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Equivariant Forms: Structure and Geometry

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Abstract. In this paper we study the notion of equivariant forms introduced in the authors' previous works. In particular, we completely classify all the equivariant forms for a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ by means of the cross-ratio, weight 2 modular forms, quasimodular forms, as well as differential forms of a Riemann surface and sections of a canonical line bundle.

1 Introduction

The notion of equivariant forms was first introduced in [6] as functions on the upper half of the complex plane $\mathbb{H}$ commuting with the modular group $\mathrm{SL}_2(\mathbb{Z})$, that is, satisfying

$$h(\gamma\cdot z)=\gamma\cdot h(z),\quad z\in\mathbb{H},\quad \gamma\in\mathrm{SL}_2(\mathbb{Z}),$$

with a specific behavior at the cusps, and where $\gamma\cdot z$ denotes the usual action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}$ by linear fractional transformations. In that paper, it was shown how to obtain equivariant forms from modular forms as well as from integrals of elliptic functions, and several connections with projective differential geometry and differential algebra were established. In [2], this notion was generalized to an arbitrary subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, and the main focus was on the so-called rational equivariant forms. More precisely, an equivariant form h for Γ is called rational if there exists a generalized modular form f for Γ of weight k and character μ such that

$$h(z)=z+k\frac{f(z)}{f'(z)}.$$

It turns out that a necessary and sufficient condition for an equivariant form to be rational, that is, to arise from a generalized modular form as above, is that all the poles of $(h(z)-z)^{-1}$ be simple with rational residues. This allows us to classify all the rational equivariant forms. In particular, if Γ has genus 0, then the generalized modular form f can be taken as a standard modular form (with trivial character). It is also shown in [2] that the rational equivariant forms are only a small class among the general equivariant forms.

In this paper, we undertake the task of classifying all the equivariant forms for an arbitrary modular subgroup. This classification will be carried out in several ways.

The first classification is done using the cross-ratio, which is projectively invariant and thus when applied to four equivariant forms for Γ, it will lead to a modular function for Γ. In particular, if one fixes three equivariant forms, then the cross-ratio

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realizes a one-to-one correspondence between the equivariant forms and the field of modular functions for Γ, in other words, with the function field of the Riemann surface $X_\Gamma = \Gamma\backslash\mathbb{H}^*$, where $\mathbb{H}^* = \mathbb{H} \cup \{\text{cusps}\}$. The Schwarz derivative, which is the infinitesimal counterpart of the cross-ratio, is also projectively invariant and interestingly, when it is applied to an equivariant form, it yields a modular form of weight 4 for Γ.

The second classification is carried out using the theory of quasimodular forms for Γ. In particular, we show that all the equivariant forms are identified with the normalized quasimodular forms of weight 2 and depth 1. This will lead to a third classification identifying the set of equivariant forms without the trivial one, $h(z) = z$, with the space of weight 2 meromorphic modular forms for Γ. In particular, this confers a structure of vector space to the set of equivariant forms. As an example, the subset of equivariant forms without fixed points forms a finite dimensional subspace that is isomorphic to the space of holomorphic weight 2 modular forms for Γ.

Finally, noting that the weight 2 modular forms correspond to differential 1-forms on X_Γ, we conclude that the equivariant forms can be looked upon as the meromorphic sections of the canonical line bundle of X_Γ.

While most of the paper can be generalized to general Fuchsian groups, we have restricted ourselves for the sake of simplicity to the subgroups of the modular group. In particular, we relied on the classical treatment of modular forms and quasi-modular forms for the modular subgroups.

2 Generalities

Let $\mathrm{SL}_2(\mathbb{R})$ be the group of 2x2 matrices with real entries and determinant 1. It acts on the upper half of the complex plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ by linear fractional transformations

$$\alpha \cdot z = \frac{az+b}{cz+d}, \quad z \in \mathbb{H}, \quad \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

The Möbius group $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm I\}$ is the full automorphism group of $\mathbb{H}$. For α as above, $z \in \mathbb{H}$, set $j_\alpha(z) = cz + d$. The map $j \colon \mathrm{SL}_2(\mathbb{R}) \times \mathbb{H} \to \mathbb{C}^*$ defines what is called an automorphic factor and satisfies the cocycle relation

$$j_{\alpha\beta}(z) = j_\alpha(\beta \cdot z)\, j_\beta(z), \quad \alpha, \quad \beta \in \mathrm{SL}_2(\mathbb{R}).$$

We now introduce two different actions of $\mathrm{SL}_2(\mathbb{R})$ on the space of meromorphic functions on $\mathbb{H}$. The classical slash operator is defined for a meromorphic function f on $\mathbb{H}$ and a positive integer k by

$$f|_k[\alpha](z) = j_\alpha(z)^{-k} f(\alpha \cdot z),$$

while the “double-slash” operator is defined for a meromorphic function f on $\mathbb{H}$ by

$$f\|[\gamma](z) = j_\gamma(z)^{-2} f(\gamma \cdot z) - r j_\gamma(z)^{-1}, \quad \gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}), \tag{2.1}$$

where again $j_\gamma(z) = rz + s$. The slash operator is usually used to define modular forms, and the double slash operator was introduced in [2] to define the notion of equivariant forms. For the sake of completeness, we show that it defines an action of $\mathrm{SL}_2(\mathbb{R})$ on the space of meromorphic functions on $\mathbb{H}$. Indeed, for elements $\beta = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\gamma = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbb{R})$, we have, on one hand,

$$f\|[\beta\gamma](z) = j_{\beta\gamma}(z)^{-2} f(\beta\gamma \cdot z) - (cp + dr)\, j_{\beta\gamma}(z)^{-1}.$$

On the other hand, we have

$$\begin{aligned}
\big(f\|[\beta]\big)\,\|[\gamma](z) &= j_\gamma(z)^{-2} f\|[\beta](\gamma \cdot z) - r\, j_\gamma(z)^{-1} \\
&= j_\gamma(z)^{-2}\big(j_\beta(\gamma \cdot z)^{-2} f(\beta\gamma \cdot z) - c\, j_\beta(\gamma \cdot z)^{-1}\big) - r\, j_\gamma(z)^{-1} \\
&= j_{\beta\gamma}(z)^{-2} f(\beta\gamma \cdot z) - c\, j_\gamma(z)^{-2} j_\beta(\gamma \cdot z)^{-1} - r\, j_\gamma(z)^{-1}.
\end{aligned}$$

One easily checks that

$$c\, j_\gamma(z)^{-2} j_\beta(\gamma \cdot z)^{-1} + r\, j_\gamma(z)^{-1} = (cp + dr)\, j_{\beta\gamma}(z)^{-1},$$

which yields

$$f\|[\beta\gamma](z) = \big(f\|[\beta]\big)\,\|[\gamma](z).$$

Let Γ be a modular subgroup, that is , a finite index subgroup of the modular group $\mathrm{SL}_2(\mathbb{Z})$. Let $\mathfrak{s}$ be a cusp of Γ, that is, $\mathfrak{s}$ is in $\mathbb{Q} \cup \{\infty\}$, and choose $\gamma = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma \cdot \mathfrak{s} = \infty$. Then the isotropy group of $\mathfrak{s}$, $\Gamma_{\mathfrak{s}} = \{\alpha \in \Gamma |\ \alpha \cdot \mathfrak{s} = \mathfrak{s}\}$, is conjugate by γ to the infinite cyclic group generated by $T^{l_{\mathfrak{s}}}$, with $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $l_{\mathfrak{s}}$ is a positive integer known as the cusp width of Γ at the cusp $\mathfrak{s}$.

Let k be a positive integer. A function f on $\mathbb{H}$ is called a meromorphic modular form or simply a modular form of weight k for a modular subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ if

(1) f is meromorphic on $\mathbb{H}$;
(2) for all $\alpha \in \Gamma$ and $z \in \mathbb{H}$, we have $f|_k[\alpha](z) = f(z)$;
(3) f is meromorphic at the cusps.

The last condition means the following: If $\mathfrak{s}$ is a cusp and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ is such that $\gamma \cdot \mathfrak{s} = \infty$, then the function $f|_k[\gamma^{-1}](z)$ is invariant under $\gamma\Gamma_{\mathfrak{s}}\gamma^{-1} = \langle T^{l_{\mathfrak{s}}}\rangle$. Hence, it has a Fourier series expansion in the local parameter at infinity $q_{\mathfrak{s}} := e^{2\pi i z/l_{\mathfrak{s}}}$ if k is even and $q_{\mathfrak{s}} = e^{\pi i z/l_{\mathfrak{s}}}$ if k is odd. The meromorphy condition means that we have the Fourier series expansion

$$f|_k[\gamma^{-1}](z) = \sum_{n=n_{\mathfrak{s}}}^{\infty} a_n^{\mathfrak{s}} q_{\mathfrak{s}}^n$$

with the integer $n_{\mathfrak{s}}$ being finite. If $n_{\mathfrak{s}} \geq 0$ for every cusp $\mathfrak{s}$ and if f is holomorphic on $\mathbb{H}$, then f is called a holomorphic modular form. A holomorphic modular form is called a cusp form if it vanishes at all cusps, in other words $n_{\mathfrak{s}} > 0$ for every cusp $\mathfrak{s}$. When $k = 0$, the modular form is called a modular function. If condition (2) is replaced by

(2′) for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathbb{H}$, we have $f|_k[\alpha](z) = \mu(\alpha) f(z)$,

where $\mu\colon \Gamma \longrightarrow \mathbb{C}^\times$ is a character not necessarily unitary, then f is called a generalized modular form (see [5]).

3 Equivariant Forms

In this section, we introduce the notion of equivariant forms as they were introduced in [6] for the modular group and generalized to an arbitrary modular subgroup in [2]. These are meromorphic functions h on $\mathbb{H}$ that commute with the action of a modular subgroup Γ; that is, we have the equivariance property

$$h(\gamma \cdot z) = \gamma \cdot h(z) \quad \text{for all} \quad z \in \mathbb{H}, \quad \gamma \in \Gamma,$$

in addition to a behavior at the cusps that will be specified below. The rigorous definition introduced in [2] involves the double-slash operator from the previous section. Obviously, the identity map $h(z) = z$ satisfies the equivariance property. If $h(z) \neq z$ is a meromorphic function on $\mathbb{H}$, we associate with it an auxiliary function

$$\widehat{h}(z) = \frac{1}{h(z) - z}.$$

Proposition 3.1 *Let h be a meromorphic function on $\mathbb{H}$ and let Γ be a modular subgroup. If $\gamma \in \Gamma$ and $z \in \mathbb{H}$, then*

$$h(\gamma \cdot z) = \gamma \cdot h(z) \quad \textit{if and only if} \quad \widehat{h}\|[\gamma](z) = \widehat{h}(z).$$

Proof For $\gamma = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in \Gamma$ we have

$$h(\gamma \cdot z) = \gamma \cdot h(z) \Leftrightarrow \widehat{h}(\gamma \cdot z) = j_\gamma(z) j_\gamma(h(z))\widehat{h}(z) \Leftrightarrow j_\gamma(z)^{-2}\widehat{h}(\gamma \cdot z) = \frac{j_\gamma(h(z))}{j_\gamma(z)}\widehat{h}(z).$$

Meanwhile, $j_\gamma(h(z)) = r(h(z) - z) + j_\gamma(z)$, so that

$$\frac{j_\gamma(h(z))}{j_\gamma(z)}\widehat{h}(z) = \widehat{h}(z) + r j_\gamma(z)^{-1}.$$

The proposition follows. ■

Let $\mathfrak{s} \in \mathbb{Q} \cup \{\infty\}$ be a cusp of Γ with cusp width $l_\mathfrak{s}$. If h is a meromorphic function on $\mathbb{H}$ that commutes with the action of Γ on $\mathbb{H}$, then $\widehat{h}\|[\gamma^{-1}](z)$ is invariant under $\gamma\Gamma_\mathfrak{s}\gamma^{-1} = \left\langle T^{l_\mathfrak{s}} \right\rangle$ and hence it is $l_\mathfrak{s}$-periodic. Therefore, it has a Fourier expansion in the local parameter $q_\mathfrak{s} = \exp(2\pi i z / l_\mathfrak{s})$ of the form

$$\widehat{h}\|[\gamma^{-1}](z) = \sum_{m \geq m_\mathfrak{s}} a_m q_\mathfrak{s}^m.$$

We say that h is meromorphic at $\mathfrak{s}$ if $\widehat{h}\|[\gamma^{-1}](z)$ is meromorphic at ∞ in the sense that the integer $m_\mathfrak{s}$ is finite. It is important to note that if this holds at a cusp $\mathfrak{s}$, then it also holds at any cusp that is Γ-equivalent to $\mathfrak{s}$.

Definition 3.2 An equivariant form for Γ is a meromorphic function on $\mathbb{H}$ that commutes with the action of Γ and is meromorphic at every cusp of Γ.

Besides the trivial example $h(z) = z$, one can attach an equivariant form to each modular form or generalized modular form. Indeed we have the following theorem.

Theorem 3.3 ([2]) *Let Γ be a modular subgroup and let f be a generalized modular form for Γ of weight k and character μ. Then the function*

$$h_f(z) = z + k\frac{f(z)}{f'(z)}$$

is an equivariant form for Γ.

The equivariance property of h_f is straightforward, while the meromorphy at the cusps of h_f as an equivariant form is equivalent to the meromorphy of f as a modular form. For the case of modular forms, this equivariance property with respect to the action of a modular subgroup also appears in [7]. The equivariant forms arising from this theorem are called *rational*. In [2], one of the main results states a necessary and sufficient condition for an equivariant form h to be rational is that $\widehat{h}$ has only simple poles on $\mathbb{H} \cup \{\infty\}$ with rational residues. Furthermore, we have the following theorem.

Theorem 3.4 ([2]) *Let Γ be a modular subgroup and let f and g be generalized modular forms of respective weights k and $k+2$ and having the same character, then*

$$h(z) = z + k\frac{f(z)}{f'(z) + g(z)}$$

is an equivariant form for Γ.

Using this theorem, one can construct infinitely many equivariant forms that are not rational by using convenient f and g in such a way that the residue of $\widehat{h}$ at a simple pole is no longer a rational number. Several simple examples are provided in [2].

4 Classification via the Cross-ratio

The cross-ratio plays an important role in projective differential geometry. It is defined for four points $z_1,\ z_2,\ z_3,\ z_4$ of the projective line $\mathbb{P}^1(\mathbb{C})$ by

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_2)(z_4 - z_3)}{(z_4 - z_2)(z_1 - z_3)}.$$

A well-known property of the cross-ratio is that it is invariant under Möbius transformations. Hence, it can be looked upon as a geometric invariant of the projective line.

In this section, we show that the cross-ratio plays an important role with regard to the equivariant forms as well. We have the following theorem.

Theorem 4.1 *Suppose we are given three equivariant forms h_1, h_2, and h_3 for a modular subgroup Γ with $h_1 \neq h_3$. The map*

$$h \longmapsto (h_1, h_2; h_3, h) = \frac{(h_1 - h_2)(h - h_3)}{(h - h_2)(h_1 - h_3)} \tag{4.1}$$

defines a bijection between the set of equivariant forms without h_1 *and the field of modular functions for* Γ *seen also as the function field of the compact Riemann surface* $X_\Gamma = \Gamma\backslash\mathbb{H}^*$*, where* $\mathbb{H}^* = \mathbb{H} \cup \{cusps\}$.

Proof Since the h_l's, $1 \leq l \leq 3$, are equivariant forms for Γ and the cross ratio is invariant under any Möbius transformations, the function f is invariant under Γ. The meromorphy property on $\mathbb{H}$ and at cusps follows from that of the equivariant forms. Clearly, the map (4.1) defines a bijection. ■

The above proposition requires the knowledge of three different equivariant forms and this can easily be achieved using the rational equivariant forms for Γ. It is worth mentioning that in [1], Brady has noted that the cross-ratio of four equivariant forms is a modular function, but without the knowledge of the existence of these equivariant forms beside one fundamental example. Indeed, motivated by the work of Heins [3] on the theory of elliptic functions, Brady, in [1], considered a lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\tau = \omega_2/\omega_1 \in \mathbb{H}$. The Weierstrass ζ-function, defined by $\zeta' = -\wp$, where $\wp$ is the Weierstrass elliptic $\wp$-function, is a pseudo-periodic function. If η_1 and η_2 are the pseudo-periods of ζ, then $h_0 = \omega_1\eta_2$, as a function of τ, commutes with the action of the modular group.

Now, the symmetric group S_4 acts on the cross-ratio $f = (h_1, h_2; h_3, h_4)$ of four equivariant forms by permuting $h_1,\ h_2,\ h_3,\ h_4$, and this produces the following symmetric relations:

$$(h_1, h_2; h_3, h_4) = f, \qquad (h_1, h_2; h_4, h_3) = \frac{1}{f},$$

$$(h_1, h_3; h_2, h_4) = 1 - f, \qquad (h_1, h_3; h_4, h_2) = \frac{1}{1-f},$$

$$(h_1, h_4; h_3, h_2) = \frac{f}{f-1}, \qquad (h_1, h_4; h_2, h_3) = \frac{f-1}{f}.$$

The modular function f is invariant under the products of disjoint transpositions (1,2)(3,4), (1,3)(2,4), (1,4)(2,3), which form the Klein four-group, and the action of any other permutation will produce one of the above transformations of f. In fact, one could only consider the action of the symmetric group S_3 as shown above by fixing one equivariant form and permuting the others. Notice that the transformations

$$z \longmapsto \frac{z}{z-1}, \quad z \longmapsto 1 - z, \quad z \longmapsto \frac{1}{z}, \quad z \longmapsto \frac{1}{1-z}, \quad z \longmapsto \frac{z-1}{z},$$

together with the identity, form a group that is isomorphic to S_3.

In what follows we will illustrate this phenomenon using classical modular functions and forms. Let $j(z)$ be the classical modular invariant, which is a Hauptmodul for $\mathrm{SL}_2(\mathbb{Z})$, and define the Jacobi theta functions by

$$\vartheta_2(z) = \sum_{n\in\mathbb{Z}} t^{(n-1/2)^2}, \quad \vartheta_3(z) = \sum_{n\in\mathbb{Z}} t^{n^2}, \quad \vartheta_4(z) = \sum_{n\in\mathbb{Z}} (-1)^n t^{n^2},$$

where $t = e^{\pi i z}$. These theta functions satisfy the Jacobi identity

$$\vartheta_2^4 + \vartheta_4^4 = \vartheta_3^4. \tag{4.2}$$

The Klein modular function

$$\lambda = \frac{\vartheta_2^4}{\vartheta_3^4} \tag{4.3}$$

is a Hauptmodul for the genus 0 principal congruence subgroup $\Gamma(2)$, which is of index 6 in $\mathrm{SL}_2(\mathbb{Z})$. It transforms under representatives of conjugacy classes of the quotient $\mathrm{SL}_2(\mathbb{Z})/\Gamma(2) \cong S_3$ as follows:

$$\lambda\big(z/(z+1)\big) = \frac{1}{\lambda}, \qquad \lambda(-1/z) = 1-\lambda,$$

$$\lambda\big(-1/(z+1)\big) = \frac{1}{1-\lambda}, \qquad \lambda(z+1) = \frac{\lambda}{\lambda-1},$$

$$\lambda\big(-(z+1)/z\big) = \frac{\lambda-1}{\lambda}.$$

Although these relations are a consequence of the transformation rules of the theta functions, we provide a proof of these relations which is a consequence of the above action on the cross-ratio of equivariant forms.

We have the following equivariant forms attached to the theta functions

$$h_{\vartheta_2}(z) = z + \frac{\vartheta_2(z)}{2\vartheta_2'(z)}, \quad h_{\vartheta_3}(z) = z + \frac{\vartheta_3(z)}{2\vartheta_3'(z)}, \quad h_{\vartheta_4}(z) = z + \frac{\vartheta_4(z)}{2\vartheta_4'(z)},$$

which are equivariant forms for $\Gamma(2)$.

Proposition 4.2 *We have* $(z,\ h_{\vartheta_2};\ h_{\vartheta_3},\ h_{\vartheta_4}) = \lambda$.

Proof One easily computes

$$(z,\ h_{\vartheta_4};\ h_{\vartheta_2},\ h_{\vartheta_3}) = \frac{\vartheta_2(\vartheta_3'\vartheta_4 - \vartheta_3\vartheta_4')}{\vartheta_3(\vartheta_2'\vartheta_4 - \vartheta_2\vartheta_4')} = \frac{\frac{\vartheta_3'}{\vartheta_3} - \frac{\vartheta_4'}{\vartheta_4}}{\frac{\vartheta_2'}{\vartheta_2} - \frac{\vartheta_4'}{\vartheta_4}}.$$

Taking the logarithmic of (4.3), we get

$$\frac{\lambda'}{\lambda} = 4\left(\frac{\vartheta_2'}{\vartheta_2} - \frac{\vartheta_3'}{\vartheta_3}\right). \tag{4.4}$$

As a consequence of the Jacobi identity (4.2), we have $1 - \lambda = \vartheta_4^4/\vartheta_3^4$, which yields, after taking the logarithmic derivative,

$$\frac{\lambda'}{1-\lambda} = 4\Big(\frac{\vartheta_3'}{\vartheta_3} - \frac{\vartheta_4'}{\vartheta_4}\Big). \tag{4.5}$$

Adding (4.4) and (4.5) we get

$$\frac{\lambda'}{\lambda(1-\lambda)} = 4\Big(\frac{\vartheta_2'}{\vartheta_2} - \frac{\vartheta_4'}{\vartheta_4}\Big). \tag{4.6}$$

Now, the proposition follows by taking the ratio of (4.5) and (4.6). ∎

Remark 4.3 As a consequence, we obtain the above transformations of λ under the action of $\mathrm{SL}_2(\mathbb{Z})/\Gamma(2)$, which is is isomorphic to S_3, and the corresponding action of S_3 on the cross-ratio.

We now introduce the classical Eisenstein series

$$E_4(z) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n, \quad E_6(z) = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n,$$

where $\sigma_k(n)$ is the sum of the k-th powers of the positive divisors of n. The Eisenstein series E_4 and E_6 are modular forms for $\mathrm{SL}_2(\mathbb{Z})$ of weight 4 and 6 respectively. We also introduce the Eisenstein series E_2 given by

$$E_2(z) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n,$$

which is not a modular form but it is rather referred to as a quasimodular form (see §5). Moreover, E_2 satisfies

$$E_2(z) = \frac{1}{2\pi i}\frac{\Delta'(z)}{\Delta(z)}, \tag{4.7}$$

where Δ is the weight 12 cusp form for $\mathrm{SL}_2(\mathbb{Z})$ (also called the modular discriminant) and is given by

$$\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24},$$

and which satisfies

$$\Delta = \frac{1}{1728}(E_4^3 - E_6^2).$$

The Eisenstein series satisfy the Ramanujan relations

$$\frac{6}{\pi i}E_2' = E_2^2 - E_4, \quad \frac{3}{2\pi i}E_4' = E_4E_2 - E_6, \quad \frac{1}{\pi i}E_6' = E_6E_2 - E_4^2.$$

The j-function is given by

$$j = 1728\frac{E_4^3}{E_4^3 - E_6^2} = \frac{E_4^3}{\Delta}$$

which is a Hauptmodul for the modular group $\mathrm{SL}_2(\mathbb{Z})$. As usual, we denote by h_f the rational equivariant form attached to the modular form f.

Proposition 4.4 *We have* $(z,\ h_{E_4}; h_\Delta, h_{E_6}) = \frac{1}{1728}j$.

Proof One easily shows that

$$(z, h_{E_4}; h_\Delta, h_{E_6}) = \frac{2E_4(\pi i E_2E_6 - E_6')}{3E_4'E_6 - 2E_4E_6'},$$

and using the Ramanujan relations, we get

$$(z,\ h_{E_4}; h_\Delta, h_{E_6}) = \frac{E_4^3}{E_4^3 - E_6^2} = \frac{1}{1728}j. \qquad \blacksquare$$

Another important tool that is projectively invariant is the Schwarz derivative defined for a meromorphic function f on a domain by

$$\{f,z\} = 2\Big(\frac{f''}{f'}\Big)' - \Big(\frac{f''}{f'}\Big)^2.$$

It is the infinitesimal counterpart of the cross-ratio and is an essential tool in projective differential geometry as well as many other fields. One can check that it satisfies the following:

- If $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C})$, then

$$\Big\{\frac{af+b}{cf+d}, z\Big\} = \{f,z\}.$$

- Chain rule: If w is a function of z, then

$$\{f,z\} = (dw/dz)^2\{f,w\} + \{w,z\}.$$

- If f is a linear fractional transform of z, then $\{f,z\} = 0$.
- Inversion formula: If $w'(z_0) \neq 0$ for some point z_0, then in a neighborhood of z_0,

$$\{z,w\} = -(dz/dw)^2\{w,z\}.$$

- The Schwarz derivative $\{f,z\}$ has a double pole at the critical points of f and is holomorphic everywhere else where it is meromorphic.

With regards to the equivariant forms, the above rules immediately yield the following proposition.

Proposition 4.5 *If f is an equivariant form for a modular subgroup Γ, then $\{f,z\}$ is a modular form of weight four.*

As an example, we have

$$\{h_{E_4}, z\} = 4\pi^2 E_4(z). \tag{4.8}$$

This follows from the fact that h_{E_4} does not have critical points, as one can show that

$$h'_{E_4}(z) = -3840\pi^2 \frac{\Delta(z)}{E_4'^2(z)},$$

and Δ does not vanish on $\mathbb{H}$. Hence, $\{h_{E_4}, z\}$ is a holomorphic modular form of weight 4 and hence is a multiple of E_4. Checking the first few coefficients of its q-expansion yields (4.8).

5 Classification via Quasimodular Forms

Quasimodular forms are a generalization of modular forms introduced by M. Kaneko and D. Zagier [4]. They turned out to be very useful tools in mathematics and mathematical physics.

A (meromorphic) quasimodular form of weight k and depth p on Γ is a meromorphic function f on $\mathbb{H}$ such that

$$f(z)|_k[\alpha] = j_\alpha(z)^{-k} f(\gamma\cdot z) = \sum_{i=0}^{p} f_i(z)\Big(\frac{c}{j_\alpha(z)}\Big)^i, \quad z\in\mathbb{H},\quad \gamma = \begin{pmatrix} a\, b \\ c\, d\end{pmatrix}\in\Gamma \tag{5.1}$$

and where the f_i are meromorphic functions on $\mathbb{H}$ with moderate growth at the cusps. The space of quasimodular forms of weight k and depth p on Γ is denoted by $\widetilde{M}_k^{(\leq p)} = \widetilde{M}_k^{(\leq p)}(\Gamma)$. The prototype of a quasimodular form is the Eisenstein series E_2 that is of weight 2 and depth 1. As a consequence of (4.7), we have the following proposition.

Proposition 5.1 *For $\alpha = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, we have*

$$E_2(\alpha\cdot z) = j_\alpha(z)^2 E_2(z) + \frac{6c}{\pi i} j_\alpha(z).$$

The following result summarizes most of the properties of quasimodular forms.

Theorem 5.2 ([4]) *Let Γ be a modular subgroup and let k and p be nonnegative integers.*

(i) *The space of quasimodular forms on Γ is closed under differentiation:*

$$D\big(\widetilde{M}_k^{(\leq p)}\big) \subset \widetilde{M}_{k+2}^{(\leq p+1)}.$$

(ii) *Every quasimodular form on Γ is a polynomial in E_2 with modular forms as coefficients. More precisely, we have*

$$\widetilde{M}_k^{(\leq p)}(\Gamma) = \bigoplus_{r=0}^{p} M_{k-2r}(\Gamma)\cdot {E_2}^r$$

for all $k,\ p\geq 0$, where $M_j(\Gamma)$ denote the space of weight j modular forms on Γ.

(iii) *Every quasimodular form on Γ can be written uniquely as a linear combination of derivatives of modular forms and of E_2. More precisely, we have*

$$\widetilde{M}_k^{(\leq p)}(\Gamma) = \begin{cases} \bigoplus_{r=0}^{p} D^r(M_{k-2r}(\Gamma)) & \text{if } p < k/2, \\ \bigoplus_{r=0}^{k/2-1} D^r(M_{k-2r}(\Gamma)) \oplus \mathbb{C}\cdot D^{k/2-1}E_2 & \text{if } p\geq k/2. \end{cases}$$

We will show that equivariant forms are closely related to quasimodular forms of weight 2 and depth 1. More precisely, we have the following proposition.

Proposition 5.3 *Let Γ be a modular subgroup and let h be a nontrivial equivariant form for Γ, then*

$$\widehat{h}(z) = \frac{1}{h(z) - z}$$

is a quasimodular form of weight 2 and depth 1.

Proof Using Proposition 3.1, if h is equivariant, then $\widehat{h}\|[\gamma](z) = \widehat{h}(z)$ and by (2.1), we have

$$\widehat{h}\|[\gamma](z) = \widehat{h}|_2[\gamma](z) - c\, j_\gamma(z)^{-1}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

It follows that

$$\widehat{h}|_2[\gamma](z) = \widehat{h}(z) + \frac{c}{cz+d}.$$

Therefore, $\widehat{h}(z)$ is a quasimodular form of weight 2 and depth 1. ■

We should notice that the quasimodular form thus obtained is of a special form. Indeed, the coefficients f_0 and f_1 as in (5.1) are given by $f_0 = \widehat{h}$, $f_1 = 1$. In fact, the first identity is expected since it follows from (5.1) by putting $c = 0$. A quasimodular form f of weight 2 and depth 1 will be called *normalized* if its corresponding f_1 from (5.1) is given by $f_1(z) = 1$. An example of such normalized weight 2 depth 1 quasimodular form is given by

$$\widetilde{E}_2(z) = \frac{\pi i}{6} E_2(z),$$

which follows from Proposition 5.1.

Conversely, we have the following proposition.

Proposition 5.4 *Let f be a normalized weight* 2 *and depth* 1 *quasimodular form for a modular subgroup Γ, then*

$$h(z) = z + \frac{1}{f(z)} \tag{5.2}$$

is an equivariant form for Γ.

Proof If f is a normalized weight 2 and depth 1 quasimodular form and h is given by (5.2), then $\widehat{h}\|[\gamma](z) = \widehat{h}(z)$. Hence, by Proposition 3.1, we have

$$h(\gamma \cdot z) = \gamma \cdot h(z), \quad z \in \mathbb{H}, \quad \gamma \in \Gamma.$$

The meromorphy of h at the cusps follows from that of E_2 and the modular forms that are involved from Theorem 5.2(ii). Therefore, h is an equivariant form for Γ. ■

Thus we have shown that the set of nontrivial equivariant forms for $SL_2(\mathbb{Z})$ are in one-to-one correspondence with the set of normalized quasimodular forms of weight 2 and depth 1. The latter is simply $\widetilde{M}_2^{(\leq 1)} \backslash \{0\}/\mathbb{C}^*$, which can be seen as a projective space with the point at infinity corresponding to the trivial equivariant form $h(z) = z$.

As an example, if g is a weight k modular form for a modular subgroup Γ, then g' is a weight $k+2$ and depth 1 quasimodular form, while g'/g is a weight 2 depth 1 quasimodular form and $f = g'/kg$ becomes a normalized one. According to the above, $h(z) = z + 1/f(z)$ is then an equivariant form. Thus we recover the rational equivariant forms of Theorem 3.3.

6 Classification via Modular Forms

In this section, we explain how the equivariant forms for a modular subgroup Γ can also be identified with the weight 2 modular forms for Γ.

Proposition 6.1 *Let h and g be two equivariant forms for Γ, then $f(z) = \widehat{h} - \widehat{g}$ is a weight 2 modular form for Γ.*

Proof Indeed since h and g are equivariant, we have for $\gamma \in \Gamma$:

$$\begin{aligned} f(z) = \widehat{h}(z) - \widehat{g}(z) &= \widehat{h}\|[\gamma](z) - \widehat{g}\|[\gamma](z) \\ &= \widehat{h}|_2[\gamma](z) - \widehat{g}|_2[\gamma](z) = f|_2[\gamma](z). \end{aligned}$$

Therefore, f is a weight 2 modular form for Γ, as the meromorphy of f at the cusps of Γ follows from that of h and g. ■

Another way to look at this fact is by noting that if f_1 and f_2 are two normalized quasimodular forms of weight 2 and depth 1, then $f = f_1 - f_2$ is a modular form of weight 2, as the quasimodular term $c/(cz+d)$ cancels out in the difference. Thus if one fixes a normalized quasimodular form of weight 2 and depth 1, say $\widetilde{E}_2$, then for every normalized quasimodular form f of weight 2 and depth 1, we have $f - \widetilde{E}_2 \in M_2(\Gamma)$, and conversely, if $g \in M_2(\Gamma)$, then $g + \widetilde{E}_2$ is a normalized weight 2 and depth 1 quasimodular form for Γ. Therefore, we have the following proposition.

Proposition 6.2 *A meromorphic function $h \neq z$ on $\mathbb{H}$ is equivariant for Γ if and only if there exists a weight 2 modular form f for Γ such that*

$$h(z) = z + \frac{1}{\widetilde{E}_2(z) + f(z)}. \tag{6.1}$$

This provides us with the above mentioned identification between nontrivial equivariant forms and the modular forms of weight 2.

We now look at the particular case when the weight 2 modular forms are holomorphic modular forms. The following is straightforward.

Proposition 6.3 *Let f be a weight 2 modular form and let h be the corresponding equivariant form as in (6.1), then f is holomorphic if and only if h does not have a fixed point.*

The space of weight 2 holomorphic modular forms for a modular subgroup Γ has dimension $g+r-1$, where g is the genus of Γ, that is of the compact Riemann surface X_Γ, and r is the number of inequivalent cusps. In our case r is always positive, since Γ has always a cusp at ∞. Hence, when $g = 0$ and $r = 1$, *e.g.*, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, then this space is trivial. Therefore, the only equivariant form $h(z)$ without fixed points corresponds to $f = 0$ in (6.1), and thus $h(z) = h_\Delta(z) = z + 1/\widetilde{E}_2(z)$ which we will denote by h_0 and refer to as the fundamental equivariant form for the rest of this paper.

In fact, Proposition 6.2 confers to the space $\mathcal{E}(\Gamma)$ of nontrivial equivariant forms a vector space structure in which h_0 plays the role of the zero element. Moreover, the equivariant forms without fixed points form a finite dimensional subspace of dimension $g + r - 1$, where g and r are as above. As for the vector space operations, they are as follows. In $\mathcal{E}(\Gamma)$, the sum $h_1 \oplus h_2$ is given by

$$\widehat{h_1 \oplus h_2} = \widehat{h}_1 + \widehat{h}_2 - \widehat{h}_0 = \widehat{h}_1 + \widehat{h}_2 - \widetilde{E}_2.$$

Recall that $\widehat{h}(z) = (h(z) - z)^{-1}$. The opposite of h is given by $2\widehat{h}_0 - \widehat{h} = 2\widetilde{E}_2 - \widehat{h}$, and if c is a scalar, then $c \odot h$ is given by

$$\widehat{c \odot h} = c\widehat{h} + (1 - c)\widehat{h}_0.$$

7 Differential Forms and Sections of a Canonical Line Bundle

The space $M_2(\Gamma)$ of weight 2 meromorphic modular forms for Γ is isomorphic to the space of meromorphic differential 1-forms on the Riemann surface X_Γ where each modular form $f(z)$ corresponds to the differential form $f(z)dz$ on $\mathbb{H}$ which is invariant under Γ. In this way, these modular forms are sections of the cotangent bundle $\Omega^1(X_\Gamma)$, that is the canonical line bundle of X_Γ, and by the previous section, the nontrivial equivariant forms for Γ can be looked at in a similar manner.

Let us proceed in a different way to connect equivariant forms to differential forms. Let h be a nontrivial equivariant form for a modular subgroup Γ with which we associate the meromorphic degree 1 differential $w = (\widehat{h}(z))dz$, where $\widehat{h}(z)$ denotes as usual $(h(z) - z)^{-1}$. Then, since h is an equivariant form and $\frac{d}{dz}(\alpha \cdot z) = j_\alpha(z)^{-2}$, we get a degree one differential satisfying

$$\alpha^* w = w + \frac{c}{j_\alpha(z)} dz, \quad \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

where $\alpha^* w = \widehat{h}(\alpha \cdot z) d(\alpha \cdot z)$. In other words,

$$\alpha^* w - w = \frac{c}{j_\alpha(z)} dz \quad \text{for all} \quad \alpha \in \Gamma. \tag{7.1}$$

Conversely, suppose we are given a degree 1 meromorphic differential w on $\mathbb{H}^*$ satisfying (7.1) for all $\alpha \in \Gamma$. Write $w = f(z)dz$ for some meromorphic function $f\colon \mathbb{H}^* \to \mathbb{C}$. Then we have

$$\alpha^* w - w = j_\alpha(z)^{-2} f(\alpha \cdot z) dz - f(z)dz = f(z)|_2[\alpha] dz - f(z)dz = \frac{c}{j_\alpha(z)} dz.$$

Hence,

$$f(z)|_2[\alpha] = f(z) + \frac{c}{j_\alpha(z)},$$

that is, f is a normalized weight 2 depth 1 quasimodular form. Therefore, $h(z) = z + 1/f(z)$ is an equivariant form by Proposition 5.4. This establishes the correspondence between equivariant forms and degree 1 meromorphic differentials on $\mathbb{H}^*$ satisfying (7.1).

Now, if $w_0,\ w_1$ are two differential forms satisfying (7.1), then the degree 1 form $w = w_1 - w_0$ is invariant under the action of Γ. Therefore, there is a weight 2 meromorphic modular form f on Γ such that $w = f(z)dz$. Hence, fixing $w_0 = \widehat{h}_0(z)dz$, we get a one-to-one correspondence between the space of degree 1 meromorphic differentials on $\mathbb{H}$ invariant under the action of Γ, which we identified with the space of degree 1 meromorphic differentials on X_Γ, and the set of degree 1 meromorphic differentials on $\mathbb{H}^*$ satisfying (7.1). As a consequence, we have the following theorem.

Theorem 7.1 *The nontrivial equivariant forms are identified with the meromorphic (global) sections of the canonical line bundle of X_Γ.*

Again, in this identification, the zero section corresponds to the fundamental equivariant form h_0, and the holomorphic sections correspond to the equivariant forms without fixed points.

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