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Equivariant functions and integrals of elliptic functions

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Abstract In this paper, we introduce the theory of equivariant functions by studying their analytic, geometric and algebraic properties. We also determine the necessary and sufficient conditions under which an equivariant form arises from modular forms. This study was motivated by observing examples of functions for which the Schwarzian derivative is a modular form on a discrete group. We also investigate the Fourier expansions of normalized equivariant functions, and a strong emphasis is made on the connections to elliptic functions and their integrals.

Keywords Equivariant functions · Schwarz derivative · Cross-ratio · Modular forms · Platonic solids · Integrals of elliptic functions

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1 Introduction

Though the problem we are studying is analytic and geometric in its nature, it can be given a general algebraic formulation as follows: Let G be a group acting on two sets X, Y and let S be a set of functions from X to Y on which G acts in the following way

$$g.f(x) = g.f(g^{-1}.x)\,, \quad \text{for all } g \in G\,,\ f \in S\,,\ x \in X.$$

An equivariant function $f : X \to Y$ (also sometimes called concomitant) is a function such that $g.f = f$, that is

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$$f(g.x) = g.f(x),\ \forall g \in G, \forall x \in X.$$

If G acts trivially on Y, then f is called an invariant. We propose to illustrate this problem in the following setting which encompasses arithmetic, analytic and geometric flavors.

A subgroup Γ of the modular group $\mathrm{SL}_2(\mathbb{Z})$ acts on the upper half-plane

$$\mathbf{H} = \{z \in \mathbb{C},\ \Im z > 0\},$$

by linear fractional transformation

$$\gamma \cdot z = \frac{az+b}{cz+d}, \quad z \in \mathbf{H},\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

We would like to investigate the class of meromorphic functions h on $\mathbf{H}$ which commute with this action. In other words, h satisfies the equivariance relation

$$h\left(\frac{az+b}{cz+d}\right) = \frac{ah(z)+b}{ch(z)+d}, \quad z \in \mathbf{H}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \tag{1.1}$$

These functions first appeared in the works [5] and [34] in the 1960s. Our interest in these functions began with the study of the modular properties of the Schwarz derivative. If f is a modular function on a Fuchsian group G of the first kind, then its Schwarz derivative $\{f, z\}$ is a weight 4 modular form on a group which is larger than G. In particular if G has genus 0, then this larger group is the normalizer of G in $\mathrm{PSL}_2(\mathbb{R})$. One formulates the converse to this statement as follows: If a meromorphic function f on the upper half-plane $\mathbf{H}$, with some favorable conditions at the cusps, is such that its Schwarz derivative $\{f, z\}$ is a weight 4 modular form on a certain group G, what can we say about f itself in terms of invariance? Is f a modular function under a nontrivial subgroup of G? How is the size of this subgroup dependent on the analytic properties of f?

This problem is equivalent to the following: Let f be a meromorphic function on $\mathbf{H}$. Let G be a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ such that for all $z \in \mathbf{H}$ and for all $\gamma \in G$ we have

$$f(\gamma \cdot z) = \Phi_\gamma \cdot f(z),$$

where Φ_γ is a certain matrix in $\mathrm{PGL}_2(\mathbb{C})$. It is clear that Φ is a group homomorphism. The problem is to determine the kernel of Φ depending on analytic properties of f and/or geometric properties of G. When the homomorphism Φ is the identity, then the function f satisfies the relation (1.1) for the group G.

If a function h satisfies the relation (1.1) and such that $h(z) - z$ is meromorphic at the cusps, then h will be called an equivariant form. The group Γ can be taken to be any discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$, but for the time being, only the modular group will be under our consideration.

It turns out that to each meromorphic modular form f of weight k for Γ, one can associate an equivariant form h as follows

$$h(z) = z + k\frac{f(z)}{f'(z)}. \tag{1.2}$$

In fact, we determine the necessary and sufficient conditions for an equivariant form to arise as in (1.2) and this will be called a rational equivariant form. We will also exhibit equivariant forms that are not rational.

Further investigations of an equivariant function h is carried out by looking at the Fourier coefficients of the periodic function $h(z) - z$, the Lambert series of $\dfrac{1}{h(z) - z}$ or at the infinite product of the modular form attached to h in the sense of Borcherds. This will be made explicit for equivariant functions attached to classical modular forms such as the Eisenstein series. The same examples are used in studying the Fourier coefficients of the reciprocal of the classical Eisenstein series as was carried out by Hardy, Ramanujan, and more recently by Berndt and Bialek among others.

Beside the modular aspect in the structure of equivariant functions, there is also a fascinating elliptic aspect to them. Indeed, the fundamental example of the equivariant function attached to the weight 12 cusp form Δ given by

$$h_1(z) = z + 12\frac{\Delta}{\Delta'} = z + \frac{6}{i\pi E_2(z)},$$

where E_2 is the weight 2 Eisenstein series, is closely related to the Weierstrass ζ-function where $\zeta' = \wp$ and $\wp$ is the classical Weierstrass elliptic function. In fact, as was observed by Heins [15], if $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is a lattice with $\tau = \omega_2/\omega_1 \in \mathbf{H}$, and if η_1 and η_2 are the pseudo-periods of ζ, then $\omega_1\eta_2$, as a function of τ, is an equivariant function for $\mathrm{SL}_2(\mathbb{Z})$ by way of the Legendre relation. It turns out that this equivariant function is nothing else but the fundamental example h_1 above. We will show that, with two exceptions, the integral of each $\wp^n$, $n \in \mathbb{Z}$, is an equivariant function.

From the differential algebra point of view, we have the important feature that each equivariant form satisfies a differential equation of degree at most 6; a fact that one expects from a function that satisfies sufficiently many functional equations. To justify this property, one goes back to the differential ring of modular forms and their derivatives, also known as the ring of quasi-modular forms, which is simply $\mathbb{C}[E_2, E_4, E_6]$, and thus has transcendence degree 3. One more time, when we specify to explicit examples of equivariant forms coming from the Eisenstein series, we find important differential properties of the reciprocal of E_2, E_4 and E_6. Namely, using a theorem of Maillet, we show that $\frac{1}{E_2}$, $\frac{1}{E_4}$ and $\frac{1}{E_6}$ satisfy algebraic differential equations over $\mathbb{Q}$. The fact that the equivariant functions are differentially algebraic enables us to use theorems, well known in transcendence theory, such as those of Maillet and Popken, to control gaps or growth coefficients, in the expansion of these functions in q-series. The same argument is also valid for the reciprocal of E_2, E_4 and E_6 completing, in some sense, the previous work of Hardy, Ramanujan and more recently of Berndt and Bialek.

Beside the modular, the elliptic and the differential aspects of equivariant functions, the bulk of this work basically articulates on three main axes: the link between the cross-ratio and the Schwarz derivative, then between the equivariance and the Schwarz derivative and finally between equivariance and the cross-ratio. This scheme is a consequence of the following intriguing facts:

- The Schwarz derivative is simply the infinitesimal counterpart of the cross-ratio.
- The Schwarz derivative of an equivariant function is a weight four modular form.
- The Riccati equation is closely related to the Schwarz differential equation.
- The cross-ratio of four solutions to the Riccati equation is a constant in the field $\mathbb{C}$.
- The cross-ration of four equivariant functions is a modular function, that is in the function field of a compact Riemann surface.

All these connections make the equivariant functions extremely rich objects to study.

Though the study of equivariant functions was undertaken since the 1960s by M. Heins and M. Brady in the framework of elliptic functions, we learned only recently from D. Zagier that the fundamental example h_1 was known by W. Nahm in connection with some physical problems. We cite in this context, in Sect. 12, some examples of a special kind of equivariance, named platonic, that appeared in the Physics literature.

2 $\mathrm{SL}_2(\mathbb{R})$, the Riccati equation, the Schwarz derivative and the cross-ratio

The Riccati equation is naturally related to the Schwarz differential equation through a change of function. In this section, we will exhibit similar relations between the Riccati equation and the functional equations satisfied by equivariant functions. Many properties of these nonlinear equations have their origins in the projective differential geometry of the special linear group $\mathrm{SL}_2(\mathbb{R})$ that we recall now. The associated Lie algebra $\mathfrak{sl}(2)$ is three dimensional with basis

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The associated infinitesimal generators are $\partial_x,\ x\partial_x,\ -x^2\partial_x$, which have the same commutation rules as A_1, A_2, A_3. The group $\mathrm{SL}_2(\mathbb{R})$ acts on the real line as the projective group

$$x \in \mathbb{R} \to \frac{ax+b}{cx+d} \in \mathbb{R}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

To give a general idea about the link between the group $\mathrm{SL}_2(\mathbb{R})$ and nonlinear differential equations, we first observe that if $f_t : \mathbb{R} \to \mathbb{R}$ is a smooth function with $f_0(x) = x$ then $f_t(x) = x + tX(x) + \mathrm{O}(t^2)$ so that X is the infinitesimal generator vector field associated to f_t. In the particular case where f_t is the projective transformation associated with

$$M(t) = \begin{pmatrix} 1+ta & tb \\ tc & 1+td \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}),$$

we have

$$f_t(x) = \frac{(1+ta)x + tb}{tcx + (1+td)} = x + t\left(b + (a-d)x - cx^2\right) + \mathrm{O}(t^2)$$

and the associated vector field is $X = \left(b + (a-d)x - cx^2\right)\dfrac{\partial}{\partial x}$. We note that for the linear differential system

$$\begin{pmatrix} \dot u(t) \\ \dot v(t) \end{pmatrix} = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$$

the quotient $x = \dfrac{u}{v}$ satisfies the Riccati equation

$$\dot x(t) = b(t) + (a(t) - d(t))x(t) - c(t)x(t)^2$$

which can be considered as a differential equation for the integral curve of the vector field $X = \left(b + (a-d)x - cx^2\right)\dfrac{\partial}{\partial x}$. It is important to notice here that the Riccati equation is the only first order nonlinear ordinary differential equation which possesses the Painlevé property, that is of not having removable singularities. Its link with the cross-ratio, and hence projective geometry, was shown by Lie [21]. We recall that the cross-ratio of four different complex numbers is defined as

$$[z_1, z_2, z_3, z_4] = \frac{(z_2 - z_1)(z_4 - z_3)}{(z_3 - z_1)(z_4 - z_2)}.$$

If u_1, u_2, u_3, u are four solutions of the Riccati equation, then their cross-ratio is constant

$$\frac{(u-u_1)(u_2-u_3)}{(u_1-u_2)(u_3-u)} = k\,,$$

leading to what is called the superposition formula that will be considered later:

$$u = \frac{u_3(u_2-u_1)+k(u_1-u_3)u_2}{u_2-u_1+k(u_1-u_3)}. \tag{2.1}$$

The Riccati equation is also the only ordinary nonlinear differential equation of first order which possesses such nonlinear superposition formula. As we will see later, this superposition formula is similar to one for the equivariant functions once we substitute the field of complex numbers by the function field of a Riemann surface.

We now introduce the Schwartz derivative (or Schwarzian). This is a differential operator introduced by Schwarz in [31] in the context of differential equations and quadratic differentials. It is defined for meromorphic functions over regions of the complex plane by

$$\{f,z\} = 2\left(\frac{f''}{f'}\right)' - \left(\frac{f''}{f'}\right)^2 = \frac{1}{f'^2}(2f'f''' - 3f''^2). \tag{2.2}$$

If w is a function of z, then the Schwarzian satisfies the chain rule:

$$\{f,z\} = \{f,w\}(dw/dz)^2 + \{w,z\}.$$

Moreover, If f is a linear fractional transformation of z, then $\{f,z\}=0$. As a consequence, if $w'(z_0)\neq 0$ for some point z_0, then in a neighborhood of this point, the inverse function $z(w)$ satisfies

$$\{z,w\} = -\{w,z\}(dz/dw)^2.$$

More significant properties of the Schwarzian follow from its close relationship with second order differential equations. Indeed, let y_1 and y_2 be two linearly independent solutions to

$$y'' + \frac{1}{4}R(z)y = 0, \tag{2.3}$$

where $R(z)$ is a meromorphic function on a certain domain. Then the quotient $f=y_1/y_2$ satisfies $\{f,z\}=R(z)$. Conversely, if f is a locally univalent function satisfying $\{f,z\}=R(z)$, then $y_1 = f/\sqrt{f'}$ and $y_2 = 1/\sqrt{f'}$ are two linearly independent solutions to (2.3). As a consequence,

Proposition 2.1 *We have*

(1) $\{f,z\}=0$ *if and only if* f *is a linear fractional transformation.*
(2) $\{f,z\}=\{g,z\}$ *if and only if each function is a linear fraction of the other.*

As a corollary of the above, we have

$$\{f,z\} = \left\{f, \frac{az+b}{cz+d}\right\}\frac{(ad-bc)^2}{(cz+d)^4}. \tag{2.4}$$

Finally, from the definition of the Schwarzian, we have

Proposition 2.2 *If* f *is a meromorphic function, then* $\{f,z\}$ *has double poles at the critical points of* f *and is holomorphic elsewhere.*

The cross-ratio is projectively invariant, that is if $f : z \to \dfrac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$ is a Möbius transformation, then

$$[f(z_1), f(z_2), f(z_3), f(z_4)] = [z_1, z_2, z_3, z_4].$$

For a general smooth function f, the Schwarz derivative measures the distortion of the cross-ratio in a certain sense. The following result is well known and is fundamental in Cartan [8].

Proposition 2.3 *Let a, b, c and d be four distinct complex numbers and f a twice-differentiable function whose second derivative is continuous in an open set $D \subset \mathbb{C}$. For $z \in D$ and small t we define the cross-ratio*

$$Sf(z,t) = [f(z+ta), f(z+tb), f(z+tc), f(z+td)],$$

then

$$Sf(z,t) = [a,b,c,d]\left(1 + \frac{1}{6}(a-b)(c-d)Sf(z)t^2 + \mathrm{o}(t^2)\right).$$

The proof of Proposition 2.3 basically uses the following expansion, valid for all x and y close to z with $x = z+h,\ y = z+k$,

$$\begin{aligned}\log \frac{f(x)-f(y)}{x-y} &= \log f'(z) + \frac{1}{2}\frac{f''(z)}{f'(z)}(h+k) \\ &\quad + \left(\frac{1}{6}\frac{f'''(z)}{f'(z)} - \frac{1}{8}\left(\frac{f''(z)}{f'(z)}\right)^2\right)(h^2+k^2) - \left(\frac{1}{6}\frac{f'''(z)}{f'(z)}\right. \\ &\quad \left. - \frac{1}{4}\left(\frac{f''(z)}{f'(z)}\right)^2\right)hk + \cdots .\end{aligned}$$

Remark 2.4 Taking the second derivative with respect to x and y gives another interpretation of the Schwarzian derivative, as a bi-differential

$$\frac{df(x)df(y)}{(f(x)-f(y))^2} = \frac{dxdy}{(x-y)^2} + \left(\frac{1}{6}\frac{f'''(z)}{f'(z)} - \frac{1}{4}\left(\frac{f''(z)}{f'(z)}\right)^2\right)dxdy + \psi(h,k)dxdy,$$

where the term $\psi(h,k)$ vanishes for $x = y = z$.

3 The case of modular functions

Let $\mathbf{H} = \{z \in \mathbb{C},\ \Im z > 0\}$ be the upper half of the complex plane. The group $\mathrm{SL}_2(\mathbb{R})$ of Möbius transformations acts on $\mathbf{H}$ in the usual manner. We restrict our attention to the modular group $\mathrm{SL}_2(\mathbb{Z})$ and its subgroups, but the general picture involves all the discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$. Let G be such a group, and let f be a modular form on G of weight k ($k \geq 0$), that is, a meromorphic function on $\mathbf{H}$ satisfying

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)\,, \quad z \in \mathbf{H}\,, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, \tag{3.1}$$

with some growth conditions at the cusps. When $k = 0$, f is called a modular function. If f is holomorphic, it will be mentioned explicitly.

The effect of the Schwarzian on modular functions is illustrated as follows. Using (2.4) and the fact that the derivative of a modular function is a modular form of weight 2, we have

Proposition 3.1 *If f is a modular function for G, then $\{z, f\}$ is a modular function and $\{f, z\}$ is a modular form of weight 4 on G.*

Suppose now that G is a discrete group of genus 0, that is the compactification of the quotient $G\backslash\mathbf{H}$ is a Riemann surface of genus 0. Any analytic embedding of this surface in the extended complex plane induces a modular function for G defined on $\mathbf{H}$. It is called a Hauptmodul, and it generates the function field of the Riemann surface. Let f be a Hauptmodul for G. Then the modular function $\{z, f\}$ is a rational function of f. As for $\{f, z\}$, we have a deeper result:

Proposition 3.2 [25] *Let G be a genus 0 discrete group and f a Hauptmodul for G. Then $\{f, z\}$ is a weight 4 modular form on the normalizer of G in $SL_2(\mathbb{R})$ and this normalizer is the maximal group with this property.*

To illustrate this proposition, Let $\Gamma(2)$ be the principal congruence group of level 2. A Hauptmodul is given by the classical Klein λ function. Since $\Gamma(2)$ has no elliptic elements, and thus λ has no critical point, we see that $\{\lambda, z\}$ is a holomorphic weight 4 modular form on the normalizer of $\Gamma(2)$ in $\mathrm{SL}_2(\mathbb{R})$ which is $\mathrm{SL}_2(\mathbb{Z})$. However, the space of weight 4 holomorphic modular forms for $\mathrm{SL}_2(\mathbb{Z})$ is one-dimensional and is generated by the weight 4 Eisenstein series E_4. In fact we have

$$\{\lambda, z\} = \pi^2 E_4(z). \tag{3.2}$$

A deeper study of this type of relations can be found in [25]. Here

$$\lambda(z) = \left(\frac{\eta(z)}{\eta(4z)}\right)^8,$$

where the eta function is defined by

$$\eta(z) = q^{\frac{1}{24}} \prod_{n\geq 1} (1-q^n), \quad q = e^{2\pi i z},$$

and the Eisenstein series E_4 is defined by

$$E_4(z) = 1 + 240 \sum_{n\geq 1} \sigma_3(n) q^n, \quad q = e^{2\pi i z},$$

where $\sigma_3(n)$ is the sum of the cubes of the positive divisors on n.
Because the normalizer of congruence subgroups is often not a subgroup of the modular group, our focus should be on a more general class of discrete groups. Namely all the discrete subgroups that are commensurable with the modular group. These groups have finite index inside their normalizers. The next section deals with the converse to the previous proposition.

4 Modular Schwarzians

In this section we look at the following question: Assume that f is a meromorphic function on $\mathbf{H}$ such that $F(z) = \{f, z\}$ is a modular form of weight 4 on a certain group G_F. The invariance group of f (on which f is a modular function) is a subgroup G_f of G_F. What is the size of G_f inside G_F? Keeping in mind the relationship between G_f and G_F for a modular function f as it was seen in the previous section. We will make explicit some instances where we have complete answers. It turns out that these answers depend on the analytic properties of f and on the structure of G_F. So far, for all kind of plausible conditions, it seems that there are always examples that provide different answers.

We shall rephrase the problem differently. Since $F(z) = \{f, z\}$ is a modular form of weight 4, then for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_F$ we have

$$F(\gamma \cdot z) = (cz+d)^4 F(z) = (cz+d)^4 \{f, z\}.$$

On the other hand, using (2.4), we have

$$F(\gamma \cdot z) = \{f(\gamma \cdot z), \gamma \cdot z\} = (cz+d)^4 \{f(\gamma \cdot z), z\}.$$

It follows that

$$\{f, z\} = \{f(\gamma \cdot z), z\}.$$

Therefore, using Proposition 2.1, there exists $\Phi_\gamma \in \mathrm{GL}_2(\mathbb{C})$ such that

$$f(\gamma \cdot z) = \Phi_\gamma \cdot f(z). \tag{4.1}$$

This defines a group homomorphism

$$\begin{aligned} \Phi \,:\, G_F &\longrightarrow \mathrm{GL}_2(\mathbb{C}) \\ \gamma &\longmapsto \Phi_\gamma \end{aligned}$$

The invariance group G_f of f is simply the kernel of Φ. It is worth mentioning that M. Kaneko and M. Yoshida have considered a similar problem in [18] where Φ is an epimorphism $r : G \longrightarrow G'$ between two Fuchsian groups of the first kind. In particular, the authors were interested in the case G and G', the kernel and the co-kernel of r, are infinite groups. They have constructed the Kappa function defined by $j(\kappa(z)) = \lambda(z)$ as an answer to this problem. We are mainly interested in the case where the co-kernel is rather finite.

We now look at a case where there is a precise answer to the above question.

Proposition 4.1 *Let f be a meromorphic function on $\mathbf{H}$ such that $f(z+1) = f(z)$. If $\{f, z\}$ is a weight 4 modular form on $SL_2(\mathbb{Z})$, then f is a modular function for $SL_2(\mathbb{Z})$.*

Proof Let $U = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. The condition $f(z+1) = f(z)$ means that $U \in \mathrm{Ker}\Phi$, and that $\langle U \rangle = \{U^n,\ n \in \mathbb{Z}\} \subseteq \mathrm{Ker}\Phi$. Since $\mathrm{Ker}\Phi$ is normal in $\mathrm{SL}_2(\mathbb{Z})$, it contains $\{L^{-1}UL,\ L \in \mathrm{SL}_2(\mathbb{Z})\}$, the normal closure of $\langle U \rangle$. Now if we set $V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then one can show that $V^{-1} = UV^{-1}UVU$. Hence V^{-1} belongs to the normal closure of $\langle U \rangle$, and so does V. Since U and V generate $\mathrm{SL}_2(\mathbb{Z})$, it follows that $\mathrm{Ker}\Phi = \mathrm{SL}_2(\mathbb{Z})$. □

Remark 4.2 The condition $f(z+1) = f(z)$ means that f has a Fourier expansion in $q = \exp(2\pi i z)$.

The above theorem provides an example where f is a modular function on the same group on which its Schwarzian is a modular form. This is equivalent to saying that the homomorphism Φ is constant. This proposition can be generalized as follows.

Theorem 4.3 *Let n be an integer such that $1 \le n \le 5$. Suppose that f is a meromorphic function on $\mathbf{H}$ such that $f(z+n) = f(z)$ and suppose that $\{f, z\}$ is a modular form of weight 4 for $SL_2(\mathbb{Z})$. Then f is a modular function for a finite index normal subgroup of $SL_2(\mathbb{Z})$.*

Proof The case $n = 1$ is settled in the above proposition. For $2 \le n \le 5$, the group $\Gamma(n)$ is of genus 0 and has no elliptic elements. Hence it is generated by parabolic elements only. Using this fact, one can show that the normal closure $\Delta(n)$ of $\langle U^n \rangle$ is simply $\Gamma(n)$. Thus f is a modular function for a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. □

As for $n \geq 6$, the argument above is no longer valid. Indeed the normal closure $\Delta(n)$ of $\langle U^n \rangle$ has infinite index in $\mathrm{SL}_2(\mathbb{Z})$.

We now provide an example where the invariance group of f is not larger than $\langle U \rangle$, although the function f has the same conditions as in the theorem. Let G be a genus 0 group having at least two cusps. Let g be a Hauptmodul for G normalized to have the value 1 at ∞ and 0 at the other cusp. Let $g(z) = \log f(z)$. Then $f(z+1) = f(z)$ and for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ not in $\langle U \rangle$ we have

$$f(\gamma \cdot z) = f(z) + 2\pi i n_\Gamma\,, \quad n_\Gamma \in \mathbb{Z},\ n_\Gamma \neq 0.$$

However, $\{f, z\}$ is a weight 4 modular form on G.

5 Equivariant forms, a first example

The principal motivation behind equivariant forms was to look for examples of meromorphic functions f where $\{f, z\}$ is a modular form of weight 4 for a discrete group G but f is not invariant under any nontrivial matrix. That is, the kernel of Φ defined in the previous section is trivial. This will be the case if, for instance, Φ =Id. In other word, f would satisfy

$$f\left(\frac{az+b}{cz+d}\right) = \frac{af(z)+b}{cf(z)+d}\,, \quad \text{for all } z \in \mathbf{H} \text{ and } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$$

On the other hand, in the paper [32], we studied and solved a family of Riccati equations of the form

$$\frac{k}{i\pi} u' + u^2 = E_4\,, \quad 1 \leq k \leq 6.$$

While the cases $2 \leq k \leq 6$ were given in terms of modular forms and functions, the solution in the case $k = 1$ turns out to satisfy the above functional equations for $G = \mathrm{SL}_2(\mathbb{Z})$.

This paper is devoted to construct and study these very special functions with surprising properties arising along the way. We will focus on the case of $G = \mathrm{SL}_2(\mathbb{Z})$ for most of the paper.

Definition 5.1 A meromorphic function h(z) on $\mathbf{H}$ is called an equivariant form for $\mathrm{SL}_2(\mathbb{Z})$ if

- For all $z \in \mathbf{H}$ and for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$h(\gamma \cdot z) = \gamma \cdot h(z). \tag{5.1}$$

- The function $h(z) - z$ is meromorphic at the cusps.

Since $h(z+1) = h(z) + 1$, the function $h(z) - z$ is 1-periodic and hence has a Fourier expansion in $q = \exp(2\pi i z)$, the local parameter of $\mathrm{SL}_2(\mathbb{Z})$ at ∞. To say that $h(z) - z$ is meromorphic at ∞ means that the Fourier expansion has finitely many negative powers of q.

The trivial example is $h_0(z) = z$ which is equivariant for any group. This example is very particular in two ways: First, it is the only Möbius transformation that is equivariant, and second, as was shown by Heins [15], h_0 is the only equivariant function that maps $\mathbf{H}$ into itself. In particular, this means that the composition of maps does not provide the set of equivariant holomorphic functions with a group structure. This result has a certain connection with the iteration of holomorphic maps from $\mathbf{H}$ into $\mathbf{H}$. We give a quick proof using the following theorem of Denjoy and Wolff [7].

Theorem 5.2 *Let $f : \mathbf{D} \to \mathbf{D}$ be a holomorphic map. We assume that f is neither an elliptic Möbius transformation nor the identity, then the successive iterates f^{on} converge, uniformly on compact subsets of $\mathbf{D}$, to a constant function $z \to c_0 \in \overline{\mathbf{D}}$.*

We recall that the elliptic Möbius transformations in $\mathbf{H}$ are all of the following form

$$z \mapsto \frac{\cos\frac{\theta}{2} z + \sin\frac{\theta}{2}}{-\sin\frac{\theta}{2} z + \cos\frac{\theta}{2}}, \quad \theta \in (0, 2\pi),$$

and they are not equivariant, for any $\theta \in (0, 2\pi)$. Thus, if h is an equivariant holomorphic function sending $\mathbf{H}$ into itself, so does any iterate h^{on}, $n \in \mathbb{N}$ which is also equivariant and holomorphic. By the Wolff-Denjoy theorem, h^{on} should tend to $c \in \overline{\mathbf{H}}$ which, by equivariance, verifies the contradictory conditions

$$c = -\frac{1}{c}, \quad c = c + 1.$$

Thus, the only equivariant holomorphic function in $\mathbf{H}$ which maps $\mathbf{H}$ into $\mathbf{H}$ is the identity map.

We now provide the first nontrivial example of equivariant functions. Let $E_2(z)$ be the classical Eisenstein series

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n}, \tag{5.2}$$

where $q = \exp(2\pi i z)$, $z \in \mathbf{H}$ and $\sigma_1(n)$ is the sum of the positive divisors of n. The series $E_2(z)$ is a holomorphic function on $\mathbf{H}$, and if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$E_2(\gamma \cdot z) = (cz + d)^2 E_2(z) + \frac{6c}{i\pi}(cz + d), \tag{5.3}$$

that is to say that E_2 is a quasi-modular form of weight 2. Meanwhile, if we define

$$E_2^*(z) = E_2(z) - \frac{3}{\pi y}, \quad y = \Im z, \tag{5.4}$$

then E_2^* is a non-holomorphic modular form of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$. Moreover, E_2 is the logarithmic derivative of the discriminant function $\Delta(z)$, the classical cusp form of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$

$$\Delta(z) = \eta(z)^{24} = q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \quad q = e^{2\pi i z}.$$

In fact, we have

$$E_2(z) = \frac{1}{2i\pi} \frac{\Delta'(z)}{\Delta(z)}. \tag{5.5}$$

Theorem 5.3 *The function h_1 given by*

$$h_1(z) = z + \frac{6}{i\pi E_2(z)} \tag{5.6}$$

is an equivariant form for $SL_2(\mathbb{Z})$.

Proof It suffices to show that h_1 is equivariant under the transformations $z \mapsto z+1$ and $z \mapsto -1/z$ which generate $\mathrm{SL}_2(\mathbb{Z})$. Since $E_2(z+1) = E_2(z)$, it is clear that $h_1(z+1) = h_1(z) + 1$. On the other hand, using (5.3), one has:

$$\begin{aligned} h_1\left(\frac{-1}{z}\right) &= \frac{-1}{z} - \frac{6i/\pi}{z^2E_2(z) - 6iz/\pi} \\ &= \frac{E_2(z)}{zE_2(z) - 6i/\pi} \\ &= \frac{-1}{h_1(z)}. \end{aligned}$$

Furthermore, one can see from (5.2) that $h_1(z) - z$ has a holomorphic q-expansion, and therefore h_1 is an equivariant function for $\mathrm{SL}_2(\mathbb{Z})$. □

The equivariant function $h_1(z)$ has no fixed points in $\mathbf{H}$ and also at ∞ in the sense that

$$\lim_{z\to i\infty} (h_1(z) - z) = \frac{6}{i\pi} \neq 0.$$

In the following, we will show that $h_1(z)$ is the unique equivariant function for $\mathrm{SL}_2(\mathbb{Z})$ with the property that it has no fixed points in $\mathbf{H} \cup \{\infty\}$ and that this example actually fits in a general construction of equivariant functions.

Theorem 5.4 *Let $h(z)$ be an equivariant function for $SL_2(\mathbb{Z})$ with no fixed points in $\mathbf{H} \cup \{\infty\}$. Then*

$$h(z) = h_1(z) = z + \frac{6}{i\pi E_2(z)}.$$

Proof Since $h(z) - z$ does not have zeros on $\mathbf{H} \cup \{\infty\}$, then the function

$$g(z) = \frac{1}{h(z) - z} - \frac{i\pi}{6} E_2(z)$$

is holomorphic in $\mathbf{H} \cup \{\infty\}$. Moreover, we have $g(z+1) = g(z)$ and using the equivariance of h and (5.3) we get

$$\begin{aligned} g(-1/z) &= \frac{zh(z)}{h(z) - z} - \frac{i\pi}{6} z^2 E_2(z) - z \\ &= \frac{z(h(z) - z) + z^2}{h(z) - z} - \frac{i\pi}{6} z^2 E_2(z) - z \\ &= z^2 g(z). \end{aligned}$$

Therefore, $g(z)$ is a weight 2 holomorphic modular form and thus $g(z) = 0$ since the space of weight 2 holomorphic modular forms for $\mathrm{SL}_2(\mathbb{Z})$ is trivial. The theorem follows. □

Remark 5.5 An alternative way to prove this theorem is to notice that under the assumptions of the theorem, the integral

$$\int_i^z \frac{dw}{h_1(w) - w}$$

does not depend on the path of integration W_z from i to z in $\mathbf{H}$ since $h_1(z)-z$ is holomorphic and non-vanishing. Therefore, the function

$$f(z) = \exp\int_i^z \frac{12dw}{h_1(w)-w}$$

is a well defined function that is holomorphic and non-vanishing on $\mathbf{H}$. Using the equivariance of h_1, one can show that $f(z)$ is a non-vanishing modular form of weight 12 and thus is a scalar multiple of $\Delta(z)$. Taking the logarithmic derivative of f and using (5.5) yield the theorem.

Proposition 5.6 *If z_0 is an elliptic fixed point of $SL_2(\mathbb{Z})$, then*

$$h_1(z_0) = \bar{z}_0.$$

Proof If $z_0 \in \mathbf{H}\cup\mathbb{Q}$ is fixed by an element γ of $\mathrm{SL}_2(\mathbb{Z})$, then $h_1(z_0)$ is also fixed by γ. Thus if z_0 is an elliptic fixed point, and since h_1 does not have fixed points in $\mathbf{H}$, we must have $h_1(z_0) = \bar{z}_0$. □

6 Analytic properties of h_1 and an application

In this section we show that there are infinitely many poles of h_1 and that they are all simple.

Proposition 6.1 *We have*

(1) *The poles of $h_1(z)$ are located at the zeros of E_2 and they are simple.*
(2) *The critical points of h_1 are located at the zeros of E_4.*

Proof It is clear that the poles of $h_1(z)$ are exactly the zeros of E_2. Now recall the following differential relation between E_2 and E_4 due to Ramanujan, [27,28]

$$\frac{1}{2\pi i}\frac{d}{dz}E_2(z) = \frac{1}{12}(E_2^2 - E_4). \tag{6.1}$$

It follows that if a zero of E_2 is not simple, then it is also a zero of E_4 and such a zero lies in the $\mathrm{SL}_2(\mathbb{Z})$-orbit of ρ, the cubic root of unity. This is impossible since Proposition 5.4 states that E_2 does not vanish at the elliptic fixed points. Therefore, the poles of $h_1(z)$ are simple. Furthermore,

$$h_1'(z) = 1 - \frac{6}{i\pi}\frac{E_2'}{E_2^2} = 1 - \frac{E_2^2 - E_4}{E_2^2} = \frac{E_4}{E_2^2}.$$

Hence $h_1'(z)$ vanishes exactly at the zeros of E_4, □

As a consequence, and using Proposition 2.2, $E_4^2\{h, z\}$ is a weight 12 holomorphic modular form and thus it is a linear combination of Δ and E_4^3 which constitute a basis of the space of weight 12 holomorphic forms for $\mathrm{SL}_2(\mathbb{Z})$. Investigation of the first two coefficients of the Fourier expansion yields

Proposition 6.2 *We have*

$$\{h_1, z\} = -2^6 3^2 \pi^2 \frac{\Delta}{E_4^2}. \tag{6.2}$$

Remark 6.3 The above proposition can be established by direct computation using similar identities to (6.1), also known as the Ramanujan identities [27], namely

$$\frac{1}{2\pi i}\frac{d}{dz}E_4(z) = \frac{1}{3}(E_2E_4 - E_6) \tag{6.3}$$

$$\frac{1}{2\pi i}\frac{d}{dz}E_6(z) = \frac{1}{2}(E_2E_6 - E_4^2), \tag{6.4}$$

together with the identity

$$E_4^3 - E_6^2 = 2^7 3^2\,\Delta.$$

Here, E_6 is the weight 6 Eisenstein series

$$E_6(z) = 1 - 504\sum_{n\geq 1}\sigma_5(n)\,q^n,$$

where $\sigma_5(n)$ is the sum of the fifth powers of the positive divisors of n.

If z_0 is a pole of h_1 then all the translates of z_0 by integers are also poles of h_1. It turns out that the only other poles of h_1 that are $\mathrm{SL}_2(\mathbb{Z})$-equivalent to z_0 are its translates by integers. Indeed,

Lemma 6.4 *If z_0 and z_1 are two poles of h_1, and if there exists $\gamma \in SL_2(\mathbb{Z})$ such that $z_1 = \gamma\cdot z_0$ then $\gamma = \left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right)$ for some integer n.*

Proof If $\gamma = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ and if z_0 and $z_1 = \gamma\cdot z_0$ are poles of h_1, then necessarily $c = 0$ and $a = d = \pm 1$. Thus $z_1 = z_0 + n$ where n is an integer. □

Using this property, we can restrict ourselves to the half strip

$$D = \left\{z = x + iy : y > 0,\ -\frac{1}{2} < x \leq \frac{1}{2}\right\}.$$

We will denote by T and S the transformations

$$Tz = z + 1\,,\quad Sz = \frac{-1}{z}.$$

Proposition 6.5 *There exists a pole z_0 of h_1 on the purely imaginary axis $\{z = iy :\ y > 0\}$.*

Proof Recall that $E_2(z) = 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n$. We see that $E_2(iy)$, $y > 0$ is real and strictly decreasing for $y \in (0,\infty)$. It takes the value 1 at ∞ and $\lim_{y\to 0} E(iy) = -\infty$. Therefore, there exists $y_0 > 0$ such that $E_2(iy_0) = 0$ and thus iy_0 is a pole of h_1. Moreover, $y_0 < 1$ since by (5.3), we have $E_2(i) = 3/\pi$. □

Recall that the fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ is

$$\mathcal{F} = \{z \in \mathbf{H} : |z| > 1,\ -1/2 < \Re z \leq 1/2\}.$$

Since $\mathrm{Im}(z_0) < 1$ we see that

$$z_0 \in S\mathcal{F}\,,\quad Sz_0 \in \mathcal{F}. \tag{6.5}$$

Theorem 6.6 *There are infinitely many poles for h_1 in D.*

Proof Let $z_1 = Sz_0 = -1/z_0$ where z_0 is as in Proposition 6.5. Then $h_1(z_1) = 0$ and z_1 is not an elliptic fixed point by Proposition 5.6. Choose an open neighborhood $U \subset \mathcal{F}$ of z_1 on which h_1 is holomorphic, not containing an elliptic fixed point and such that no two points of U are $\mathrm{SL}_2(\mathbb{Z})-$equivalent. Then U is mapped by h_1 onto an open neighborhood V of 0. There are infinitely many rational numbers in the open set V. If $x = h_1(z_x)$, $z_x \in U$, is such a rational number, let γ_x be such that $\gamma_x \cdot x = \infty$. Then $h_1(\gamma_x \cdot z_x) = \gamma_x \cdot h_1(z_x) = \gamma_x \cdot x = \infty$. Therefore, $\gamma_x \cdot z_x$ is a pole of h_1. In the meantime, no two such poles $\gamma_x \cdot z_x$ and $\gamma_y \cdot z_y$ are $\mathrm{SL}_2(\mathbb{Z})-$equivalent because z_x and z_y, which are in U, are not. □

Corollary 6.7 *The Eisenstein series E_2 has infinitely many non-equivalent zeros.*

Remark 6.8 As a corollary, the Eisenstein series E_2 has infinitely many non-equivalent zeros; a result that has been established without the notion of equivariant functions in [12].

7 Rational equivariant functions, the general case

In this section, from each modular form we construct an equivariant function. Using (5.5), one can rewrite the equivariant function h_1 as

$$h_1(z) = z + 12\frac{\Delta}{\Delta'}.$$

It turns out that this expression can be generalized as follows

Theorem 7.1 [34] *Let f be a modular form on $SL_2(\mathbb{Z})$ of weight k. Then the function*

$$h(z) = z + k\,\frac{f(z)}{f'(z)} \tag{7.1}$$

is equivariant for $SL_2(\mathbb{Z})$.

Proof Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. We have $f(\gamma \cdot z) = (cz+d)^k f(z)$, hence

$$f'(\gamma \cdot z) = kc(cz+d)^{k+1} f(z) + (cz+d)^{k+2} f'(z).$$

Therefore,

$$\begin{aligned} h(\gamma \cdot z) &= \frac{az+b}{cz+d} + \frac{k(cz+d)^k f(z)}{kc(cz+d)^{k+1} f(z) + (cz+d)^{k+2} f'(z)} \\ &= \frac{kf(z)(abz+bc+1) + (az+b)(cz+d)f'(z)}{(cz+d)(kcf(z) + (cz+d)f'(z))}. \end{aligned}$$

On the other hand, we have

$$\gamma \cdot h(z) = \frac{(az+b)f'(z) + akf(z)}{(cz+d)f'(z) + kcf(z)}.$$

Since $ad - bc = 1$, we have $(az+b)c + 1 = a(cz+d)$. The identity $h(\gamma \cdot z) = \gamma \cdot h(z)$ follows. □

Proposition 7.2 *If f is a modular form of weight k, then scalar multiples of f and integral powers of f give rise to the same equivariant function h. The modular functions correspond to the trivial equivariant function $h_0(z) = z$.*

Proof This is straightforward keeping in mind that, for an integer m, the weight of f^m is km. □

In what follows, we will find sufficient conditions for an equivariant function to arise from a modular form.

Proposition 7.3 *If h is equivariant for $SL_2(\mathbb{Z})$, then the set of residues of the meromorphic function $1/(h(z)-z)$ at the simple poles is finite.*

Proof Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Differentiating $h(\gamma z) = \gamma h(z)$ yields

$$\frac{h'(\gamma z)}{(cz+d)^2} = \frac{h'(z)}{(ch(z)+d)^2}.$$

Thus, if $h(z_0)=z_0$, then $h'(\gamma z_0)=h'(z_0)$, that is, h' takes the same value at the orbit of a fixed point of h. Hence, the set of values $h'(z_0)$ when z_0 describes the set of fixed points of h is completely determined if we restrict ourselves to the fundamental domain $\mathcal{F}$. Moreover, since $h(z)-z$ is meromorphic at $i\infty$, there is a neighborhood of $i\infty$ of the form $\{z \in \mathbf{H} : \Im z > y_0\}$ on which $h(z)-z$ does not vanish (except possibly at $i\infty$). Therefore, all the zeros of $h(z)-z$ in $\mathcal{F}$ are within the closure of $\{z \in \mathcal{F} : \Im z \leq y_0\}$ which is compact and thus we have only finitely many zeros. In the meantime, the residue of $1/(h(z)-z)$ at a simple z_0 of h is simply $1/(h'(z_0)-1)$. The proposition follows. □

Theorem 7.4 *Let h be an equivariant function satisfying the following conditions:*

(1) *The poles of $1/(h(z)-z)$ in $\mathbf{H}$ are simple and their residues are rational numbers.*
(2) *At ∞ we have:*

$$\lim_{z\longrightarrow\infty} \frac{1}{h(z)-z} \in 2\pi i\mathbb{Q}.$$

Then there exists a modular form f of integer weight k for $SL_2(\mathbb{Z})$ such that

$$h(z) = z + k\,\frac{f(z)}{f'(z)}.$$

Proof Define the function $f(z)$ by

$$f(z) = \exp \int_i^z \frac{kdz}{h(z)-z},$$

where k is a positive integer to be chosen conveniently. The path of integration Σ_z is chosen to lie in $\mathbf{H} \setminus S$, where S is the set of (simple) zeros of $h(z)-z$. By assumption, the residues of $1/(h(z)-z)$ are rational numbers, and using Proposition 7.3, these rational numbers have bounded denominator. Therefore, there exists $k \in \mathbb{Z}^+$ such that for each such residue r, kr is an integer and

$$\lim_{z\longrightarrow\infty} \frac{k}{h(z)-z} \in 2\pi i\mathbb{Z},$$

and we also suppose $k \equiv 0 \mod 4$. If we choose a different path of integration Σ_z' from i to z lying in $\mathbf{H} \setminus S$, then

$$\int_{\Sigma_z} \frac{kdz}{h(z)-z} - \int_{\Sigma_z'} \frac{kdz}{h(z)-z} = 2\pi ik \sum \text{Residues} \in 2\pi i\mathbb{Z},$$

where the sum of the residues is taken over the finite number of poles within the closed path $\Sigma_z - \Sigma'_z$. Therefore, $f(z)$ is well defined on $\mathbf{H} \setminus S$. We extend f to a meromorphic function on S in the following way. Let m (an integer) be the residue of $k/(h(z) - z)$ at z_0. If $r > 0$ we define $f(z_0) = 0$ to make f holomorphic at z_0 and the order of f at z_0 is r. If $r < 0$ then z_0 is a pole of f of order $-r$. Thus f is a well-defined meromorphic function of $\mathbf{H}$. Furthermore,

$$f(z+1) = \exp \int_i^{z+1} \frac{kdz}{h(z) - z} = f(z)g(z)$$

where

$$g(z) = \exp \int_z^{z+1} \frac{kdz}{h(z) - z}.$$

Since h is an equivariant function, it is clear that $g'(z) = 0$ and hence g is constant. Taking the limit $z \to i\infty$ yields $g(z) = 1$ since by assumption,

$$\frac{k}{h(z) - z} = ka_0 + \sum_{n \geq 1} a_n q^n , \quad q = e^{2\pi i z},$$

and $ka_0 \in 2\pi i \mathbb{Z}$. Therefore,

$$f(z+1) = f(z).$$

On the other hand,

$$\begin{aligned} f(-1/z) &= \exp \int_i^{-1/z} \frac{kdw}{h(w) - w} \\ &= \exp \int_i^{z} \frac{kh(t)dt}{t(h(t) - t)} \\ &= f(z) \exp \int_i^{z} \frac{kdt}{t} \\ &= z^k f(z) \quad \text{since } k \equiv 0 \mod 4. \end{aligned}$$

Thus f is a meromorphic modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$. □

Motivated by the above theorem, we have

Definition 7.1 An equivariant function that arises from a modular form as in (7.1) is called a rational equivariant function.

Remark 7.5 If f is a weight k modular form, then the corresponding equivariant function $h(z) = z + kf(z)/f'(z)$ satisfies the conditions of the above theorem; that is to say that the two conditions are necessary and sufficient conditions for h to be of that form. Moreover the conditions are optimal as we will see that there are indeed examples of equivariant functions that do not satisfy them and thus they are not rational. Indeed, if we take

$$\tilde{h}_1(z) = z + \frac{6E_2^2E_6}{i\pi E_2E_4^2E_6 + \Delta},$$

then one can show that $\tilde{h}_1$ is equivariant but $\tilde{h}_1(z) - z$ has a double pole at the cubic root of unity ρ. Also,

$$\tilde{h}_2(z) = z + \frac{6E_4}{i\pi(E_2E_4 + E_6)}$$

is equivariant having poles only at the zeros of E_4 but the residue of $1/(\tilde{h}_2(z) - z)$ at ρ is irrational. Finally,

$$\tilde{h}_3(z) = z + \frac{6\Delta}{i\pi E_2\Delta + E_{14}}$$

is equivariant, but $\lim_{z\to i\infty}(\tilde{h}_3(z) - z) = 0$. These three examples show that one cannot remove the conditions of the converse theorem above.

8 Lambert series and Borcherds products

For a rational equivariant function $h(z) = z + kf(z)/f'(z)$, we will investigate the Fourier coefficients of the periodic function $h(z) - z$, the Lambert series of $\dfrac{1}{h(z) - z}$. We also point out a possible link with a theorem of Borcherds at least by considering some examples. We begin by recalling some classical analytic facts [36], p. 147.

Lemma 8.1 *Given any sequence $(a_n)_{n\geq 0}$, $a_0 \neq 0$, we formally have*

$$\frac{1}{\sum_{n=0}^{\infty} a_nq^n} = \frac{1}{a_0} + \sum_{1}^{\infty} \frac{(-1)^n G_n}{n!a_0^{n+1}} q^n$$

with

$$G_n = \begin{vmatrix} 2a_1 & a_0 & 0 & 0 & \cdots & 0 \\ 4a_2 & 3a_1 & 2a_0 & 0 & \cdots & 0 \\ 6a_3 & 5a_2 & 4a_1 & 3a_0 & \cdots & 0 \\ . & . & . & . & \cdots & . \\ (2n-2)a_{n-1} & . & . & . & \cdots & (n-1)a_0 \\ na_n & (n-1)a_{n-1} & . & . & . & a_1 \end{vmatrix}$$

Corollary 8.2 *Let $E_2(q)$ be the weight 2 Eisenstein series (5.2) seen as a function of q and set*

$$\frac{1}{E_2(q)} = \sum_{n=0}^{\infty} \alpha_n q^n, \tag{8.1}$$

then $\alpha_0 = 1$ *and*

$$n!(24)^{-n}\alpha_n = \begin{vmatrix} 2\sigma_1(1) & 1 & 0 & 0 & \cdots & 0 \\ 4\sigma_1(2) & 3\sigma_1(1) & 2 & 0 & \cdots & 0 \\ 6\sigma_1(3) & 5\sigma_1(2) & 4\sigma_1(1) & 3 & \cdots & 0 \\ . & . & . & . & \cdots & . \\ (2n-2)\sigma_1(n-1) & . & . & . & \cdots & n-1 \\ n\sigma_1(n) & (n-1)\sigma_1(n-1) & . & . & . & \sigma_1(1) \end{vmatrix}.$$

Lemma 8.3 *Consider a formal power series* $\sum_{n\geq 1} b_n x^n$ *with complex coefficients, then a sequence* $(a_n)_{n\geq 1}$ *can be found such that the following expansion in Lambert series holds*

$$\sum_{n\geq 1} b_n x^n = \sum_{n\geq 1} a_n \frac{x^n}{1-x^n} \tag{8.2}$$

with

$$b_n = \sum_{m|n} a_m, \quad a_m = \sum_{d|m} \mu\left(\frac{m}{d}\right) b_d = \sum_{d|m} \mu(d) b_{\frac{m}{d}}, \tag{8.3}$$

μ *being the Möbius function defined as usual by the inverse of the Riemann zeta function*

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}, \quad \frac{1}{\zeta(s)} = \sum_{n\geq 1} \frac{\mu(n)}{n^s}, \quad \Re s > 1.$$

We now consider an equivariant function h and the associated function g, defined by

$$\frac{k}{2i\pi} g(z) = \frac{1}{h(z)-z} = \sum_{n\geq 0} b_n e^{2i\pi nz}.$$

It is a meromorphic periodic function on the upper half plane **H**. Let, with the notation of Lemma 8.3,

$$a(n) = -\frac{1}{n}\sum_{d|n} \mu(d) b_{\frac{n}{d}} \quad \text{and} \quad f(z) = e^{2i\pi b_0 z} \prod_{n\geq 1} (1-e^{2i\pi nz})^{a(n)}. \tag{8.4}$$

Applying the theta differential operator $\theta = \frac{1}{2i\pi}\frac{d}{dz}$ and using the Möbius inversion formula, we have

$$\frac{\theta f}{f} = b_0 - \sum_{n=1}^{\infty}\sum_{d|n} d a(d) e^{2i\pi nz} = \frac{k}{2i\pi} g.$$

We define the slash operator of weight k as usual: Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, then

$$F|_k[\gamma](z) = (cz+d)^{-k} f(\gamma z).$$

According to [19] we have the following

Definition 8.1 Let $\Gamma \subset \Gamma(1)$ containing $-I$. A meromorphic function $F : \mathbf{H} \mapsto \mathbb{C}$ is said a generalized modular form of weight k and a character ν if

(1) F satisfies the modular transformation law

$$F|_k[M](z) = \nu(M)F(z)$$

for all $M \in \Gamma$, with $\nu(M)$ independent of $z \in \mathbf{H}$.

(2) F has a left-finite Fourier expansion at each parabolic cusp in a fundamental region $\mathcal{R}$ of Γ.

The generalized modular forms differ from the classical modular forms with a character by the fact the multiplier system ν need not be unitary.

Proposition 8.4 *For h equivariant, the infinite product $f(z)$, given in (8.4), is a generalized modular form*

Proof We have

$$\frac{\theta f|_0[\gamma]}{f|_0[\gamma]} = \frac{\theta f}{f}|_2[\gamma] = \frac{k}{2i\pi} g|_2[\gamma] = \frac{k}{2i\pi} g + \frac{k}{2i\pi} \frac{c}{cz+d}$$

where the last equality is a consequence of the equivariance of h. Therefore,

$$\frac{\theta f|_0[\gamma]}{f|_0[\gamma]} - \frac{\theta f}{f} = \frac{k}{2i\pi} \frac{c}{cz+d}, \tag{8.5}$$

which is a fundamental equation to study the modular properties of f. It follows that

$$\frac{\theta\left(\frac{f|_0[\gamma]}{f}\right)}{\frac{f|_0[\gamma]}{f}} = \frac{\theta f|_0[\gamma]}{f|_0[\gamma]} - \frac{\theta f}{f} = \frac{k}{2i\pi} \frac{c}{cz+d} = \frac{\theta(cz+d)^k}{(cz+d)^k}.$$

Hence, $\dfrac{f|_k[\gamma]}{f(z)} = \dfrac{f|_0[\gamma]}{f(z)(cz+d)^k}$ is a non-zero constant $\nu(\gamma)$ on $\mathbf{H}$. Using the cocycle relation of the slash operator, it is easily seen that ν is actually a character of $SL_2(\mathbb{Z})$ yielding the multiplier system of the generalized modular form f of weight k. □

Let us give some examples. For the basic equivariant function h_1 given by (5.6), the associated infinite product is given by the discriminant function

$$\Delta(z) = q \prod_{n=1}^{\infty} (1-q^n)^{24}, \quad q = e^{2\pi i z}.$$

According to (5.5)

$$E_2(z) = \frac{1}{2i\pi} \frac{\Delta'(z)}{\Delta(z)},$$

and thus the fundamental Eq. (8.5) reduces to (5.3).

We can also consider, as in [4], the following infinite products

$$\begin{aligned} j(\tau) &= q^{-1} \prod_{n>0} (1-q^n)^{3c_0(n^2)} \\ &= q^{-1}(1-q)^{-744}(1-q^2)^{80256}(1-q^3)^{-12288744}\cdots, \\ E_6(\tau) &= \prod_{n>0}(1-q^n)^{a(n^2)} \\ &= (1-q)^{504}(1-q^2)^{143388}(1-q^3)^{51180024}\cdots. \end{aligned}$$

and define the corresponding equivariant functions and associated Lambert series expansions. More generally, in [4], Borcherds gives a striking description of the exponents in the infinite product expansion of several modular forms in terms of the Fourier coefficients of some half integer meromorphic modular forms.

The q-expansion of j starts as

$$J(z) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots, \quad q = e^{2i\pi z} \tag{8.6}$$

and we introduce a sequence of modular functions

$$J_0(z) = 1, \ J_1(z) = J(z) - 744$$

and for $m \geq 2$, we define

$$j_m(z) = J_1(z)|T_0(m)$$

where $T_0(m)$ is the normalized mth Hecke operator defined by

$$g(z)|T_0(m) = \sum_{d|m,\, ad=m} \sum_{b=0}^{d-1} g\left(\frac{az+b}{d}\right)$$

Then we have [4]

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n = \prod_{n=1}^{\infty} (1-q^n)^{c(n^2)} \tag{8.7}$$

where the $c(n)s$ denote the coefficients of a weight 1/2-modular form on $\Gamma_0(4)$ whose expansion starts as

$$f(z) = q^{-3} + 4 - 240q + 26760q^4 - 85995q^5 + 1707264q^8 + \cdots$$

or more explicitly,

$$c(n) = 8 + \frac{1}{3n} \sum_{d|n} \mu\left(\frac{n}{d}\right) j_d(\omega), \ \omega = \frac{1+\sqrt{-3}}{2};$$

a formula which can be compared with the identity $J = \frac{E_4^3}{\Delta}$. Borcherds theorem shows at once that the knowledge of the sequence of exponents $c(n)$ with the use of Lemma 8.1 gives, in principal, the full Fourier coefficients of the periodic part $h(z) - z$ of an equivariant function h. However, this not an easy task in practice, even for basic example such as h_1.Indeed, due to the lack of modularity, the Fourier series of $\frac{1}{E_2}$ has not been investigated before, contrary to $\frac{1}{E_4}$ and $\frac{1}{E_6}$. The Fourier series of $\frac{1}{E_6}$ has been studied by Hardy and Romanian [14], and very recently the Fourier series of $\frac{1}{E_4}$ has been studied by Bernie, Bialys [3]. The results are deep and for comparison and later discussion we quote them and give an immediate consequence using Lemma 8.1.

Theorem 8.5 (Hardy–Ramanujan) *Let* $C = \frac{3\pi^2}{4\Gamma^8(\frac{3}{4})}$. *Define the coefficients* p_n *by*

$$\frac{1}{E_6(q)} = \sum_{n=0}^{\infty} p_n q^n.$$

Then, for $n \geq 0$,

$$p_n = \sum_{\mu} T_\mu(n),$$

where μ runs over all integers of the form

$$\mu = 2^a \prod_{j=1}^{r} p_j^{a_j}, \tag{8.8}$$

where $a = 0$ or 1, p_j is a prime of the form $4m+1$, and a_j is a nonnegative integer and where

$$T_1(n) = \frac{2}{C^2} e^{n\pi}, \quad T_2(n) = \frac{2}{C^2} \frac{(-1)^n}{2^4} e^{n\pi},$$

and for for $\mu > 2$,

$$T_\mu(n) = \frac{2}{C^2} \frac{e^{\frac{2n\pi}{\mu}}}{\mu^4} \sum_{c,d} 2\cos\left((ac+bd)\frac{2n\pi}{\mu} + 8\tan^{-1}\frac{c}{d}\right),$$

where the sum is over all pairs (c, d) satisfying $\mu = c^2 + d^2$ and (a, b) is any solution to $ad - bc = 1$.

Comparing with Lemma 8.1, we obtain at once, with $\sigma_5(k) = \sum_{d|n} d^5$

$$n!(504)^{-n} p_n = \begin{vmatrix} 2\sigma_5(1) & 1 & 0 & 0 & \cdots & 0 \\ 4\sigma_5(2) & 3\sigma_5(1) & 2 & 0 & \cdots & 0 \\ 6\sigma_5(3) & 5\sigma_5(2) & 4\sigma_5(1) & 3 & \cdots & 0 \\ . & . & . & . & \cdots & . \\ (2n-2)\sigma_5(n-1) & . & . & . & \cdots & n-1 \\ n\sigma_5(n) & (n-1)\sigma_5(n-1) & . & . & . & \sigma_5(1) \end{vmatrix}.$$

In a similar way, we have for E_4

Theorem 8.6 (Berndt–Bialek) *Let $\rho = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and set*

$$\frac{1}{E_4(q)} = \sum_{n=0}^{\infty} \beta_n q^n$$

and

$$G = E_6(\rho),$$

then

$$\beta_n = (-1)^n \frac{3}{G} \sum_{(\lambda)} \frac{h_\lambda(n)}{\lambda^3} e^{n\pi \frac{\sqrt{3}}{\lambda}}.$$

Here λ runs over the integers of the form (8.8),

$$h_1(n) = 1, \quad h_3(n) = -1,$$

and, for $\lambda \geq 1$

$$h_\lambda(n) = 2\sum_{c,d} \cos\left((ad + bc - 2ac - 2bd + \lambda)\frac{n\pi}{\lambda} - 6\tan^{-1}\left(\frac{c\sqrt{3}}{2d-c}\right)\right),$$

where the sum is over all pairs (c, d) *satisfying* $\lambda = c^2 - cd + d^2$ *and* (a, b) *is any solution to* $ad - bc = 1$.

Comparing with the Lemma 8.1 we obtain once again, with $\sigma_3(k) = \sum_{d|n} d^3$,

$$n!(240)^{-n}\beta_n = \begin{vmatrix} 2\sigma_3(1) & 1 & 0 & 0 & \cdots & 0 \\ 4\sigma_3(2) & 3\sigma_3(1) & 2 & 0 & \cdots & 0 \\ 6\sigma_3(3) & 5\sigma_3(2) & 4\sigma_3(1) & 3 & \cdots & 0 \\ . & . & . & . & \cdots & . \\ (2n-2)\sigma_3(n-1) & . & . & . & \cdots & n-1 \\ n\sigma_3(n) & (n-1)\sigma_3(n-1) & . & . & . & \sigma_3(1) \end{vmatrix}.$$

Remark 8.7 It would be interesting to have another formulation for the determinant in (8.1) similar to those given for the coefficients of $\frac{1}{E_4(q)}$ and $\frac{1}{E_6(q)}$ despite of the lack of the modularity for E_2.

9 $\wp^n$ as an elliptic function and a differential algebra

In this section, we give some properties of the meromorphic elliptic function $\wp^n$, $n \in \mathbb{Z}$ and with two exceptions, we will associate to each $\wp^n$ an equivariant function. As was observed by Heins, the basic example of equivariant functions is related to the Weierstrass functions $\zeta' = -\wp$ by way of the Legendre relation. Our main task is to find integrals of $\wp^n$, $n \in \mathbb{Z}$. For later use, let us recall the essential idea. If ω_1, ω_2 are two complex numbers with $\Im(\omega_2/\omega_1) > 0$ and $\Omega = 2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z}$, the Weierstrass functions are

$$\wp(z; \omega_1, \omega_2) = \wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Omega\setminus\{0\}}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right),$$

$$\zeta(z; \omega_1, \omega_2) = \zeta(z) = \frac{1}{z} + \sum_{\omega\in\Omega\setminus\{0\}}\left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right),$$

$$\sigma(z; \omega_1, \omega_2) = \sigma(z) = \prod_{\omega\in\Omega\setminus\{0\}}\left(1 - \frac{z}{\omega}\right)\exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right).$$

Hence $\wp = -\zeta'$, $\frac{\sigma'}{\sigma} = \zeta$. In some situations, it is better to use the odd Jacobi theta function

$$\theta_1(z|\tau) = i\sum_{n\in\frac{1}{2}+\mathbb{Z}} (-1)^{n-\frac{1}{2}} e^{2i\pi nz} e^{i\pi n^2\tau}.$$

The zero divisors of this function form the lattice $\mathbb{Z} + \tau\mathbb{Z}$. The general principal that we follow and which goes back to Liouville and Hermite is that if the principal part of an elliptic meromorphic function at each of its poles is known, then this function is determined up to an additive constant. More precisely we have the decomposition theorem [36].

Theorem 9.1 *If $a_k, 1 \leq k \leq n$, is the set of poles of an elliptic function f, of periods $2\omega_1, 2\omega_2$ and if at a_k the principal part is*

$$\sum_{s=1}^{r_k} c_{ks}(z-a_k)^{-s},$$

then there exists constant A such that

$$f(z) = A + \sum_{k=1}^{n}\sum_{s=1}^{r_k} \frac{(-1)^{s-1}}{(s-1)!} c_{ks}\zeta^{(s-1)}(z-a_k).$$

Consequently, a primitive of f is given by

$$\int f(z)dz = Az + B + \sum_{k=1}^{n}\left[c_{k1}\log\sigma + \sum_{s=2}^{r_k}\frac{(-1)^{(s-1)}}{(s-1)!}c_{ks}\zeta^{(s-2)}(z-a_k)\right],$$

with B being an arbitrary constant. Moreover, with Jacobi theta function , we have

$$f(z) = C + \sum_{k=1}^{n}\sum_{s=1}^{r_k}\frac{(-1)^{(s-1)}}{(s-1)!}c_{ks}\frac{d^s}{dz^s}\log\theta_1\left(\frac{\pi z - \pi a_k}{2\omega_1}|\frac{2\omega_2}{2\omega_1}\right)$$

and

$$\int f(z)dz = Cz + D + \sum_{k=1}^{n}\sum_{s=1}^{r_k}\frac{(-1)^{(s-1)}}{(s-1)!}c_{ks}\frac{d^{s-1}}{dz^{s-1}}\log\theta_1\left(\frac{\pi z - \pi a_k}{2\omega_1}|\frac{2\omega_2}{2\omega_1}\right).$$

The coefficient c_{k1} is the residue of f at the pole a_k, hence

$$\sum_{i=1}^{n} c_{k1} = 0.$$

In particular $c_{11} = 0$ if there is only one pole. This theorem is very deep in the sense that it gives all the differential relations that will be considered below and also all the known relations between Weierstrass elliptic functions and Jacobi elliptic functions. In addition, it essentially says that if $\mathcal{D} = \mathbb{C}\left[\frac{d}{dz}\right]$ is the ring of differential operators with constant coefficients and $\mathcal{M}$ the $\mathbb{C}$-vector space of elliptic meromorphic functions with a pole at 0, then $\mathcal{M}$ is a left $\mathcal{D}$-module, that is $(\mathcal{M}, \frac{d}{dz})$ is a differential graded algebra

$$\mathcal{M} = \mathbb{C} \oplus \mathbb{C}\wp \oplus \mathbb{C}\wp' \oplus \cdots \mathbb{C}\wp^{(n)} \oplus \cdots.$$

As a $\mathcal{D}$-module, $\mathcal{M}$ is generated by two elements $1, \wp$, hence we a have the free resolution

$$0 \to \mathcal{D} \xrightarrow{\phi} \mathcal{D}^2 \xrightarrow{\psi} \mathcal{M} \to 0$$

where $\psi(D_1, D_2) = D_1.1 + D_2\wp$ and $\phi(D) = (D, -D.1)$.

Perhaps the most fascinating examples of applications of this theorem are the two following identities of Frobenius and Stickelberger

$$(-1)^{\frac{n(n-1)}{2}}\prod_{k=1}^{n}k!\frac{\sigma(z_0+z_1+\cdots+z_n)\prod_{0\le i<j\le n}(z_i-z_j)}{\sigma^{n+1}(z_0)\cdots\sigma^{n+1}(z_n)}$$
$$=\begin{vmatrix} 1 & \wp(z_0) & \wp'(z_0) & \cdots & \wp^{(n-1)}(z_0)\\ 1 & \wp(z_1) & \wp'(z_1) & \cdots & \wp^{(n-1)}(z_1)\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 1 & \wp(z_n) & \wp'(z_n) & \cdots & \wp^{(n-1)}(z_n)\end{vmatrix}$$

and of Kiepert

$$(-1)^{(n-1)}\left(\prod_{k=1}^{n-1}k!\right)^2\frac{\sigma(nu)}{\sigma^{n^2}(z)}=\begin{vmatrix} \wp'(z) & \wp''(z) & \cdots & \wp^{(n-1)}(z)\\ \wp''(z) & \wp'''(z) & \cdots & \wp^{(n)}(z)\\ \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots\\ \wp^{(n-1)}(z) & \wp^{(n)}(z) & \cdots & \wp^{(2n-3)}(z)\end{vmatrix}.$$

In the identity of Frobenius and Stickelberger, the left hand side considered as function of z_0, is an elliptic function having $z_0=0$ as a pole of order at most n. Its decomposition according to the theorem (9.1) is given by the right hand side. The coefficients of the decomposition are obtained by developing the determinant with respect to the elements of the first row. The identity of Kiepert can be obtained from the one of Frobenius and Stickelberger by a limiting process.

The Weierstrass function $\wp$ is homogeneous of degree -2 and is a generating function of the classical Eisenstein series

$$\wp(z;\omega_1,\omega_2)=\frac{1}{z^2}+\sum_{k=1}^{\infty}(2k+1)G_{2k+2}z^{2k},\quad G_{2k+2}=\sum_{\omega\in\Omega\setminus\{0\}}\omega^{-2k}.$$

For $\tau=\frac{\omega_2}{\omega_1}$ fixed, $\wp$ and its derivative $\wp'$ are elliptic functions for $\mathbb{Z}+\tau\mathbb{Z}$. The zeros of $\wp'$ in $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ occur at the points of order 2, namely $\frac{1}{2},\frac{\tau}{2}$ and $\frac{1+\tau}{2}$. On the other hand, the zeros of $\wp$ were described in [11] and more recently in [10]. Since $\wp$ assumes every value in $\mathbb{C}\cup\{\infty\}$ exactly twice in $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, it follows that $\wp$ has two zeros therein which can be written as $\pm z_0$ since $\wp$ is even.

Proposition 9.2 *The zeros of the $\wp$-function are given by $\pm z_0$ where, by the Eichler–Zagier formula,*

$$z_0=m+\frac{1}{2}+n\tau\pm\left(\log\frac{5+2\sqrt{6}}{2i\pi}+144i\pi\sqrt{6}\int_{\tau}^{i\infty}(\sigma-\tau)\frac{\Delta(\sigma)}{E_6(\sigma)^{\frac{3}{2}}}d\sigma\right)$$

for all $m,n\in\mathbb{Z}$, where the integral is to be taken over the vertical line $\sigma=\tau+i\mathbb{R}^+$ or by the Duke-Imamoglu formula

$$z_0=\frac{1+\tau}{2}+\frac{c_2x^{\frac{1}{4}}F\left(\frac{1}{3},\frac{2}{3},1;\frac{3}{4},\frac{5}{4}|x\right)}{F\left(\frac{1}{12},\frac{5}{12}|1-x\right)},\quad c_2=\frac{-i\sqrt{6}}{3\pi}$$

where $x = 1 - \dfrac{1728}{j}$ *and where the generalized hypergeometric series defined for* $|x| < 1$ *by*

$$F(a_1, \ldots, a_m; b_1, \ldots, b_{m-1}|x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_m)_n}{(b_1)_n \cdots (b_{m-1})_n} \frac{x^n}{n!}.$$

and $(a)_n = a(a+1)\cdots(a+n-1)$.

The pseudo-periods $\eta_1,\ \eta_2$ of the Weierstrass ζ-function are defined by

$$\zeta(u + 2\omega_\alpha) = \zeta(u) + 2\eta_\alpha, \quad \zeta(\omega_\alpha) = \eta_\alpha, \quad \alpha = 1, 2.$$

The Legendre relation is $\eta_1\omega_2 - \eta_2\omega_1 = \frac{1}{2}i\pi$. The periods and pseudo-periods of the Weierstrass $\wp$-function are related to Ramanujan Eisenstein series by

$$E_2(z) = \frac{12}{\pi^2}\eta_1\omega_1, \qquad E_4(z) = 12\left(\frac{\omega_1}{\pi}\right)^4 g_2, \qquad E_6(z) = 216\left(\frac{\omega_1}{\pi}\right)^6 g_3.$$

In particular if $2\omega_1 = 1,\ 2\omega_2 = \tau,\ \tau = \dfrac{\omega_2}{\omega_1}$, then

$$h_1(\tau) = \tau + \frac{6}{i\pi E_2(\tau)} = \frac{\eta_2}{\eta_1}. \tag{9.1}$$

Thus the basic equivariant function is a quotient of pseudo-periods. This interpretation will reveal important differential properties. We would like to extend this construction to the powers $\wp^n, n \in \mathbb{Z}$ with two exceptions.

Let $\mathbf{L}$ be the set of lattices in the $\mathbb{R}$-vector space $\mathbb{C}$ and

$$\mathbf{M} = \{(\omega_1, \omega_2) \in \mathbb{C}^{*2} \ :\ \tau = \Im(\omega_2/\omega_1) > 0\}.$$

Then $\mathbf{L}$ can be identified with the quotient $\mathbf{M}/\mathrm{SL}_2(\mathbb{Z})$. Moreover, $\mathbb{C}^*$ acts on $\mathbf{L}$ and on $\mathbf{M}$ yielding two more identifications

$$\mathbf{M}/\mathbb{C}^* \approx \mathbf{H}, \quad \mathbf{R}/\mathbb{C}^* \approx \mathbf{H}/\mathrm{PSL}_2(\mathbb{Z}).$$

Following Brady [5], we introduce

Definition 9.1 A function $f : \mathbb{C} \times \mathbf{L} \to \mathbb{P}$ is called pseudo-periodic if it is meromorphic in z and for each $\omega \in \Omega = 2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z},\ (\omega_1, \omega_2) \in \mathbf{M}$ there is a constant $\eta(\omega|\Omega)$ such that for each $z \in \mathbb{C}$

$$f(z + \omega, \omega) = f(z, \omega) + \eta(\omega|\Omega).$$

In [5], Brady observed first that if f is a homogeneous pseudo-periodic function with pseudo-periods $\eta(\omega|\Omega)$, then the map $\tau \to \dfrac{\eta(\tau|\Omega_\tau)}{\eta(1|\Omega_\tau)}$ is an equivariant function, provided that $\tau \to \eta(1|\Omega_\tau), \tau \to \eta(\tau|\Omega_\tau)$ are meromorphic and $\eta(1|\Omega_\tau)$ does not vanish identically.

In this paper, we are studying a similar question, namely the zeta function associated to the elliptic function $\wp^n,\ (n \in \mathbb{Z}^*)$. This function is an even elliptic function, homogeneous of degree $-2n$, of periods ω_1, ω_2. We look for a primitive of $\wp^n$ giving rise to pseudo-periodic function. We recall the following definition [11]

Definition 9.2 Let k and m be two fixed integers. A function $\phi : \mathbf{H} \times \mathbb{C} \to \mathbb{P}$ is called a meromorphic Jacobi form of weight k and index m if

(i) ϕ is meromorphic on $\mathbb{C} \times \mathbf{H}$,

(ii) ϕ satisfies

$$\phi\left(\frac{z}{\gamma\tau+\delta}, \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) = (\gamma\tau+\delta)^k \exp\left(2i\pi m \frac{\gamma z^2}{\gamma\tau+\delta}\right)\phi(z,\tau),$$

for every $\begin{pmatrix}\alpha\ \beta\\ \gamma\ \delta\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

(iii) ϕ has a meromorphic q-expansion of the form

$$\phi(z,\tau) = \sum_{n\geq h} c_n(z)q^n, \quad 0 < |\xi| < A,\ 0 < |q| < B|\xi|^N,$$
$$\xi = e^{2i\pi z}, A > 0, B > 0, N \in \mathbb{N}, ,$$

where the coefficients $c_n(z)$ are in the function field $\mathbb{C}(\xi)$.

The function $\wp$ is an example of a Jacobi form of weight 2 and index 0, that is a meromorphic function that satisfies

$$\wp\left(\frac{z}{\gamma\tau+\delta}; \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) = (\gamma\tau+\delta)^2\,\wp(z;\tau).$$

Thus $\wp^n,\ n \in \mathbb{N}^*$ is a Jacobi form of weight $2n$. The origin is the unique pole in a fundamental domain, of order $2n$.

In general, let $\Phi(z,\tau)$ be a meromorphic periodic function in z with respect to the lattice $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$. Assume that, as function of τ, it satisfies

$$\Phi\left(\frac{z}{\gamma\tau+\delta}; \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) = (\gamma\tau+\delta)^m\,\Phi(z,\tau)$$

for every

$$M = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Let $g_p(\tau)$ be the p-th coefficient of the Taylor expansion of $\Phi(z;\tau)$ at $x_0 = x\tau + y$ for some $x, y \in \mathbb{R}$. Then for any $M \in \mathrm{SL}_2(\mathbb{Z})$ such that $(x', y') = (x, y)M \equiv (x, y) \in \mathbb{Z}^2$, we have

$$g_p\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) = (\gamma\tau+\delta)^{m+p}\,g_p(\tau).$$

This means that for a fixed integer $n \neq 0$, the Taylor coefficient $g_p^n,\ p \in \mathbb{N}$ at the origin of the function $\wp^n$ is a modular form on $\mathrm{SL}_2(\mathbb{Z})$ of weight $p + 2n$.

As is well known, many analytic properties of the $\wp$-function come from the differential equation

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3, \quad g_2 = 4\frac{\pi^4}{3}E_4, \quad g_3 = \frac{8\pi^6}{27}E_6. \tag{9.2}$$

From the decomposition theorem (9.1) or from (9.2) we obtain

$$\wp^2 = \frac{1}{6}\wp'' + \frac{1}{12}g_2, \quad \wp\wp' = \frac{1}{12}\wp''',$$
$$\wp^3 = \frac{1}{10}\left(\frac{1}{12}\wp^{(4)} + \frac{3}{2}g_2\wp + g_3\right), \ldots,$$

and more generally, we have

Proposition 9.3 *For each positive integer,*

$$\wp^n = \frac{B_0^{(n)}}{(2n-1)!}\wp^{(2n-2)} + \frac{B_1^{(n)}}{(2n-3)!}\wp^{(2n-4)} + \cdots + \frac{B_r^{(n)}}{(2n-2r-1)!}\wp^{(2n-2r-2)} + \cdots + B_{n-2}^{(n)}\wp'' + B_{n-1}^{(n)}\wp + B_n^{(n)}, \tag{9.3}$$

where, for $0 \le r \le n+1$

$$B_r^{(n+1)} = \tfrac{(2n-2r)(2n-2r+1)}{2n(2n+1)} B_r^{(n)} + \tfrac{2n-1}{4(2n+1)} B_{r-2}^{(n-1)} g_2 + \tfrac{n-1}{2(2n+1)} B_{r-3}^{(n-2)} g_3 \tag{9.4}$$

and $B_r^{(n)} = 0$ *for* $r < 0$ *and* $r > n$.

As a consequence of (9.3), we obtain

$$\wp^n \wp' = \frac{1}{n+1}\frac{B_0^{(n+1)}}{(2n+1)!}\wp^{(2n+1)} + \cdots + \frac{1}{n+1}\frac{B_r^{(n+1)}}{(2n-2r+1)!}\wp^{(2n-2r+1)} + \cdots + \frac{1}{n+1}B_{n-1}^{(n+1)}\wp''' + \frac{1}{n+1}B_n^{(n+1)}\wp'. \tag{9.5}$$

B_{n-1}^n are constants and $B_1^n, B_2^n, \ldots, B_{n-1}^n$ are homogeneous polynomials in g_2 and g_3. In particular $B_{n-2}^n = 0$.

Lemma 9.4 *For every* $n \in \mathbb{N} \setminus \{0\}$, *we have:*

$$\wp^{n+1} = \frac{1}{2n(2n+1)}(\wp^n)'' + \frac{(2n-1)}{4(2n+1)} g_2 \wp^{n-1} + \frac{n-1}{2(2n+1)} g_3 \wp^{n-2},$$

with $\wp^{-1}$ *taken as* 0.

The proof of this lemma is a straightforward computation from (9.2) and (9.3). We conclude from (9.3) that lower order coefficients and pseudo-periods are given by the four-term relations

$$B_{n+1}^{(n+1)} = \frac{2n-1}{4(2n+1)} g_2 B_{n-1}^{(n-1)} + \frac{n-1}{2(2n+1)} g_3 B_{n-2}^{(n-2)}, \tag{9.6}$$

$$B_n^{(n+1)} = \frac{2n-1}{4(2n+1)} g_2 B_{n-2}^{(n-1)} + \frac{n-1}{2(2n+1)} g_3 B_{n-3}^{(n-2)}, \tag{9.7}$$

$$\eta_{n+1} = \frac{2n-1}{4(2n+1)} g_2 \eta_{n-1} + \frac{n-1}{2(2n+1)} g_3 \eta_{n-2}. \tag{9.8}$$

By inversion of these relations or by taking successive derivations of (9.2) we obtain

$$\begin{aligned} \wp'' &= 6\wp^2 - \frac{1}{2} g_2, \\ \wp''' &= 12\wp\wp', \\ \wp^{(4)} &= 12\left(10\wp^3 - \frac{3}{2} g_2 \wp - g_3\right), \\ &\text{etc.} \end{aligned} \tag{9.9}$$

More generally, we have for each $n \in \mathbb{N}$

$$\wp^{(2n)} = P_{n+1}(\wp), \quad \wp^{(2n+1)} = Q_n(\wp)\,\wp', \tag{9.10}$$

where P_{n+1}, Q_n are polynomials in $\wp$ of degree $n+1$ and n, respectively and their coefficients are polynomials in g_2 and g_3 with rational coefficients. For $l < n+1$, the coefficient of $\wp^l$ is a modular form of weight $2n+2-2l$. Looking at one of these polynomials

$$A\wp^m + A_1\wp^{m-1} + \cdots,$$

we notice that it must be a homogeneous polynomial of degree $-2m$ in u, ω_1, ω_2 so that its coefficients must be of the following form

$$A_1 = 0, \quad A_2 = ag_2, \quad A_3 = bg_3,$$
$$A_4 = cg_2^2, \quad A_5 = dg_2g_3, \quad A_6 = eg_2^3 + fg_3^2, \cdots$$

where $a, b, \ldots, f$ are numerical constants. A precise analysis of the polynomials P_n, Q_n can also be done using the following results due to Feldman [2] and which are very useful in transcendence theory.

Lemma 9.5 *The jth derivative $\wp^{(j)}(z)$ of $\wp(z)$ can be expressed in the form*

$$\sum u(t, t', t'')(\wp(z))^t(\wp'(z))^{t'}(\wp''(z))^{t''}$$

where the summation is over all non-negative integers t, t', t'' with $2t + 3t' + 4t'' = j + 2$ and $u(t, t', t'', j, k)$ denotes rational integers with absolute values at most $3^j(j+7)!$.

Lemma 9.6 *For any positive integer k, the jth derivative of $(\wp(z))^k$ can be expressed in the form*

$$\sum u(t, t', t'', j, k)(\wp(z))^t(\wp'(z))^{t'}(\wp''(z))^{t''}$$

where the summation is over all non-negative integers t, t', t'' with $2t + 3t' + 4t'' = j + 2k$ and $u(t, t', t'', j, k)$ denote rational integers with absolute values at most $j!48^j(7!2^8)^k$.

We recall that for a field k and a non-zero polynomial $P \in k[X_1, X_2, \cdots, X_n]$, the weight $w(P)$ is defined by

$$w(P) = \deg_t P\left(tX_1, t^2X_2, \ldots, t^nX_n\right).$$

The polynomial P is called isobaric of weight $w(P)$ if for any monomial $X_1^{i_1}\ldots X_n^{i_n}$ of $P(X_1, \ldots, X_n)$, we have

$$w(P) = \sum_{r=1}^{n} ri_r.$$

It is a natural problem to investigate the polynomial in three indeterminate variables

$$Q(X, Y, Z) = \sum u(t, t', t'') X^t Y^{t'} Z^{t''}.$$

It is isobaric if one attributes to X, Y and Z the weights 2, 3 and 4, respectively.

To give a similar formula for $\wp^n$ for a negative integer n, we use the precise analysis of Zagier and Eichler concerning the zeros of the Weierstrass $\wp$-function, quoted in proposition (9.2). They gave two proofs of their result which reveal that the zero $z_0(\tau)$ of $\wp(z, \tau)$ is, as function of τ, a (multi-valued) modular form of weight -1 and, in addition, its second derivative is given by

$$z_0''(\tau) = \pm 144 i\pi\sqrt{6}\frac{\Delta(\tau)}{E_6(\tau)}$$

and hence

$$z_0(\tau) = m + \frac{1}{2} \pm \frac{1}{2i\pi}\log\epsilon + n\tau \pm 144i\pi\sqrt{6}\left(e^{2i\pi\tau)} + 183e^{4i\pi\tau)} + \cdots\right).$$

We will consider two cases:

The case $\tau \neq i$: The function $\wp(z) = \wp(z; 1, \tau)$ has two simple zeros in L, the fundamental parallelogram of the periods 1 and τ, of opposite residues. Now for a zero $z_0 \in L$ of $\wp$

$$\begin{aligned} \wp^{-1}(z) &= \left[\wp'(z_0)(z - z_0) + \frac{\wp''(z_0)}{2}(z - z_0)^2 + \cdots\right]^{-1} \\ &= \frac{1}{\wp'(z_0)(z - z_0)} - \frac{\wp''(z_0)}{2\wp'^2(z_0)} + \cdots. \end{aligned}$$

Since $\wp$ is even, the other zero in L is $z_1 = -z_0 + 1 + \tau$. From the decomposition theorem (9.1) we obtain

$$\frac{1}{\wp(z)} = A + \frac{1}{\wp'(z_0)}\left[\zeta(z - z_0) - \zeta(z + z_0 - 1 - \tau)\right]. \tag{9.11}$$

The case $\tau = i$ (Lemniscatic case [30]): We first recall some known facts about the lemniscatic case. The lattice $\Lambda = \mathbb{Z} + i\mathbb{Z}$ admits a complex multiplication in the sense that the multiplication is an automorphism of Λ. Hence all the Eisenstein series G_p vanish except when p is divisible by 4. In this situation G_{4k} are real because Λ is also invariant under complex conjugation. The $\wp$-function verifies in this case

$$\begin{aligned} \wp'^2(z) &= 4\wp(z)^3 - g_2\wp(z), \\ \wp(z) &= \frac{1}{z^2} + \sum_{k=1}^{\infty}(4k - 1)G_{4k}z^{4k-2} \end{aligned}$$

from which we obtain the classical formulas $g_2 = 60G_4$, $g_3 = 140G_6$. We set $g_2 = 4w^4$, $w > 0$ and we obtain for the Lattice $L = w\wedge$:

$$30G_4 = 2, \quad G_8 = \frac{6}{7}G_4^2, \quad G_{12} = \frac{7}{22}G_4G_8,$$

and more generally the following recursion formula

$$((4k-1)(4k-2)(4k-3) - 12(4k-1))\, G_{4k} = 6G_{l+m=k}(4l-1)(4m-1)G_{4l}G_{4m}$$

which shows that all G_{4p}, $p = 1, 2, \ldots$, are positive. For our purpose, we retain that for $\tau = i$, we have from the theory of complex multiplication (Chowla-Selberg formula)

$$g_2(i) = 60G_4(i) = 60 \sum_{(m,n)\in\mathbb{Z}\setminus(0,0)} \frac{1}{(m+ni)^4} = 60\frac{\Gamma(\frac{1}{4})^8}{2^6 3.5.\pi^2}.$$

The number $\Gamma(\frac{1}{4})$ has very interesting arithmetical properties. It was shown to be transcendental by Chudnovsky, and Nesterenko showed that $\Gamma(\frac{1}{4}), \pi, e^{\pi}$ are algebraically independent. Moreover, several of its infinite product expansions are known, from which we conclude that some relations exist between the exponents in (8.7) and the Catalan constant for example, or the Glaisher–Kinkelin constant as well [1]. More precisely, the Fourier series expansion shows that for the lattice $\mathbb{Z} + i\mathbb{Z}$ we have $G_4(i) = \frac{\pi^4}{45} E_4(i)$, and hence

$$E_4(i) = 3\frac{\Gamma(\frac{1}{4})^8}{(2\pi)^6},$$

and from (8.7), we obtain the infinite product expansion

$$\Gamma\left(\frac{1}{4}\right) = \left(\frac{1}{3}\right)^{\frac{1}{8}} (2\pi)^{\frac{3}{4}} \prod_{n=1}^{\infty} (1 - e^{-2\pi n})^{\frac{c(n)}{8}}.$$

In the meantime, $\tau = i$, $z_0 = (1+i)/2$ is a double zero for $\wp$ and by (9.9) we have

$$\frac{1}{\wp(z)} = \frac{1}{\wp''(z_0)(z-z_0)^2} - \frac{2}{3}\frac{\wp'''(z_0)}{\wp''(z_0)}(z-z_0) + \cdots = \frac{1}{\wp''(z_0)(z-z_0)^2} + \cdots .$$

Finally, from the decomposition theorem (9.1) we obtain

$$\frac{1}{\wp(z)} = -2\frac{1}{\wp''(\frac{1+i}{2})}\zeta'\left(z - \frac{1+i}{2}\right) = \frac{4}{g_2(i)}\wp\left(z - \frac{1+i}{2}\right). \tag{9.12}$$

10 Algebraic differential equations, a special Lie algebra and hypergeometric equations

In this section we study various differential equations related to equivariant functions. We first prove the following result

Theorem 10.1 *All equivariant functions on* **H** *satisfy (nonlinear) differential relations of order at most 6.*

This result will emerge from the full functional Eq. (1.1). We recall that, in general, one functional equation is not enough to insure the existence of a differential equation. The Euler gamma function Γ verifies essentially one functional equation and a theorem of Hölder asserts that it is differentially transcendental over the field of rational functions $\mathbb{R}(X)$, which means that there is no polynomial $P(X_1, X_2, \ldots, X_n) \in \mathbb{R}(x)[X_1, X_2, \ldots, X_n]$ such that

$$P\left(\Gamma, \Gamma', \Gamma'', \ldots, \Gamma^{(n)}\right) = 0,$$

or, in other words, the transcendence degree

$$\mathrm{Tr}_{\deg}\left(x, \Gamma'(x), \Gamma''(x), \ldots, \Gamma^{(n)}(x), \ldots\right) = \infty.$$

We have at least two specific methods to check the asserted statement for the basic example $h_1(z) = z + \dfrac{6}{i\pi E_2(z)}$. One has been given in Sect. 9 and the second one follows from the fact that the Eisenstein series $y = E_2(z) = \dfrac{6}{i\pi}\dfrac{1}{h_1(z) - z}$ satisfies the Chazy equation [37]

$$D^3 y - yD^2 y + \frac{3}{2}(Dy)^2 = 0, \quad Dy = \frac{1}{2i\pi}\frac{dy}{dz}. \tag{10.1}$$

This Chazy equation for E_2 is converted, for $y = \frac{1}{E_2}$, into a differential equation as follows

Lemma 10.2 *We have*

$$-y^2 D^3 y + 6yDyD^2 y - 6(Dy)^3 + yD^2 y - \frac{1}{2}(Dy)^2 = 0.$$

As a consequence, we obtain, after a lengthy but elementary computation,

Proposition 10.3 *The derivative h' of h satisfies the following differential equation*

$$\begin{aligned}
&3h'^4 h^{(4)2} - 24h'^3 h'' h^{(3)} h^{(4)} + 8h'^3 h^{(3)3} - 6h'^3 h^{(4)2} \\
&+18h'^2 h^{(2)3} h^{(4)} + 12h'^2 h^{(2)2} h^{(3)2} - 12h'^2 h'' h^{(3)} h^{(4)} \\
&+32h'^2 h^{(3)3} + 3h'^2 h^{(4)2} - 18h' h''^4 h^{(3)} + 54h' h''^3 h^{(4)} - 48h' h''^2 h^{(3)2} \\
&-36h' h^{(2)} h^{(3)} h^{(4)} + 32h' h^{(3)3} - 36h''^4 h^{(3)} + 36h''^3 h^{(4)} - 36h^{(2)2} h^{(3)2} = 0.
\end{aligned}$$

To prove Theorem 10.1 in its full generality we use the fundamental fact that the Schwarz derivative h of an equivariant holomorphic function f is given by a differential expression of order 3 and is a modular form of weight 4. It is a classical fact that the graded ring of modular forms for $SL_2(\mathbb{Z})$ and their derivatives is generated over $\mathbb{C}$ by the three basic Eisenstein series E_2, E_4, E_6. In the meantime, $\mathbb{C}(E_2, E_4, E_6)$ has, according to [22], a transcendence degree 3 over $\mathbb{C}$. Hence, there exists a polynomial $P(X_1, X_2, X_3, X_4) \in \mathbb{C}[X_1, X_2, X_3, X_4]$ such that $P(h, h', h'', h''') = 0$. This establishes Theorem 10.1.

As we already pointed out, one needs to know the Fourier coefficients of $\frac{1}{E_2}$ in order to determine the periodic part of the fundamental example h_1. Though no elegant description seems to emerge (except what we said in Lemma 8.1), these coefficients can also be given by induction from Lemma 10.2. However, some arithmetical properties are guaranteed by the following general result of Hurwitz [17] concerning the analytic solutions of algebraic differential equations. Before we proceed further, let us recall the general context mentioned earlier in this paper regarding the algebraic setting.

Theorem 10.4 (Eisenstein [26]) *A series*

$$\sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{Q},$$

of radius of convergence 1, has the property that if the sum is algebraic, then there exists an integer N such that the quantities $a_n N^n$ are integers.

The converse is not true, for in any series with a finite radius of convergence, it suffices to change the signs of an infinite number of coefficients so that the circle of convergence of the new series shall be reduced to the extent that the new series cannot be algebraic although its coefficients have the required property. Taking into account that an algebraic equation is a zero order differential equation, Hurwitz, in [17], used an argument similar to Heine's proof of Eisenstein's theorem to prove the following result

Theorem 10.5 *Suppose that a holomorphic function f around the origin is given by a power series expansion*

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots, \quad a_k = \frac{p_k}{q_k} \in \mathbb{Q}$$

and is a solution of an algebraic differential equation

$$P(x, f, f', \dots, f^{(l)}) = 0, \quad P \in \mathbb{Z}(x)[X_1, X_2, \dots, X_l].$$

Then there exists a polynomial $T(X) = \gamma_0 + \gamma_1 X + \cdots + \gamma_l X^l \in \mathbb{Z}[X]$ *and an integer* N *such that every prime which divides* $q_N, q_{N+1}, q_{N+2} \dots$ *must divide* $T(N), T(N)T(N+1), T(N)T(N+1)T(N+2), \dots$ *respectively.*

This result knew several refinements and extensions, including results of Popken and Ostrowski [23], p. 206. They have been used in transcendence problems [23,33]. They show, in particular, that if p_m denotes the largest prime in the denominator of the coefficient a_m, then

$$\limsup_{m\to\infty} \frac{p_m}{\log m} < \infty.$$

Moreover, the following result of Maillet [24] gives an idea on the gaps in power series solutions of an algebraic differential equation:

Theorem 10.6 (Maillet) *Let* $\sum_{m=0}^{\infty} b_m z^m$ *be a given formal solution to a differential equation of order* k *and degree* μ, *we can find a fixed number* τ, *independent of* m *such that for large* m, *if* b_m *is non-zero, the previous non-zero coefficient has an index greater than or equal to* $\dfrac{(m+\tau)}{\mu}$.

This will enable us to establish a different approach to study the coefficients of $\dfrac{1}{E_2}, \dfrac{1}{E_4}$ and $\dfrac{1}{E_6}$ as power series in $q = e^{2i\pi z}$ solutions to algebraic differential equations.

The three Eisenstein series E_2, E_4, E_6 all satisfy algebraic differential equations; a fact that is well known [29,37]. As we had already observed in Lemma 10.2, E_2 and $\dfrac{1}{E_2}$ are solutions to algebraic differential equations, but we could not find any reference giving the needed explicit algebraic differential equation for E_6. We will provide it here after giving a detailed summary of the necessary ideas following [29,37].

Let $M_k(\Gamma, \nu)$ be the space of all automorphic forms $f(z)$ of weight k with respect to a Fuchsian group of the first kind Γ with a multiplier system ν, and set $M_k(\Gamma, 1) = M_k(\Gamma)$. The classical Bol operator [29] is the differential operator D such that for each integer $m > 1$,

$$D^m : M_k(\Gamma, \nu) \mapsto M_{m(k+2)}(\Gamma, \nu)$$

and is defined as follows: Let $f \in M_k(\Gamma, \nu)$, $k \neq 0$ and recall the differential operator $\theta = \dfrac{1}{(2i\pi)} \dfrac{d}{dz} = q \dfrac{d}{dq}$, $q = e^{2i\pi z}$, then

$$\theta^m \left((f)^{((1-m)/k} \right) = (f)^{((1-(k+1)m)/k} D_m(f)$$

where $D_m(f)$ is a polynomial in f and its derivatives $f, f', \dots, f^{(m)}$. On the other hand, the Rankin-Cohen product is defined [37] for $f \in M_k(\Gamma)$ and $g \in M_l(\Gamma)$ as follows

Definition 10.1

$$[f, g]_n = \sum_{r=0}^{n} (-1)^r \binom{k+n-1}{n-r} \binom{l+n-1}{r} f^{(r)} g^{(n-r)},$$

where $f^{(r)}$ stands for $D^r f = \dfrac{1}{(2i\pi)^r} \dfrac{d^r f}{dz^r}$.

In this way $[f, g]_0 = fg$ and

$$[f, g]_1 = kfg' - lf'g,\ [f, g]_2 = \frac{k(k+1)}{2} fg'' - (k+1)(l+1) f'g' + \frac{l(l+1)}{2} f''g.$$

The Rankin–Cohen product sends $M_k(\Gamma) \times M_l(\Gamma)$ into $M_{k+l+2n}(\Gamma)$. Furthermore, the iterated Rankin-Cohen products are related in a remarkable way to the Bol operator. In fact it is easily seen that if f is of weight k, then

$$D_2(f) = -\frac{1}{k^2(k+1)} [f, f]_2,$$
$$D_3(f) = \frac{2}{k^3(k+1)} [f, [f, f]_1]_1.$$

On the other hand and very generally, if Γ is a discrete group acting on $\mathbf{H}$ such that a suitably chosen fundamental domain for Γ in $\mathbf{H}$ is finite with respect to the invariant metric, then the quotient $\mathbf{H}/\Gamma$ can be given a structure of a compact Riemann surface $\mathcal{R}$. As a consequence, any two meromorphic functions on $\mathcal{R}$ are algebraically dependent over $\mathbb{C}$. This is the main reason for a reasonably well behaved automorphic function to satisfy an algebraic differential equation over $\mathbb{C}$. More precisely [29].

Theorem 10.7 *Let Γ be a discrete group such that the quotient space $\mathbf{H}/\Gamma$ is of finite volume. Let $f \in M_k(\Gamma, \nu)$ be an analytic automorphic form with a multiplier system ν satisfying $\nu^s \equiv 1$ for some positive integer s. Then if the product ks is a positive integer, there exists a polynomial $P \in \mathbb{C}[X_1, X_2, X_3]$ such that on $\mathbf{H}$*

$$P\left(f, D^2(f), D^3(f)\right) = 0.$$

In [29] (Proposition 4), the following differential equation for E_4 is given. We give it here together with the corresponding differential equation for E_6.

Theorem 10.8 *The three Eisenstein series E_2, E_4 and E_6, as $q-$series, are all solutions to differential equations over $\mathbb{Q}$. The equation for E_2 is the Chazy equation, given in proposition (10.1) and for E_4 and E_6, we have*

$$5\left(D_3(E_4)\right)^2 - 576 D_2(E_4)^3 + 20 E_4^3 D_2(E_4)^2 = 0,$$
$$2744 D_2(E_6)^3 E_6^{12} - 343 D_3(E_6)^3 E_6^8 + 226492416 D_2(E_6)^6 E_6^4$$
$$-3096576 D_3(E_6)^2 D_2(E_6)^3 E_6^4 + 10584 D_3(E_6)^4 E_6^4 = 0.$$

Corollary 10.9 *The three inverses $\frac{1}{E_2}, \frac{1}{E_4}, \frac{1}{E_6}$, as q-series, are all solutions to algebraic differential equations over $\mathbb{Q}$. Their coefficients, as given in* Sect. 8*, satisfy the conclusions given in Hurwitz's theorem* (10.5) *and Maillet's theorem* (10.6).

On the other hand, we can show that the function h_1 is also related to a monodromy problem by considering it as a function of the modular invariant J rather than the variable $z \in \mathbf{H}$. This will be undertaken later in this section.

Another aim of the present section is to consider a special Lie algebra related to elliptic functions, well suited to study the powers $\wp^n$. We also show here that the basic equivariant function h_1 verifies a third order differential equation, with respect to the coordinate j, arising from a hypergeometric equation of Picard-Fuchs type. This shows that h_1 is also intimately related to uniformization theory. We first recall some classical facts on periods associated to elliptic curves. We begin with the two differential forms

$$\frac{dx}{\sqrt{4x^3 - g_2x - g_3}}, \quad \frac{xdx}{\sqrt{4x^3 - g_2x - g_3}}$$

and introduce the periods

$$\omega = \int \frac{dx}{\sqrt{4x^3 - g_2x - g_3}}, \quad \eta = \int \frac{xdx}{\sqrt{4x^3 - g_2x - g_3}}, \tag{10.2}$$

where the integrals are taken over suitable cycles on the elliptic curve $y^2 = 4x^3 - g_2x - g_3$. We normalize the periods in the following way

$$\Omega = \sqrt{\frac{g_3}{g_2}}\,\omega, \quad H = \sqrt{\frac{g_3}{g_2}}\,\eta.$$

As usual, we introduce $\Delta = g_2^3 - 27g_3^2$ that we suppose is non-vanishing and we let $J = \frac{g_2^3}{\Delta}$. We thus have the connexion formula

$$\begin{pmatrix} \frac{d\Omega}{dJ} \\ \frac{dH}{dJ} \end{pmatrix} = \begin{pmatrix} \frac{J+2}{12J(J-1)} & \frac{1}{18J} \\ \frac{-1}{8(J-1)} & -\frac{J+2}{12J(J-1)} \end{pmatrix} \begin{pmatrix} \Omega \\ H \end{pmatrix}. \tag{10.3}$$

A short calculation shows that a general differential system (whatever the variable J and the functions Ω, H) of the form

$$\frac{d}{dJ}\begin{pmatrix} \Omega \\ H \end{pmatrix} = \begin{pmatrix} a(J) & b(J) \\ c(J) & d(J) \end{pmatrix} \begin{pmatrix} \Omega \\ H \end{pmatrix}$$

leads to the second order differential equation

$$\frac{d^2\Omega}{dJ^2} - \left(a + d + \frac{b'}{b}\right)\frac{d\Omega}{dJ} + (ad - bc - a').$$

Moreover, the periods satisfy the Legendre relation

$$\begin{pmatrix} \omega_1 & \omega_2 \\ \eta_1 & \eta_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{pmatrix} = \frac{-i\pi}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Recall that the $J-$function given by (8.6) realizes a complex analytic isomorphism of Riemann surfaces

$$J : \mathbf{H}/\mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}, \quad \tau \to J(\tau).$$

If $\mathbf{H}^* = \mathbf{H} \cup \mathbb{P}^1(\mathbb{Q})$, then the function J extends as a complex analytic isomorphism of compact Riemann surfaces

$$J : \mathbf{H}^*/\mathrm{SL}_2(\mathbb{Z}) \to \mathbb{P}^1(\mathbb{C}).$$

It is also related to Klein's absolute invariant λ and to Eisenstein series E_4, E_6 by

$$J = \frac{4}{27}\frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2} = \frac{E_4^3}{E_4^3 - E_6^2}.$$

The link to uniformization theory is illustrated by the hypergeometric equation

$$J(J-1)\frac{d^2y}{dJ^2}+\left(\frac{2}{3}-\frac{7}{6}J\right)\frac{dy}{dJ}-\frac{1}{144}y=0,$$

in the sense that J is the inverse of the (multi-valued) function obtained by taking the quotient of two linearly independent solutions to this differential equation. A general fact, shown in [13], concerning elliptic functions is that if ϕ is an elliptic function of periods ω_1, ω_2, then the following functions

$$4g_2\frac{\partial\phi}{\partial g_2}+6g_3\frac{\partial\phi}{\partial g_3}-u\frac{\partial\phi}{\partial u}, \quad 18g_3\frac{\partial\phi}{\partial g_2}+g_2^2\frac{\partial\phi}{\partial g_3}-3\zeta(u)\frac{\partial\phi}{\partial u} \tag{10.4}$$

are also elliptic with the same periods. This leads, for the function $\wp(u;\omega_1,\omega_2)$, to the precise identities

$$\begin{aligned} 4g_2\frac{\partial\wp}{\partial g_2}+6g_3\frac{\partial\wp}{\partial g_3}-u\frac{\partial\wp}{\partial u}&=2\wp, \\ 18g_3\frac{\partial\wp}{\partial g_2}+g_2^2\frac{\partial\wp}{\partial g_3}-3\zeta(u)\frac{\partial\wp}{\partial u}&=6\wp^2-g_2. \end{aligned} \tag{10.5}$$

Following [13] or the more recent treatment [6], we introduce the differential operators

$$\begin{aligned} L_0&=4g_2\frac{\partial}{\partial g_2}+6g_3\frac{\partial}{\partial g_3}-u\frac{\partial}{\partial u}, \\ L_1&=\frac{\partial}{\partial u}, \\ L_2&=6g_3\frac{\partial}{\partial g_2}+\frac{1}{3}g_2^2\frac{\partial}{\partial g_3}-\zeta(u;g_2,g_3)\frac{\partial}{\partial u}. \end{aligned} \tag{10.6}$$

The Lie brakets are

$$[L_0,L_k]=kL_k, \quad [L_1,L_2]=\wp(u,g_2,g_3)L_1.$$

These operators generate the ring $Der\,F$ of the derivations of the ring F of elliptic functions. We introduce the fundamental operator

$$D=12g_3\frac{\partial}{\partial g_2}+\frac{2}{3}g_2^2\frac{\partial}{\partial g_3}=2L_2+2\zeta(u;g_2,g_3)\frac{\partial}{\partial u}, \tag{10.7}$$

then $[L_0, D]=0$ and

$$L_0\wp=2\wp, \quad L_0\wp^n=2n\wp^n, \ n\in\mathbb{N}. \tag{10.8}$$

This eigenfunction character of $\wp^n$ could be used to compute the coefficients in the relations (9.3). A more important property is that the operator operator D connects the periods to pseudo-periods by the relation

$$D\omega_i=-2\eta_i, \quad i=1,2$$

and the discriminant function Δ belongs to Ker$D$ as $D\Delta=0$. A consequence of this description is that the fundamental operator gives a correspondence between different types of hypergeometric differential equations

$$J(J-1)\frac{d^2\omega_i}{dJ^2}+\left(\frac{7}{6}J-\frac{2}{3}\right)\frac{d\omega_i}{dJ}+\frac{1}{144}\omega_i=0 \quad i=1,2 \tag{10.9}$$

and

$$J(J-1)\frac{d^2\eta_i}{dJ^2}+\left(\frac{5}{6}J-\frac{1}{3}\right)\frac{d\eta_i}{dJ}+\frac{1}{144}\eta_i=0 \quad i=1,2. \tag{10.10}$$

A variant of (2.3) is the fact that if $f=\frac{y_1}{y_2}$, with two independent solutions y_1, y_2 to the differential equation

$$\frac{d^2y}{dx^2}+2Q_1\frac{dy}{dx}+Q_2y=0,$$

then the Schwarz derivative $\{f,x\}$ satisfies

$$\{f,x\}=2\left(Q_2-Q_1^2-\frac{dQ_1}{dx}\right),$$

and hence, from (9.1) and (10.10) we obtain, for the fundamental equivariant function, a third order differential equation, as was pointed out in the proof of the theorem (10.1).

Corollary 10.10 *The basic equivariant function h_1 satisfies*

$$\{h_1,J\}=-\frac{3}{8}\frac{1}{(J-1)^2}+\frac{4}{9}\frac{1}{J^2}-\frac{23}{72}\frac{1}{J(J-1)}.$$

This equation is the same as that of Proposition 6.2 except that it is now given in connection to a monodromy problem for a linear second order differential equation. That the two equations are the same is due to the following equations

$$J=\frac{g_2^3}{\Delta},\quad J-1=27\frac{g_3^2}{\Delta},\quad \Delta=g_2^3-27g_3^2.$$

11 Equivariant functions and Binary forms

In this section we revisit the notion of cross-ratio with more details. Let Δ be the diagonal of $(\mathbb{P}^1(\mathbb{C}))^4$, the set of 4-tuples with at least two equal coordinates and set $X=(\mathbb{P}^1(\mathbb{C}))^4\setminus\Delta$. The group $G=GL(2,\mathbb{C})$ acts naturally on X by $g.(z_1,z_2,z_3,z_4)=(g.z_1,g.z_2,g.z_3,g.z_4)$. The map

$$R:(z_1,z_2,z_3,z_4)\mapsto\frac{(z_2-z_1)(z_4-z_3)}{(z_3-z_1)(z_4-z_2)}$$

is an invariant function under this action inducing a bijection on the orbit space

$$R:X/GL(2,\mathbb{C})\to\mathbb{C}.$$

This is strongly connected with the theory of invariants of binary quartics

$$F(x,y)=f_0y^4+4f_1xy^3+6f_2x^2y^2+4f_3x^3y+f_4x^4.$$

The algebra of invariants $\mathcal{A}=\mathbb{C}[f_0,\ldots,f_4]^{SL_2(\mathbb{C})}$ is freely generated by two invariants of weight 4 and 6, respectively

$$g_2=\begin{vmatrix}f_0&f_2\\f_2&f_4\end{vmatrix}-4\begin{vmatrix}f_1&f_2\\f_2&f_3\end{vmatrix},\quad g_3=\begin{vmatrix}f_0&f_1&f_2\\f_1&f_2&f_3\\f_2&f_3&f_4\end{vmatrix}. \tag{11.1}$$

We can identify the binary quartic F with its roots $z_1, \ldots, z_4 \in \mathbb{P}^1(\mathbb{C})$ and consider the discriminant $\Delta = g_2^3 - 27g_3^2$ of weight 12 and the absolute rational invariant J, related to the cross-ratio

$$\lambda = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}, \qquad J = \frac{g_2^3}{\Delta} = \frac{4(\lambda^2 - \lambda + 1)^3}{27\lambda^2(\lambda - 1)^2}.$$

The symmetric group $\mathbf{S}_4$ acts on rational functions of the roots z_i by permuting the indices. The alternating group $\mathbf{A}_4$ has the Klein 4-group $\mathbf{V}$ as a proper normal subgroup consisting in double transpositions Id, $(12)(34)$, $(13)(24)$, $(14)(23)$. The Klein 4-group $\mathbf{V}$ fixes the cross-ratio λ and induces an action of $\mathbf{S}_3 = \mathbf{S}_4/\mathbf{V}$ under which the orbit of λ is

$$\lambda, \quad \frac{1}{\lambda}, \quad 1 - \lambda, \quad \frac{1}{1 - \lambda}, \quad \frac{\lambda}{\lambda - 1}, \quad \frac{\lambda - 1}{\lambda}.$$

This means that the permutation group $\mathbf{S}_3$ acts on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ by Möbius transformations and the orbit space $\mathbb{P}^1(\mathbb{C})/\mathbf{S}_3$ is a 2-sphere [20]. The map $\lambda \to J$ is a rational map on the Riemann sphere. It commutes with the action of $\mathbf{S}_3$. The orbit space of $\mathbf{S}_3$ may be identified with the Riemann sphere, and the quotient map with $\lambda \to J$ too. The inverse map is exactly the Kappa function considered by Kaneko and Yoshida in [18]. There are three exceptional values of J whose

(i) $J = 1;\ \lambda = -1, \frac{1}{2}, 2$. Here $g_3 = 0$ and the roots lines from a harmonic range (Lemniscatic case).

(ii) $J = \infty,\ \lambda = 0, 1, \infty$. The form is degenerate, and $\Delta = 0$.

(iii) $J = 0;\ \lambda = -\omega, \omega^2, \omega = e^{2i\pi/3}$. Here $g_2 = 0$ and the roots lines from a equi-anharmonic range.

In [5], Brady gives a parametrization of equivariant functions by using the projective invariance of the cross-ratio:

$$[f(z_1), f(z_2), f(z_3), f(z_4)] = [z_1, z_2, z_3, z_4],$$

where $f(z)$ is a Möbius transformation.

Let $\mathcal{A}$ the class of all meromorphic equivariant functions for $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbf{H}$ and let $\mathcal{M}$ the field of all meromorphic modular functions on $\mathbf{H}$.

Theorem 11.1 [5] *Let F_1, F_2 and F_3 be three mutually distinct elements in $\mathcal{A}$. The function*

$$\phi : \mathcal{A} \to \mathcal{M}, \quad \phi(F) = \frac{F_1 - F_2}{F_1 - F_3} \cdot \frac{F_3 - F}{F_2 - F} = [F_1, F_2, F_3, F] \tag{11.2}$$

is a $(1, 1)$-map of $\mathcal{A} \setminus \{F_2\}$ onto $\mathcal{M}$, whose inverse map, as in (2.1), is

$$G \to \frac{F_2(F_1 - F_3)G - (F_1 - F_2)F_3}{G(F_1 - F_3) - (F_1 - F_2)}.$$

We give an example: let h_i, $1 \le i \le 4$ four rational equivariant functions as in theorem (7.4), $h_i = z + k_i \dfrac{f_i'}{f_i}$, f_i being a modular form of weight k_i, then

$$[h_1, h_2, h_3, h_4] = [f_1, f_2, f_3, f_4]_1$$

where

$$[h_1, h_2, h_3, h_4] = \frac{(h_1 - h_2)(h_4 - h_3)}{(h_1 - h_3)(h_4 - h_2)}$$

is the usual cross-ratio and

$$[f_1, f_2, f_3, f_4]_1 = \frac{[f_1, f_2]_1 [f_4, f_3]_1}{[f_1, f_3]_1 [f_4, f_2]_1}$$

with $[f_i, f_j]_1$ is the first Rankin-Cohen bracket (10.1) of f_i, f_j.

This result is similar to what we have encountered in Sect. 2: the constants in the Riccati equation are replaced by modular functions. The (1, 1)-map ϕ can serve to provide the class $\mathcal{A}$ with many properties. We give one example here. The functions that are meromorphic on the upper half plane $\mathbf{H}$ can be seen as functions taking their values in the Riemann sphere, or the extended complex plane, by attributing the value ∞ to the poles. Usually a topology on the set of meromorphic functions is defined as follows. We consider a sequence of compacts sets $(K_n)_{n\in\mathbb{N}}$ such that $K_n \subset K^o_{n+1}$, $\cup_{n\geq 0} K_n = \mathbf{H}$. Let σ the spherical metric on the Riemann sphere. For each $n \in \mathbb{N}$ and two meromorphic functions f, g on $\mathbf{H}$, we define $\rho_n(f, g) = \sup_{z\in K_n} \sigma(f(z), g(z))$. In this way the set of meromorphic functions on $\mathbf{H}$ is endowed with a structure of metric space (not complete) with the distance

$$d(f, g) = \sum_{n\geq 0} \frac{\rho_n(f, g)}{1 + \rho_n(f, g)}.$$

As a consequence of (11.2), we obtain that the pace $\mathcal{A}$ is connected. It is possible to define paths in $\mathcal{A}$ by defining paths on the space of modular functions and considering their images by the map (11.2). In addition, it would be worthwhile to transfer to equivariant functions the following stability properties of modular functions:

(1) Action of Hecke operators.
(2) If g is modular for Γ so is $g'^{-2}\{g, z\}$ and $\{z, g\}$.
(3) If k is a positive integer and $F(\tau)$ is a modular form of weight $2k$ with respect to Γ, then $F(\tau)\left(\frac{dJ(\tau)}{d\tau}\right)^{-k}$ is a modular function with respect to Γ.

These operations induce corresponding operations on the set of equivariant functions.

12 Special cases: Platonic equivariance

We would like to give some examples, borrowed from geometry and physics [16], where some special equivariant functions are considered. We emphasize the examples of equivariant functions which are rational and associated to platonic solids and with lattice maps. These examples appeared recently in some physical literature in connection with monopoles and Nahm equations or in dynamical systems. We provide few examples:

- Example 1: Tetrahedral symmetry. The rational function

$$R(z) = \frac{i\sqrt{3}z^2 - 1}{z(z^2 - i\sqrt{3})}$$

verifies

$$R(-z) = -R(z),\ \ R\left(\frac{1}{z}\right) = \frac{1}{R(z)},\ \ R\left(\frac{iz+1}{-iz+1}\right) = \frac{iR(z)+1}{-iR(z)+1}.$$

Hence it is equivariant with respect to the sub-group of $\mathrm{PSL}_2(\mathbb{C})$ generated by

$$z \to z,\ z \to \frac{1}{z},\ z \to \frac{iz+1}{-iz+1}.$$

The Wronskian of $R(z) = \frac{p}{q}$, $W = p'q - pq'$, is given by

$$W(z) = -i\sqrt{3}(z^4 + 2i\sqrt{3}z^2 + 1)$$

which is proportional to a tetrahedral Klein polynomial.

- Example 2: Octahedral symmetry. The rational function

$$R(z) = \frac{z(z^4 - 5)}{-5z^4 + 1}$$

verifies

$$R\left(\frac{iz+1}{-iz+1}\right) = \frac{iR(z)+1}{-iR(z)+1}$$

and the Wronskian is $W(z) = -5(z^8 + 14z^4 + 1)$ which is proportional to a octahedral Klein polynomial.

- Example 3: Icosahedral symmetry [9]. We identify the Riemann sphere with a sphere in $\mathbb{R}^3$ so that 0 and ∞ are poles and the circle $|z| = 1$ is the equator. We inscribe a regular icosahedron in the sphere normalized so that one vertex is at 0 and the adjacent vertex lies on the positive real axis in the Riemann sphere $\mathbb{P}^1(\mathbb{C})$. The isometries of the icosahedron act on $\mathbb{P}^1(\mathbb{C})$ via a group $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$. More precisely,

$$\Gamma \approx \mathrm{PSL}(4) \approx \mathrm{PSL}(5) \approx A_5,$$

the latter being the alternating group. We have [9]

Proposition 12.1 *There are exactly four rational maps of degree* < 31 *which commute with the icosahedral group. These four maps, of degree 1, 11, 19 and 29 are given respectively by:*

$$\begin{aligned}
f_1(z) &= z,\\
f_{11}(z) &= \frac{z^{11} + 66z^6 - 11z}{-11z^{10} - 66z^5 + 1},\\
f_{19}(z) &= \frac{-57z^{15} + 247z^{10} + 171z^5 + 1}{-z^{19} + 171z^{14} - 247z^9 - 57z^4},\\
f_{29}(z) &= \frac{-87z^{25} - 3335z^{20} - 6670z^{10} - 435z^5 + 1}{-z^{29} - 435z^{24} + 6670z^{19} + 3335z^9 + 87z^4}.
\end{aligned}$$

These facts can be checked by direct computations. The generators of the icosahedron are [20], p. 47

$$z \to \epsilon^\mu z, \quad z \to \frac{-\epsilon^{4\mu}}{z},\ z \to \epsilon^\nu \frac{-(\epsilon - \epsilon^4)\epsilon^\mu z + (\epsilon^2 - \epsilon^3)}{(\epsilon^2 - \epsilon^3)\epsilon^\mu z + (\epsilon - \epsilon^4)},$$
$$z \to -\epsilon^{4\nu} \frac{(\epsilon^2 - \epsilon^3)\epsilon^\mu z + (\epsilon - \epsilon^4)}{-(\epsilon - \epsilon^4)\epsilon^\mu z + (\epsilon^2 - \epsilon^3)}; \quad \epsilon = e^{\frac{2i\pi}{5}},\ \mu, \nu = 0, 1, 2, 3, 4.$$

The following table, from Tannery and Molk [35] (volume IV, pp. 108–109) gives the integrals $\int \wp^n(u)du;\ \wp(u) = \wp(u; 1, z),\ \zeta(u) = \zeta(u; 1, z),\ \Im z > 0$:

	J_n
J_1	$-\zeta(u)$
J_2	$\frac{1}{3!}\wp'(u) + \frac{g_2}{2^2 3}u$
J_3	$\frac{1}{5!}\wp'''(u) - \frac{3g_2}{2^2 5}\zeta(u) + \frac{g_3}{2.5}u$
J_4	$\frac{\wp^{(5)}(u)}{37!} + \frac{g_2}{5}\frac{\wp'(u)}{3!} - \frac{g_3}{7}\zeta(u) + \frac{5g_2^2}{2^4.3.7}u$
J_5	$\frac{\wp^{(7)}(u)}{9!} + \frac{g_2}{2^2}\frac{\wp'''(u)}{5!} + \frac{5g_3}{2^2.7}\frac{\wp'(u)}{3!} - \frac{7g_2^2}{2^4.3.5}\zeta(u) + \frac{g_2g_3}{2.3.5}u$
J_6	$\frac{\wp^{(9)}(u)}{11!} + \frac{3g_2}{2.5}\frac{\wp^{(5)}(u)}{7!} + \frac{3g_3}{2.7}\frac{\wp'''(u)}{5!} + \frac{17g_2^2}{2^4.5^2}\frac{\wp'(u)}{3!} - \frac{3.29g_2g_3}{2^2.5.7.11}\zeta(u) + \left(\frac{3.5g_2^3}{2^6.7.11} + \frac{g_3^2}{5.11}\right)u$
J_7	$\frac{\wp^{(11)}(u)}{13!} + \frac{7g_2}{2^2.5}\frac{\wp^{(7)}(u)}{9!} + \frac{g_3}{2^2}\frac{\wp^{(5)}(u)}{7!} + \frac{7g_2^2}{2^3.3.5}\frac{\wp'''(u)}{5!} + \frac{3.23.g_2g_3}{2^4.5.11}\frac{\wp'(u)}{3!}$
	$-\left(\frac{7.11g_2^3}{2^6.3.5.13} + \frac{5g_3^2}{2.7.13}\right)\zeta(u) + \frac{433g_2^2g_3}{2^5.3.5.7.13}u$
J_8	$\frac{\wp^{(13)}(u)}{15!} + \frac{2g_2}{5}\frac{\wp^{(9)}(u)}{11!} + \frac{2g_3}{7}\frac{\wp^{(7)}(u)}{9!} + \frac{23g_2^2}{2^2.3.5^2}\frac{\wp^{(5)}(u)}{7!} + \frac{2^3g_2g_3}{7.11}\frac{\wp'''(u)}{5!}$
	$\left(\frac{61g_2^3}{2^2.5^3.13} + \frac{3.31g_3^2}{2^2.7^2.13}\right)\frac{\wp'(u)}{3!} - \frac{167g_2^2g_3}{2^3.3.5.7.11}\zeta(u) + \left(\frac{13g_2^4}{2^8.7.11} + \frac{7g_2g_3^2}{2^2.3.5.11}\right)u$
J_9	$\frac{\wp^{(15)}(u)}{17!} + \frac{3^2g_2}{2^2.5}\frac{\wp^{(11)}(u)}{13!} + \frac{3^2g_3}{2^2.7}\frac{\wp^{(9)}(u)}{11!} + \frac{3.13g_3^2}{2^4.5^2}\frac{\wp^{(7)}(u)}{9!} + \frac{3^2.13g_2g_3}{2^4.5.11}\frac{\wp^{(5)}(u)}{7!}$
	$+\left(\frac{3.47g_2^3}{2^5.5^2.13} + \frac{3^2.53g_3^2}{2^4.7^2.13}\right)\frac{\wp'''(u)}{5!} + \frac{3^2.181.g_2^2g_3}{2^5.5^2.7.11}\frac{\wp'(u)}{3!}$
	$-\left(\frac{7.11g_2^4}{2^8.13.17} + \frac{3^3.223g_2.g_3^2}{2^2.5.7.11.13.17}\right)\zeta(u) + \left(\frac{7g_3^3}{2.5.7.11.17} + \frac{383g_2^3g_3}{2^3.7.11.13.17}\right)u$
J_{10}	$\frac{\wp^{(17)}(u)}{19!} + \frac{g_2}{2}\frac{\wp^{(13)}(u)}{15!} + \frac{5g_3}{2.7}\frac{\wp^{(11)}(u)}{13!} + \frac{29g_2^2}{2^4.3.5}\frac{\wp^{(9)}(u)}{11!} + \frac{3.17g_2g_3}{2^2.7.11}\frac{\wp^{(7)}(u)}{9!}$
	$+\left(\frac{587g_2^3}{2^5.3.5^2.13} + \frac{5.17g_3^2}{2^4.7.13}\right)\frac{\wp^{(5)}(u)}{7!} + \frac{137g_2^2g_3}{2\cdot 3.7.11}\frac{\wp'''(u)}{5!}$
	$+\left(\frac{31.1453g_2^4}{2^8.3.5^3.13.17} + \frac{3.15817g_2g_3^2}{2^4.7^2.11.13.17}\right)\frac{\wp'(u)}{3!}$
	$-\left(\frac{3251g_2^3g_3}{2^5.7.11.13.19} + \frac{2.5.g_3^3}{7.13.19}\right)\zeta(u) + \left(\frac{1357g_2^2g_3^2}{2^4.7.11.13.19} + \frac{13.17g_2^5}{2^{10}.7.11.19}\right)u$

The equivariant functions $h_n(z)$

From the previous table, we deduce the first ten equivariant functions,

	h_n
h_1	$\frac{\eta_2}{\eta_1}$
h_2	z
h_3	$\dfrac{-\frac{3g_2}{2^2 5}\eta_2+\frac{g_3}{2.5}z}{-\frac{3g_2}{2^2 5}\eta_1+\frac{g_3}{2.5}}$
h_4	$\dfrac{-\frac{g_3}{7}\eta_2+\frac{5g_2^2}{2^4.3.7}z}{-\frac{g_3}{7}\eta_1+\frac{5g_2^2}{2^4.3.7}}$
h_5	$\dfrac{-\frac{7g_2^2}{2^4.3.5}\eta_2+\frac{g_2g_3}{2.3.5}z}{-\frac{7g_2^2}{2^4.3.5}\eta_1+\frac{g_2g_3}{2.3.5}}$
h_6	$\dfrac{-\frac{3.29g_2g_3}{2^2.5.7.11}\eta_2+\left(\frac{3.5g_2^3}{2^6.7.11}+\frac{g_3^2}{5.11}\right)z}{-\frac{3.29g_2g_3}{2^2.5.7.11}\eta_1+\left(\frac{3.5g_2^3}{2^6.7.11}+\frac{g_3^2}{5.11}\right)}$
h_7	$\dfrac{-\left(\frac{7.11g_2^3}{2^6.3.5.13}+\frac{5g_3^2}{2.7.13}\right)\eta_2+\frac{433g_2^2g_3}{2^5.3.5.7.13}z}{-\left(\frac{7.11g_2^3}{2^6.3.5.13}+\frac{5g_3^2}{2.7.13}\right)\eta_1+\frac{433g_2^2g_3}{2^5.3.5.7.13}}$
h_8	$\dfrac{-\frac{167g_2^2g_3}{2^3.3.5.7.11}\eta_2+\left(\frac{13g_2^4}{2^8.7.11}+\frac{7g_2g_3^2}{2^2.3.5.11}\right)z}{-\frac{167g_2^2g_3}{2^3.3.5.7.11}\eta_1+\left(\frac{13g_2^4}{2^8.7.11}+\frac{7g_2g_3^2}{2^2.3.5.11}\right)}$
h_9	$\dfrac{-\left(\frac{7.11g_2^4}{2^8.13.17}+\frac{3^3.223g_2.g_3^2}{2^2.5.7.11.13.17}\right)\eta_2+\left(\frac{7g_3^3}{2.5.7.11.17}+\frac{383g_2^3g_3}{2^3.7.11.13.17}\right)z}{-\left(\frac{7.11g_2^4}{2^8.13.17}+\frac{3^3.223g_2.g_3^2}{2^2.5.7.11.13.17}\right)\eta_1+\left(\frac{7g_3^3}{2.5.7.11.17}+\frac{383g_2^3g_3}{2^3.7.11.13.17}\right)}$
h_{10}	$\dfrac{-\left(\frac{3251g_2^3g_3}{2^5.7.11.13.19}+\frac{2.5.g_3^3}{7.13.19}\right)\eta_2+\left(\frac{1357g_2^2g_3^2}{2^4.7.11.13.19}+\frac{13.17g_2^5}{2^{10}.7.11.19}\right)z}{-\left(\frac{3251g_2^3g_3}{2^5.7.11.13.19}+\frac{2.5.g_3^3}{7.13.19}\right)\eta_1+\left(\frac{1357g_2^2g_3^2}{2^4.7.11.13.19}+\frac{13.17g_2^5}{2^{10}.7.11.19}\right)}$

The equivariant functions h_n can be given in terms of E_2, E_4, E_6 by means of the relations (9.2).

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