are two straight lines with internal angle $\alpha_1\pi$. A similar transformation will bring $\tau(0)$ to 0 with one side on the real axis and the other side inside $\mathfrak{H}$. Since these two transformations do not affect $\{\tau, f\}$, we will still denote the resulting function by τ. The map $w = \tau^{\frac{1}{\alpha}}$ maps the angular sector onto $\mathfrak{H}$. By the Schwarz reflection principle, w is regular at $f = 0$ and can be continued analytically throughout a neighbourhood of 0. It follows that w is a power series in f, and so is $\{w, f\}$. Moreover, $w'(0) \neq 0$ and $\{\tau, w\} = (1-\alpha_1^2)/w^2$. It follows that in a neighbourhood of $f = 0$ we have

$$\{\tau, f\} = \frac{1-\alpha_1^2}{f^2} + \frac{c_1}{f} + \cdots, \tag{5.1}$$

where the dots represent a regular expression at $f = 0$. Similarly at $f = 1$ we have

$$\{\tau, f\} = \frac{1-\alpha_2^2}{(f-1)^2} + \frac{c_2}{f-1} + \cdots. \tag{5.2}$$

It is not difficult to see that the same formula holds if the angle is zero. At ∞, we have

$$\{\tau, f\} = \frac{1-\alpha_3^2}{f^2} + \text{ higher powers in } \frac{1}{f}. \tag{5.3}$$

This means that $\{\tau, f\}$ has a zero of multiplicity at least 2 at ∞. It follows that the expression

$$R(f) := \{\tau, f\} - \frac{1-\alpha_1^2}{f^2} - \frac{1-\alpha_2^2}{(f-1)^2} - \frac{c_1}{f} - \frac{c_2}{f-1}$$

has no poles in the upper half-plane. By the Schwarz reflection principle, we can extend τ to the lower half-plane through the segment between 0 and 1. The image of the lower half-plane is an triangle adjacent to the previous one. By Liouville's theorem, $R(f)$ is a constant, and this constant is 0 because of the vanishing at ∞. Moreover, since $\{\tau, f\}$ has a zero of at least second degree at ∞, this imposes relations among c_1 and c_2, and we obtain

Proposition 5.1. *We have*

$$\{\tau, f\} = \frac{1-\alpha_1^2}{f^2} + \frac{1-\alpha_2^2}{(f-1)^2} + \frac{\alpha_1^2+\alpha_2^2-\alpha_3^2-1}{f(f-1)}. \square \tag{5.4}$$

Let us assume that $\alpha_i = 1/m_i$ for $i = 1, 2, 3$ with m_i a positive integer, and if $\alpha_i = 0$, $m_i = \infty$. Let r_i be the reflection in the side opposite to the angle $\alpha_i\pi$. Now r_1, r_2 and r_3 generate a group G^* of isometries of $\mathfrak{H}$ and the images of the initial triangle tessellate $\mathfrak{H}$. The products of an even number of reflections r_i form an orientation-preserving subgroup G of index 2 in G^*. The group G is Fuchsian since it is discrete, and is generated by the elements $x_1 = r_2r_3$, $x_2 = r_1r_3$ and $x_3 = r_1r_2$ which are rotations around the vertices through angles $2\alpha_1\pi$, $2\alpha_2\pi$ and $2\alpha_3\pi$ and satisfying

$$x_1^{m_1} = x_2^{m_2} = x_3^{m_3} = x_1x_2x_3 = 1.$$

By the reflection principle, the function f can be continued through one side to a reflected triangle which is mapped to the lower half-plane. Another reflection will give a triangle which is mapped again to the upper half-plane. Thus f is an automorphic function for the group G, and any triangle together with an adjacent one will form a fundamental domain for G. The element x_i is an elliptic element when m_i is finite. When $m_i = \infty$, x_i is a parabolic element and the corresponding vertex (fixed point) is called a cusp and lies on the boundary of $\mathfrak{H}$ with internal angle zero. Notice that G is of genus zero and f is a Hauptmodul for G.

Examples. Let $\alpha_1 = 1/2$, $\alpha_2 = 1/3$ and $\alpha_3 = 0$. The group G is $\Gamma := \mathrm{PSL}_2(\mathbb{Z})$ and f is a normalized form of the elliptic modular function j that we denoted by J. The equation (5.4) gives

$$\{\tau, J\} = \frac{36J^2 - 41J + 32}{36J^2(J-1)^2}.$$

Another important case is when all the angles are zero. In this case all the vertices are cusps and they lie on the boundary of $\mathfrak{H}$. The associated group is $G = \Gamma(2)$, the principal congruence subgroup of level 2 which is defined in general for any positive integer n by

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \ , \ b \equiv c \equiv 0 \mod n \ , \ a \equiv d \equiv 1 \mod n \right\}.$$

The elliptic modular function λ is a Hauptmodul for $\Gamma(2)$ which sends ∞, 0 and 1 respectively to 0, 1, and ∞. From (5.4) we have

$$\{\tau, \lambda\} = \frac{\lambda^2 - \lambda + 1}{\lambda^2(\lambda - 1)^2}.$$

It is clear that λ does not take the values 0 and 1 on $\mathfrak{H}$ since they are taken on the boundary, so that $\{\tau, \lambda\}$ is holomorphic on $\mathfrak{H}$. This may also be seen from the fact that, with the above choice of triangle, there are no ramification points (elliptic fixed points) on $\mathfrak{H}$ as $\{\tau, \lambda\}$ has singularities only at these points.

Now we turn our attention to the function

$$\{\lambda, \tau\} = -\lambda'^2 \, \frac{\lambda^2 - \lambda + 1}{\lambda^2(\lambda - 1)^2}, \tag{5.5}$$

which is an automorphic form of weight 4 for the normalizer of $\Gamma(2)$ which is Γ. It would be interesting to know the behaviour of this form at the cusps. One way to do this is to write the q-expansion at the cusps explicitly , but since λ has various connections with elliptic curves and elliptic function theory, we take a different approach using theta functions.

Let ω and ω' be two primitive periods of a Weierstrass $\mathfrak{p}$ function. We set $\tau = \omega'/\omega$ and assume that $\mathrm{Im}(\tau) > 0$. We also set $t = \exp(\pi i \tau)$ so that $t^2 = q$. The elliptic curve $\mathbb{C}/(\omega\mathbb{Z} + \omega'\mathbb{Z})$ has the following Weierstrass equation

$$y^2 = 4x^3 - g_2 x - g_3 = 4(x - e_1)(x - e_2)(x - e_3),$$

where g_2 and g_3 are the classical elliptic functions (Eisenstein series):

$$g_2 = \left(\frac{2\pi}{\omega}\right)^4 \left(\frac{1}{12} + 20q + \cdots\right) , \quad g_3 = \left(\frac{2\pi}{\omega}\right)^6 \left(\frac{1}{216} - \frac{7}{3} q + \cdots\right).$$

And

$$e_1 = \mathfrak{p}\left(\frac{\omega}{2}\right) , \quad e_2 = \mathfrak{p}\left(\frac{\omega + \omega'}{2}\right) , \quad e_3 = \mathfrak{p}\left(\frac{\omega'}{2}\right).$$

The modular discriminant Δ is given by

$$\Delta = g_2^3 - 27 g_3^2 = 16(e_1 - e_2)^2(e_2 - e_3)^2(e_3 - e_1)^2.$$

The function λ is given by

$$\lambda = \frac{e_2 - e_3}{e_1 - e_3}.$$

Since Δ does not vanish on $\mathfrak{H}$, we see again that λ does not take the values 0 or 1 on $\mathfrak{H}$.

All the above can be expressed in terms of the Jacobi null theta functions defined by

$$\vartheta_2 = \sum_{-\infty}^{\infty} t^{(n+\frac{1}{2})^2}\ , \quad \vartheta_3 = \sum_{-\infty}^{\infty} t^{n^2}\ , \quad \vartheta_4 = \sum_{-\infty}^{\infty} (-1)^n t^{n^2},$$

with the following transformations:

$$\vartheta_2(\tau+1) = e^{\frac{\pi i}{4}}\,\vartheta_2(\tau) \quad \vartheta_3(\tau+1) = \vartheta_4(\tau)\ , \quad \vartheta_4(\tau+1) = \vartheta_3(\vartheta).$$

$$\begin{aligned} &\vartheta_2\left(\frac{-1}{\tau}\right) = (-i\tau)^{\frac{1}{2}}\vartheta_4(\tau),\ \vartheta_3\left(\frac{-1}{\tau}\right) = (-i\tau)^{\frac{1}{2}}\vartheta_3(\tau),\\ &\vartheta_4\left(\frac{-1}{\tau}\right) = (-i\tau)^{\frac{1}{2}}\vartheta_2(\tau), \end{aligned} \tag{5.6}$$

and the fundamental relation

$$\vartheta_2^4 + \vartheta_4^4 = \vartheta_3^4. \tag{5.7}$$

In term of these functions, the elliptic functions introduced above are given by:

$$e_1 = \frac{\pi^2}{3\omega^2}(\vartheta_3^4 + \vartheta_4^4)\ , \quad e_2 = \frac{\pi^2}{3\omega^2}(\vartheta_2^4 - \vartheta_4^4)\ , \quad e_3 = \frac{-\pi^2}{3\omega^2}(\vartheta_2^4 + \vartheta_3^4)\ ,$$

$$\Delta = 16\left(\frac{\pi}{\omega}\right)^{12}\ \vartheta_2^8\vartheta_3^8\vartheta_4^8\ ,$$

$$g_2 = \frac{2}{3}\left(\frac{2\pi}{\omega}\right)^4 (\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8)\ ,$$

$$J + 744 = 32\,\frac{(\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8)^3}{\vartheta_2^8\vartheta_3^8\vartheta_4^8},$$

$$\lambda = \frac{\vartheta_2^4}{\vartheta_3^4} = 1 - \frac{\vartheta_4^4}{\vartheta_3^4}.$$

Recall that the normalized Eisenstein series of weight 4 is given by

$$E_4(\tau) = \frac{12}{(2\pi)^4}\,g_2(\tau) = 1 + 240\sum_{n\geq 1}\sigma_3(n)q^n,$$

where $\sigma_3(n)$ is sum of the cubes of the positive divisors of n. Then

Proposition 5.2. *We have the formula*

$$\frac{1}{\pi^2}\{\lambda, \tau\} = E_4(\tau). \tag{5.8}$$

Proof. Since the expression for $\{\lambda, \tau\}$ involves λ', we need to evaluate the latter in terms of theta functions. From (5.6) we deduce that $\lambda(\tau+1) = \lambda(\tau)/(\lambda(\tau)-1)$ and $\lambda(-1/\tau) = 1 - \lambda(\tau)$. This allows us to determine the t-expansion at any cusp. In fact we can show that the weight 2 modular form λ'/λ is holomorphic at the cusps 0, 1 and ∞ for $\Gamma(2)$. The space of such forms has dimension two, and a basis is given by any two of the forms ϑ_2^4, ϑ_3^4 and ϑ_4^4. Let us write $\lambda'/\lambda = \alpha\vartheta_2^4 + \beta\vartheta_4^4$, taking this at $\tau + 1$, we obtain $\lambda'/(\lambda - 1) = \alpha\vartheta_4^4 + \beta\vartheta_2^4$. Since λ, λ' and ϑ_2 vanish at ∞, we find $\alpha = 0$. Evaluating $\lambda'/\lambda = \beta\vartheta_4^4$ at ∞ we get $\beta = \pi i$. Therefore we obtain simultaneously

$$\frac{\lambda'}{\lambda} = \pi i \vartheta_4^4\,, \quad \frac{\lambda'}{1-\lambda} = \pi i \vartheta_2^4. \tag{5.9}$$

Using these identities and the relation (5.7) in (5.4) we obtain

$$\{\lambda, \tau\} = \frac{\pi^2}{2}\,(\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8). \tag{5.10}$$

This is a weight 4 modular form Γ holomorphic at ∞. The space of such forms is 1-dimensional generated by the Eisenstein series E_4. Since the value of $\{\lambda, \tau\}$ at ∞ is π^2 the proposition follows. □

6. Expansion at the vertices

Let G be a genus zero Fuchsian group and f a Hauptmodul for G, we assume that G is finitely generated, or equivalently, that a Dirichlet region for G has a finite number of sides. Each vertex belongs to a cycle of vertices that are conjugate under the action of G. If a vertex τ_0 corresponds to an elliptic fixed point which is not a pole for f and if $f(\tau_0) = a_0$, then

$$f(\tau) = a_0 + a_1(\tau - \tau_0)^n + \cdots\,,$$

where $a_1 \neq 0$ and n is the order of the elliptic transformation fixing τ_0. Using the auxiliary function $t = (\tau - \tau_0)^n$, one can show easily that, in a neighbourhood of τ_0,

$$\{\tau, f\} = \left(1 - \frac{1}{n^2}\right)\frac{1}{(f - a_0)^2} + \frac{c_0}{f - a_0}.$$

If τ_0 is a pole, one can show that

$$\{\tau, f\} = \left(1 - \frac{1}{n^2}\right)\frac{1}{f^2} + \frac{c}{f^3} + \cdots\,,$$

which means that $\{\tau, f\}$ has a zero of order 2 at ∞.

If τ_0 is a finite cusp fixed by the transformation A in G, then in a neighbourhood of τ_0, A acts as [4,6]:

$$\frac{1}{A\tau - \tau_0} = \frac{1}{\tau - \tau_0} + c,$$

and f becomes a power series in the variable w where

$$\frac{1}{\tau - \tau_0} = \frac{c}{2\pi i} \log w.$$

This transformation essentially sends the parabolic sector at τ_0 onto an infinite strip. It follows that

$$\{\tau, f\} = \frac{1}{(f-a)^2} + \frac{c}{f-a} + \cdots, \quad a = f(\tau_0).$$

If f has a pole at a cusp, or if the pole is inside a fundamental region, it is not difficult to see that $\{\tau, f\}$ has a zero of order 2 or 4 respectively. In the same way as in the triangular case in Sect. 5, we can show that

$$\{\tau, f\} = \sum_{k=1}^{n} \left(1 - \frac{1}{{n_k}^2}\right) \frac{1}{(f-a_k)^2} + \sum_{k=1}^{n} \frac{c_k}{f-a_k}, \tag{6.1}$$

where the sum is taken over a set of vertices, one from each cycle, and a_k are the values of f at these cycles. If one of the vertices is a pole, then n must be replaced by $n-1$. Each integer n_k is the order of the transformation fixing the corresponding vertex with the convention that $n_k = \infty$ at a cusp. Also, n_k can be taken such that $2\pi/n_k$ is the sum of the internal angles of the given cycle, which is in accord with the triangular case of the previous section.

In the following we focus on the behaviour of the form $\{f, \tau\}$ at the cusps. This will be based on the identity $\{f, \tau\} = -(f')^2 \{\tau, f\}$ and on what we have learnt about $\{\tau, f\}$. Near a cusp, τ_0, for G, we have a local uniformizer of the form

$$w = \exp \frac{2\pi i}{c(\tau - \tau_0)}$$

if τ_0 is a finite vertex, and

$$w = \exp \frac{2\pi i}{c}\, \tau$$

if the cusp is at ∞. In either case, the Hauptmodul f is holomorphic or has a pole at $w = 0$. If we assume that f is holomorphic at τ_0, then near 0, we have the expansion

$$f(\tau) = f(\tau_0) + a_1 w + a_2 w^2 + \cdots,$$

whereas

$$f'(\tau) = \frac{dw}{d\tau}\, (a_1 + 2a_2 w + \cdots) = \frac{-2\pi i}{c(\tau - \tau_0)^2}\, \left(a_1 w + 2a_2 w^2 + \cdots\right).$$

It follows that $(\tau - \tau_0)^4 \{f, \tau\}$ is a power series in w. This is just the growth condition for the holomorphy of the weight 4 automorphic form at the cusp τ_0 [6]. If the cusp is at ∞, then the local parameter is

$$w = \exp \frac{2\pi i}{c} \tau$$

at which the holomorphy of $\{f, \tau\}$ is easily seen (and can be seen also from (4.8)).

At elliptic points the form $\{f, \tau\}$ has a pole of order 2 which comes from $f'^2/(f-a)^2$. Summing up, we have

Proposition 6.1. *At elliptic points the weight 4 form $\{f, \tau\}$ has a pole of order two and is holomorphic elsewhere including at parabolic points.* □

7. Congruence subgroups with no elliptic elements and lattices

The congruence groups $\Gamma_0(n)$ $(n \geq 1)$ are defined by

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \ , \quad c \equiv 0 \mod n \right\} .$$

Only their images in $\mathrm{PSL}_2(\mathbb{R})$ are used in this note; we use the same notation to designate the corresponding group of Möbius transformations.

We will restrict this section to the family of Fuchsian groups G which satisfy the following conditions

(1) G is a genus zero group.
(2) G contains some $\Gamma_0(n)$ with finite index.
(3) The stabilizer of the cusp at ∞ is generated by $\tau \to \tau + 1$.

According to [11], there are only finitely many Fuchsian groups satisfying (1), (2) and (3). From the previous section, the Schwarzian of a Hauptmodul for such a group G is a holomorphic weight 4 form if and only if

(4) G contains no elliptic elements.

Our aim is to first identify the groups satisfying the four conditions. Next we identify these holomorphic weight 4 forms as classical theta functions for some rank 8 lattices.

In this section we deal with the groups of the form $\Gamma_0(n)$.

Proposition 7.1. *The integers n for which $\Gamma_0(n)$ has genus zero and does not contain any elliptic elements are $n = 4, 6, 8, 9, 12, 16$ and 18.*

Proof. According to [5], there are only 15 positive integers n for which the groups $\Gamma_0(n)$ have genus zero, namely for $n = 1, \ldots, 10, 12, 13, 16, 18, 25$. We now look at condition (4). Let

$$\begin{pmatrix} a & b \\ nc & d \end{pmatrix} \in \Gamma_0(n)\,, \quad ad - nbc = 1\,,$$

be an elliptic element. If the order of this element is 2 then the trace is 0, thus $a + d = 0$ and $a^2 \equiv -1 \mod n$. If the order is 3 then the trace is ± 1, and from $(a+d)^2 = 1$ and $ad \equiv 1 \mod n$ we deduce that $(a-d)^2 \equiv -3 \mod n$. It follows that if -1 and -3 are not squares mod n, then there are no elliptic elements in $\Gamma_0(n)$. Conversely, if -1 or -3 is a square it is clear how to construct elliptic elements; for example, if $a^2 = -1$, take $c = 1$, $d = -a$ and $b = 0$, and the matrix obtained is an elliptic element of order 2. It is easy to see from the genus zero list that $\Gamma_0(n)$ has no elliptic elements if and only if $n = 4, 6, 8, 9, 12, 16$ and 18. □

For $n = 2, 3, 4$, the transformation $\tau \to \tau/n$ conjugates the group $\Gamma_0(n^2)$ to $\Gamma(n)$, the principal congruence subgroup, and in fact, by examining the indices of these groups inside the full modular group Γ, these values are the only one for which $\Gamma_0(n^2)$ is conjugate to $\Gamma(n)$, except for $n = 6$ in which case $\Gamma(6)$ (or $\Gamma_0(36)$) is no longer of genus zero. If $f_n(\tau)$ is a Hauptmodul for $\Gamma_0(n^2)$ for $n = 2, 3, 4$, then $\tilde{f}_n(\tau) = f_n(\tau/n)$ is a Hauptmodul for $\Gamma(n)$. The Schwarzian $\{\tilde{f}_n, \tau\}$ is a weight 4 holomorphic modular form for the normalizer, Γ, of $\Gamma(n)$. The space of such forms is 1-dimensional, generated by the Eisenstein series E_4. Using (4.8) with the appropriate width at ∞, we have

Proposition 7.2. *For $n = 2, 3, 4$, we have*

$$\frac{n^2}{4\pi^2}\,\{\tilde{f}_n, \tau\} \;=\; E_4(\tau). \tag{7.1}$$

$$\frac{1}{4\pi^2}\,\{f_n, \tau\} \;=\; E_4(n\tau). \qquad \square \tag{7.2}$$

These formulas are presented as illustrations of (4.8) and as a generalization of (5.8). Notice that explicit knowledge of the Hauptmoduls is not necessary. In the normalized form (4.1), we have the Hauptmoduls:

$$\begin{aligned}
f_2(\tau) &= \left(\frac{\eta(\tau)}{\eta(4\tau)}\right)^8 + 8 = \frac{16}{\lambda(2\tau)} - 8 \\
&= q^{-1} + 20q - 62q^3 + 216q^5 - 641q^7 + \cdots\,, \\
f_3(\tau) &= \left(\frac{\eta(\tau)}{\eta(9\tau)}\right)^3 + 3 \\
&= q^{-1} + 5q - 7q^2 + 3q^5 + 15q^8 + \cdots\,, \\
f_4(\tau) &= \frac{\eta(8\tau)^6}{\eta(4\tau)^2\eta(16\tau)^4} \\
&= q^{-1} + 2q^3 - q^7 - 2q^{11} + 3q^{15} + 2q^{19} - 4q^{23} + \cdots\,,
\end{aligned}$$

where η is the Dedekind eta-function:

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n\geq 1}(1-q^n).$$

The Schwarzian of each of these functions is invariant for a conjugate of Γ, and it is given in terms of E_4 which is nothing but the theta function of the root lattice $\mathbf{E}_8$. We will see next that all the other groups involve different root lattices. We need the following

Lemma 7.3. *Let $\Gamma_0(n)^+$ denotes the group $\Gamma_0(n)$ extended by all its Atkin-Lehner involutions. Let $\mathfrak{M}_k(n)$ denote the space of weight k holomorphic automorphic forms for $\Gamma_0(n)^+$. Then*

$$dim\, \mathfrak{M}_k(n) = \frac{1}{12}n(k-1) - \frac{1}{3}\chi_3(n(k-1)) - \frac{1}{4}\chi_4(n(k-1)) + \frac{1}{2}a, \quad (7.3)$$

where $n = a^2 b$ with b square-free, and χ_3 and χ_4 are the primitive Dirichlet characters modulo 3 *and* 4.

Proof. See formula (5) in [10]. □

To obtain the normalizer of $\Gamma_0(n)$ we let h be the largest divisor of 24 for which h^2 divides n, then the normalizer is the conjugate of $\Gamma_0(n/h)+$ by $\tau \to h\tau$. The group $\Gamma_0(n/h)+$ is referred to as the Helling group corresponding to $\Gamma_0(n)$.

Proposition 7.4. *Let f_{12} be a Hauptmodul for $\Gamma_0(12)$, then*

$$\frac{1}{4\pi^2}\{f_{12}, \tau\} = \theta_{4A_2}(2\tau), \quad (7.4)$$

where $\theta_{4A_2}(\tau)$ is the theta function of four copies of the hexagonal lattice $\mathbf{A}_2$.

Proof. The theta function of the $\mathbf{A}_2$ lattice is [3]: $\vartheta_3(\tau)\vartheta_3(3\tau) + \vartheta_2(\tau)\vartheta_2(3\tau)$, therefore

$$\theta_{4A_2}(2\tau) = (\vartheta_3(2\tau)\vartheta_3(6\tau) + \vartheta_2(2\tau)\vartheta_2(6\tau))^4. \quad (7.5)$$

On the other hand, the normalizer of $\Gamma_0(12)$ is $\Gamma_0(3)^+$. Using Lemma 7.3., we have $\dim \mathfrak{M}_k(3) = 1$. Using (5.6), we can see easily that $\theta_{4A_2}(2\tau)$ given by (7.5), belongs to $\mathfrak{M}_4(3)$ and has in the q-expansion a constant term (value at ∞) equal to 1. If f_{12} is a Hauptmodul for $\Gamma_0(12)$ then, by (4.8), $\{f_{12}, \tau\}$ has constant term $4\pi^2$. The proposition follows. □

The normalized Hauptmodul (in the form (4.1)) for $\Gamma_0(12)$ is given by

$$\begin{aligned} f_{12}(\tau) &= \frac{\eta(4\tau)^4\eta(6\tau)^2}{\eta(2\tau)^2\eta(12\tau)^4} \\ &= q^{-1} + 2q + q^3 - 2q^7 - 2q^9 \\ &\quad + 2q^{11} + 4q^{13} + 3q^{15} - 4q^{17} + \cdots, \end{aligned}$$

and

$$\frac{1}{4\pi^2}\{f_{12},\tau\} = (\vartheta_3(4\tau)\vartheta_3(12\tau) + \vartheta_2(4\tau)\vartheta_2(12\tau))^4$$
$$= 1 + 24q^2 + 216q^4 + 888q^6 + 1752q^8$$
$$+ 3024q^{10} + 7992q^{12} + \cdots .$$

Proposition 7.5. *Let f_8 and f_{18} be Hauptmoduls for $\Gamma_0(8)$ and $\Gamma_0(18)$ respectively, and let θ_{2D_4} be the theta function of two copies of the (Hurwitz quaternionic) root lattice $\mathbf{D}_4$, then*

$$\frac{1}{4\pi^2}\{f_8,\tau\} = \theta_{2D_4}(2\tau), \tag{7.6}$$

and

$$\frac{1}{4\pi^2}\{f_{18},\tau\} = \theta_{2D_4}(3\tau), \tag{7.7}$$

Proof. The groups $\Gamma_0(8)$ and $\Gamma_0(18)$ both correspond to the same Helling group $\Gamma_0(2)^+$ and are both normal in it, in fact their normalizers are both conjugate to $\Gamma_0(2)^+$ via the maps $\tau \to 2\tau$ and $\tau \to 3\tau$ respectively. The theta function of the root lattice $\mathbf{D}_4$ is $\frac{1}{2}(\vartheta_3^4(\tau) + \vartheta_4^4(\tau))$, hence the theta function of $\mathbf{D}_4 \oplus \mathbf{D}_4$ is

$$\theta_{2D_4}(\tau) = \frac{1}{4}(\vartheta_3^4(\tau) + \vartheta_4^4(\tau))^2,$$

which belongs $\mathfrak{M}_4(2)$. This space is 1-dimensional by Lemma 7.3.. Taking into account the conjugating maps between $\Gamma_0(2)^+$ and the normalizers of $\Gamma_0(8)$ and $\Gamma_0(18)$, we deduce the relations (7.6) and (7.7). □

The normalized Hauptmodul for $\Gamma_0(8)$ is

$$f_8(\tau) = \frac{\eta(4\tau)^{12}}{\eta(2\tau)^4\eta(8\tau)^8}$$
$$= q^{-1} + 4q + 2q^3 - 8q^5 - q^7 + 20q^9 - 2q^{11} + 9q^{14} + \cdots ,$$

and

$$\frac{1}{4\pi^2}\{f_8,\tau\} = \frac{1}{4}(\vartheta_3^4(2\tau) + \vartheta_4^4(2\tau))^2$$
$$= 1 + 48q^2 + 624q^4 + 1344q^6 + 5232q^8 + \cdots .$$

The normalized Hauptmodul for $\Gamma_0(18)$ is

$$f_{18}(\tau) = \frac{\eta(6\tau)\eta(9\tau)^3}{\eta(3\tau)\eta(18\tau)^3}$$
$$= q^{-1} + q^2 + q^5 - q^8 - q^{11} + q^{17} + 2q^{20} + \cdots ,$$

and

$$\frac{1}{4\pi^2}\{f_{18},\tau\} = \frac{1}{4}(\vartheta_3^4(3\tau)+\vartheta_4^4(3\tau))^2$$
$$= 1+48q^3+624q^6+1344q^9+5232q^{12}+\cdots .$$

Proposition 7.6. *Let f_6 be a Hauptmodul for $\Gamma_0(6)$, and let $\theta_{A_2\otimes D_4}$ be the theta function of the lattice* $\mathbf{A}_2\otimes\mathbf{D}_4$*, then*

$$\frac{1}{4\pi^2}\{f_6,\tau\} = \theta_{A_2\otimes D_4}(\tau). \tag{7.8}$$

Proof. According to [7], the theta function of $\mathbf{A}_2\otimes\mathbf{D}_4$ is given by

$$\theta_{A_2\otimes D_4}(\tau) = h_1^2(1-4h_2-16h_2^3+16h_2^4) = \theta_{G_2\otimes F_4}(\tau), \tag{7.9}$$

where

$$h_1(\tau) = (\vartheta_3(\tau)\vartheta_3(2\tau)\vartheta_3(3\tau)\vartheta_3(6\tau))^2, \quad \text{and}$$
$$h_2(\tau) = \frac{\eta(\tau/2)\eta(3\tau/2)\eta(4\tau)\eta(12\tau)}{\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)}.$$

Neither h_1 nor h_2 is invariant under $\Gamma_0(6)^+$ but the product in (7.9) is. Using Lemma 7.3., we see that the space $\mathfrak{M}_4(6)$ is 2-dimensional and a basis of this space is obtained from $\theta_{A_2\otimes D_4}$ and the cusp form $(\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2$ which has the following q-expansion $q-2q^2-3q^3+4q^4+6q^5+\cdots$. On the other hand, the q-expansions of $\frac{1}{4\pi^2}\{f_6,\tau\}$ and $\theta_{A_2\otimes D_4}$ do not contain any term in q, see (4.8), and both of them have constant term 1, the proposition follows. □

The normalized Hauptmodul for $\Gamma_0(6)$ is

$$f_6(\tau) = 5+\frac{\eta(\tau)^5\eta(3\tau)}{\eta(2\tau)\eta(6\tau)^5}$$
$$= q^{-1}+6q+4q^2+-3q^3-12q^4-8q^5+12q^6+\cdots ,$$

and

$$\frac{1}{4\pi^2}\{f_6,\tau\} = 1+72q^2+192q^3+504q^4+576q^5+2280q^6+\cdots$$

Remark 7.1. We have exhausted the list of all the groups $\Gamma_0(n)$ which are of genus zero and contain no elliptic elements, in other words, those which can be generated by parabolic elements. The Schwarzians of their Hauptmoduls are theta functions of rank 8 root lattices with various minimal norms. In the next section, we will see that this property holds only for these groups as we will exhibit what we believe to be a complete list of all the Fuchsian groups satisfying the conditions (1)–(4).

8. Some more groups

For each n in the list given in Proposition 7.1., the group $\Gamma_0(n)$ has a single conjugate which also satisfies the conditions (1)–(4) of the previous section. These groups have Hauptmoduls whose Schwarzian is given by a theta function with a shifted argument. We use the notation $\Gamma_0'(n)$ for the conjugate of $\Gamma_0(n)$. The conjugation maps, the resulting groups, their Hauptmoduls, and their Schwarzians are given in the following table:

Conjugation	*Group*	*Hauptmodul f*	$\frac{1}{4\pi^2}\{f,\tau\}$
$\tau \to \tau + \frac{1}{4}$	$\Gamma_0'(4)$	$\left(\frac{\eta(\tau+\frac{1}{4})}{\eta(4\tau)}\right)^8 + 8i$	$E_4\left(2\tau + \frac{1}{2}\right)$
$\tau \to \tau + \frac{1}{2}$	$\Gamma_0'(6)$	$\frac{\eta(\tau)^4\eta(4\tau)^4\eta(6\tau)^4}{\eta(2\tau)^4\eta(3\tau)^4\eta(12\tau)^4} + 4$	$\theta_{A_2\otimes D_4}\left(\tau + \frac{1}{2}\right)$
$\tau \to \tau + \frac{1}{4}$	$\Gamma_0'(8)$	$\frac{\eta(2\tau)^4}{\eta(8\tau)^4}$	$\theta_{2D_4}\left(2\tau + \frac{1}{2}\right)$
$\tau \to \tau + \frac{1}{2}$	$\Gamma_0'(9)$	$\frac{\eta(2\tau)^9\eta(9\tau)^3\eta(36\tau)^3}{\eta(\tau)^3\eta(4\tau)^3\eta(18\tau)^9} - 3$	$E_4\left(3\tau + \frac{1}{2}\right)$
$\tau \to \tau + \frac{1}{4}$	$\Gamma_0'(12)$	$\frac{\eta(2\tau)^2\eta(8\tau)^2\eta(12\tau)^2}{\eta(4\tau)^2\eta(6\tau)^2\eta(24\tau)^2}$	$\theta_{4A_2}\left(2\tau + \frac{1}{2}\right)$
$\tau \to \tau + \frac{1}{8}$	$\Gamma_0'(16)$	$\frac{\eta(4\tau)^2}{\eta(16\tau)^2}$	$E_4\left(4\tau + \frac{1}{2}\right)$
$\tau \to \tau + \frac{1}{2}$	$\Gamma_0'(18)$	$\frac{\eta(3\tau)\eta(12\tau)\eta(18\tau)^6}{\eta(6\tau)^2\eta(9\tau)^3\eta(36\tau)^3}$	$\theta_{2D_4}\left(3\tau + \frac{1}{2}\right)$

There are three groups which are not conjugate to any $\Gamma_0(n)$ but still satisfy the conditions (1)–(4). Namely, the group that we denote by $G_{27|3}$ which is the invariance group of the Hauptmodul $f_{27|3}(\tau) = \eta(3\tau)/\eta(27\tau)$; it is conjugate to a subgroup of index 3 in $\Gamma_0(9)$ (containing $\Gamma_0(81)$). The second group is $G_{32|8}$ which is the invariance group of $f_{32|8}(\tau) = \eta(8\tau)/\eta(32\tau)$; it is conjugate to a subgroup of index 8 in $\Gamma_0(4)$. The third group is $G'_{27|3}$, a conjugate to $G_{27|3}$ via the map $\tau \to \tau + 1/2$ which also satisfies the conditions (1)–(4); its normalized Hauptmodul is given by:

$$f_{27|3'}(\tau) = \frac{\eta(6\tau)^3\eta(27\tau)\eta(108\tau)}{\eta(3\tau)\eta(12\tau)\eta(54\tau)^3}.$$

Proposition 8.1. *We have*

$$\frac{1}{4\pi^2}\{f_{32|8},\tau\} = \frac{21}{5}E_4\left(8\tau+\frac{1}{2}\right) - \frac{16}{5}E_4(16\tau). \tag{8.1}$$

Proof. The normalizer of the group $G_{32|8}$ is $\Gamma_0(4)^+$, which is conjugate to $\Gamma_0(2)$ via the map $\tau \to 8\tau + 1/2$. On the other hand, the space of weight 4 holomorphic modular forms for $\Gamma_0(2)$ is 2-dimensional generated by $E_4(\tau)$ and $E_4(2\tau)$. The proposition follows from the knowledge of the first two q-coefficients of the series in (8.1). □

Proposition 8.2. *We have*

$$\frac{1}{4\pi^2}\{f_{27|3},\tau\} = E_4(3\tau) - 48\eta(3\tau)^8 - 216\,\frac{\eta(\tau)^6\eta(9\tau)^6}{\eta(3\tau)^4}. \tag{8.2}$$

Proof. The normalizer of $G_{27|3}$ is $\Gamma_0(9)^+$ in which $\Gamma(3)$ has index 2. The space of weight 4 holomorphic modular forms for $\Gamma_0(9)^+$ is 3-dimensional by Lemma 7.3., and a basis is given by the three forms in the right side of (8.2). The proposition follows from the knowledge of the few first q-coefficients. □

We have the following q-expansions:

$$\begin{aligned} f_{27|3}(\tau) &= \frac{\eta(3\tau)}{\eta(27\tau)} \\ &= q^{-1} - q^2 - q^5 + q^{14} + q^{20} + \cdots, \\ f_{27|3'}(\tau) &= \frac{\eta(6\tau)^3\eta(27\tau)\eta(108\tau)}{\eta(3\tau)\eta(12\tau)\eta(54\tau)^3} \\ &= q^{-1} + q^2 - q^5 - q^{14} - q^{20} + \cdots, \\ f_{32|8}(\tau) &= \frac{\eta(8\tau)}{\eta(32\tau)} \\ &= q^{-1} - q^7 - q^{15} + \cdots, \end{aligned}$$

and

$$\begin{aligned} \frac{1}{4\pi^2}\{f_{27|3},\tau\} &= 1 - 48q^3 - 216q^6 + 1536q^9 \\ &\quad - 1560q^{12} - 3024q^{15} + \cdots, \\ \frac{1}{4\pi^2}\{f_{27|3'},\tau\} &= 1 + 48q^3 - 216q^6 - 1536q^9 \\ &\quad - 1560q^{12} + 3024q^{15} + \cdots, \\ \frac{1}{4\pi^2}\{f_{32|8},\tau\} &= 1 - 1008q^8 + 8304q^{16} - 28224q^{24} + \cdots. \end{aligned}$$

We mention that these Schwarzians are not lattice theta functions since some coefficients are negative and there is no other group which is conjugate to one of the above three groups giving a theta function (or at least making all the coefficients nonnegative). This means that the property that the Schwarzian is given by a theta function of a rank 8 lattice holds only for those groups satisfying (1)–(4) that are $\Gamma_0(n)$ up to conjugacy. This completes our description of the Schwarzian of Hauptmoduls of Fuchsian groups satisfying the conditions (1)–(4), namely Fuchsian groups of genus zero and with noelliptic elements, containing some $\Gamma_0(n)$ with finite index and such that the width of the cusp at ∞ is 1. The above 17 groups are characterized by the fact that they are the only ones, among those satisfying the conditions (1)-(4), with normalized Hauptmoduls having rational Fourier coefficients. Indeed, as we have seen, they are all given by eta-products. A complete classification of the groups satisfying the conditions (1)-(4) will appear elsewhere.

Acknowledgements. We thank Simon Norton, Oliver Atkin and the members of CICMA for helpful discussions.

References

1. Atkin, A.O.L., Lehner, J.: Hecke operators on $\Gamma_0(m)$. Math. Ann. **185** (1970) 134–160
2. Conway, J.H., Norton, S.P.: Monstrous moonshine. Bull. London Math. Soc. **11** (1979), no. 3, 308–339
3. Conway, J.H., Sloane, N.J.A.: Sphere Packings, Lattices and Groups. Grundlehren der Mathematischen Wissenschaften Springer, New York, 1993
4. Ford, L.R.: Automorphic functions. McGraw-Hill 1929
5. Ogg, Andrew P.: Hyperelliptic modular curves. Bull. Soc. Math. France **102** (1974), 449–462
6. Rankin, Robert A.: Modular forms and functions. Cambridge University Press, 1977
7. Rains, E., Sloane, N.J.A.; Shadow theory of modular and unimodular lattices (1998) (preprint)
8. Schwarz, H.A.: Gesammelte Mathematische Abhandlungen, vol. 2, Berlin, 1880
9. Sansone, G., Gerretsen, J.: Lectures on the theory of functions of a complex variable. II: Geometric theory. Wolters-Noordhoff Publishing, Groningen 1969
10. Skoruppa, N.-P., Zagier, D.: Jacobi forms and a certain space of modular forms. Invent. Math. **94** (1988), no. 1, 113–146
11. Thompson, J.G.: A finiteness theorem for subgroups of $\mathrm{PSL}(2, \mathbf{R})$ which are commensurable with $\mathrm{PSL}(2, \mathbf{Z})$. The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979), pp. 533–555, Proc. Symp. Pure Math. **37**, Amer. Math. Soc., Providence, R.I., 1980