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On the critical points of modular forms

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ABSTRACT

In this paper, we study the critical points of classical modular forms. In particular, we prove that for each modular form f for a subgroup of $\mathrm{SL}_2(\mathbb{Z})$, its derivative f' has infinitely many inequivalent zeros and all, but a finite number, are simple.

1. Introduction

Let Γ be a modular subgroup, that is, a subgroup of finite index of the modular group $\mathrm{SL}_2(\mathbb{Z})$. The group Γ acts on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C},\ \mathrm{Im}(z) > 0\}$ by linear fractional transformations. Then f is a weight k modular form for Γ, k being a positive integer, if f is a holomorphic function on $\mathbb{H}$ satisfying

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \quad z \in \mathbb{H},\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \tag{1.1}$$

in addition to some growth conditions of f at the cusps of Γ, making f holomorphic at these cusps.

The study of the zeros and poles of modular forms is well developed and lies around the valence formula. However, the critical points of modular forms have not received any attention. The purpose of this paper is to show that for any modular form of any weight for Γ, the derivative f' of f

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has infinitely many zeros that are inequivalent relative to Γ. In fact, infinitely many such zeros lie within a vertical strip of width equal to the cusp width at infinity for Γ. Moreover, all but a finite number of these zeros are simple. In fact, the multiple zeros happen to be the zeros of an auxiliary modular form constructed using the Rankin–Cohen bracket and thus there is only a finite number of equivalence classes of such multiple zeros. The main result is a generalization of a result in [3], where it is shown that the Eisenstein series E_2 has infinitely many inequivalent zeros which are also the zeros of the derivative Δ' of the discriminant cusp form Δ.

Our main tool is the equivariance of the function

$$h(z) = z + k\frac{f(z)}{f'(z)},$$

where f is a modular form of weight k. Indeed, the function h has the intriguing property that it commutes with the action of Γ. Using classical theorems from complex analysis such as the Schwarz lemma and the theorem of Denjoy–Wolff, we will prove that in fact h has infinitely many inequivalent poles in $\mathbb{H}$.

It should be noted that some ideas in Section 3 were inspired from the work of M. Heins [4]. In this paper a particular function commuting with the action of the modular group has been constructed as the quotient of two pseudo-periods of the Weierstrass ζ-function. However, the proofs of these results in Section 3 are made much simpler since we deal only with modular subgroups and thus avoiding the intricate geometry of a general Fuchsian group of the first kind.

The paper is organized as follows: In Section 2, we attach, to each modular form, an equivariant function that commutes with the action of the modular subgroup. In Section 3, we study mapping properties of the equivariant functions with the aim that if such equivariant function is not the identity, then the image of the upper half-plane must meet the real line. In Section 4, we prove the main result of this paper that the derivative of every modular forms has infinitely many inequivalent zeros in the upper half-plane. We also provide an interesting application about the vanishing of the Fourier coefficients of modular forms. In Section 5, we extend the main result to the case of quasimodular forms of weight 2 and depth 1 having monomial quasimodular polynomial. Finally, in the last section, we study few examples involving Eisenstein series and theta functions.

2. From modular forms to equivariant functions

While, for most of this note, the group Γ can be any discrete group commensurable with the modular group, we will suppose, for the sake of clarity, that Γ is a modular subgroup. Let f be a modular form for Γ of weight k, $k \geqslant 0$. We attach to f a meromorphic function

$$h(z) = h_f(z) = z + k\frac{f(z)}{f'(z)}. \tag{2.1}$$

We have:

Proposition 2.1. *(See [9,8].) The function $h(z)$ satisfies*

$$h(\gamma \cdot z) = \gamma \cdot h(z) \quad \text{for all } \gamma \in \Gamma,\ z \in \mathbb{H}. \tag{2.2}$$

This important statement follows from a straightforward calculation. A complete classification of all functions satisfying the relation (2.2) as well as many of their properties is carried out in [8] and [2]. Following [8], we call such function an *equivariant function* for Γ. If $k = 0$, that is f is a modular function, then $h(z) = h_0(z) := z$.

Proposition 2.2. *Let Γ be a modular subgroup, then the only Möbius transformation that is equivariant for Γ is $h_0(z)$.*

Proof. Let $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Since Γ has finite index in $\mathrm{SL}_2(\mathbb{Z})$, and since T and S have infinite orders, a certain power T^n of T and a certain power S^m of S must belong to Γ. If $h(z) = \gamma \cdot z$, $\gamma \in \mathrm{GL}_2(\mathbb{C})$, is equivariant, then γ must commute with all the elements of Γ. In particular, if γ commutes with both T^n and S^m, then $\gamma = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, $\lambda \in \mathbb{C}$, that is, $h(z) = h_0(z) = z$. □

3. Mapping properties

In the following, we will study the image of the upper half-plane $\mathbb{H}$ under an equivariant function h. The ultimate goal is to show that if $h \neq h_0$ is holomorphic, then $h(\mathbb{H}) \cap \mathbb{R}$ is not empty.

Proposition 3.1. *Let $h \neq h_0$ be a holomorphic equivariant function for a modular subgroup Γ such that h maps the upper half-plane $\mathbb{H}$ to itself. If h has a fixed point z_0, then it is unique.*

Proof. Let M be a Möbius transformation that maps $\mathbb{H}$ conformally onto the unit disc D such that $M(z_0) = 0$. The map $g = MhM^{-1} : D \to D$ is a holomorphic map such that $g(0) = 0$. If h has another fixed point $z_1 \neq z_0$ in $\mathbb{H}$, it will correspond to a nonzero fixed point of g in D. Using the Schwarz lemma, we must have $g(z) = \lambda z$, and thus $h(z)$ is a Möbius transformation. According to Proposition 2.2, we must have $h(z) = h_0(z) = z$ which is excluded by assumption. □

Proposition 3.2. *Let $h \neq h_0$ be a holomorphic equivariant function for a modular subgroup Γ such that h maps the upper half-plane $\mathbb{H}$ to itself. Then h does not have a fixed point in $\mathbb{H}$.*

Proof. Let us assume that there exists $z_0 \in \mathbb{H}$ such that $h(z_0) = z_0$. For $\gamma \in \Gamma$, we have $h(\gamma \cdot z_0) = \gamma \cdot h(z_0) = \gamma \cdot z_0$ and hence $\gamma \cdot z_0$ is also a fixed point of h. Therefore, using Proposition 3.1, $\gamma \cdot z_0 = z_0$ for all $\gamma \in \Gamma$, which is impossible if one takes $\gamma = T^n$, a power of the translation T that belongs to Γ. Thus, h cannot have a fixed point in $\mathbb{H}$. □

Proposition 3.3. *Let h be a holomorphic equivariant function for a modular subgroup Γ mapping the upper half-plane $\mathbb{H}$ to itself, then $h(z) = h_0(z) = z$.*

Proof. Let us assume that $h \neq h_0$. Then, using Proposition 3.2, h does not have a fixed point in $\mathbb{H}$. As above, let M be a Möbius transformation that maps $\mathbb{H}$ conformally onto the unit disc D. The map $g = MhM^{-1} : D \to D$ is a holomorphic map. According to the theorem of Denjoy–Wolff, [1], g is either a conformal rotation about some fixed point z_0, or the successive iterates $g_n = g^{\circ n}$ converge, uniformly on compact subsets of D, to a constant function $z \mapsto c_0 \in \bar{D}$. The former situation is excluded since g does not have a fixed point. It follows that the holomorphic maps $h_n = h^{\circ n} : \mathbb{H} \to \mathbb{H}$ also converge to a constant function $z \mapsto c_1 = M^{-1}c_0$. Suppose $c_1 \neq \infty$ and let n_0 be a positive integer such that T^{n_0} is in Γ. Since the functions h_n are also equivariant, we have $h_n(z + n_0) = h_n(z) + n_0$. For a fixed $z \in \mathbb{H}$, taking n to infinity yields $c_1 = c_1 + n_0$ which is a contradiction since $n_0 > 0$. If $c_1 = \infty$, let n_1 be a positive integer such that $S^{n_1}(z) = z/(n_1 z + 1)$ is in Γ. For a fixed $z \in \mathbb{H}$, taking n to ∞ in $h_n(S^{n_1} z) = S^{n_1} h_n(z)$ yields a contradiction since $n_1 > 0$. □

We shift our attention to the case when an equivariant function maps the upper half-plane to the lower half-plane $\mathbb{H}^-$.

Proposition 3.4. *An equivariant function for a modular subgroup Γ cannot map the upper half-plane $\mathbb{H}$ to the lower half-plane $\mathbb{H}^-$.*

Proof. Suppose that such h exists. We extend h to a mapping $\tilde{h}$ defined on $\mathbb{H} \cup \mathbb{H}^-$ by setting $\tilde{h}(z) = \overline{h(\bar{z})}$ for $z \in \mathbb{H}^-$. Then the restriction of $\tilde{h} \circ \tilde{h}$ maps $\mathbb{H}$ to $\mathbb{H}$ and it is equivariant for Γ. According to Proposition 3.3, we must have $\tilde{h} \circ \tilde{h}(z) = z$. This, in particular, implies that $h(z)$ is bijective. Therefore, the function $g(z) = 1/h(z)$ is an automorphism of $\mathbb{H}$ and so is an element of $\mathrm{PSL}_2(\mathbb{R})$. Therefore h

is a Möbius transformation that is equivariant. According to Proposition 2.2, $h(z) = h_0(z)$. This is a contradiction since h_0 is supposed to map $\mathbb{H}$ to $\mathbb{H}^-$. $\square$

4. Critical points of modular forms

In this section we prove that for any nonzero modular form f of positive weight k for a modular subgroup Γ, the derivative f' has infinitely many zeros in $\mathbb{H}$ that are inequivalent relative to Γ. We recall that, using the valence formula for modular forms, f has only $k\mu/12$ inequivalent zeros counted with multiplicity where $\mu = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$.

Theorem 4.1. *If f is a nonzero modular form of weight $k > 0$ for a modular subgroup Γ, then f' has infinitely many zeros in $\mathbb{H}$ that are inequivalent relative to Γ.*

Proof. Let f be such a modular form. The equivariant function

$$h(z) = z + k\frac{f(z)}{f'(z)}$$

satisfies $h(z) \neq h_0(z)$. Moreover, there exists $z_0 \in \mathbb{H}$ such that $h(z_0) \in \mathbb{R}$. Indeed, if h has a pole z, then for any $\gamma \in \Gamma$ not a translation,

$$h(\gamma \cdot z) = \gamma \cdot h(z) = \gamma \cdot \infty \in \mathbb{Q}.$$

On the other hand, if $h(z)$ is holomorphic in $\mathbb{H}$, then by Proposition 3.3 $h(\mathbb{H})$ is not contained in $\mathbb{H}$ and by Proposition 3.4, $h(\mathbb{H})$ is not contained in $\mathbb{H}^-$. Since $h(\mathbb{H})$ is connected, we must have $h(\mathbb{H}) \cap \mathbb{R} \neq \emptyset$. In either case, there is always $z_0 \in \mathbb{H}$ at which h takes a real value. In fact, z_0 cannot be an elliptic fixed point for Γ, otherwise, there exists an elliptic element $\gamma \in \Gamma$ such that $\gamma \cdot z_0 = z_0$, and hence $h(z_0)$ is also fixed by γ and thus $h(z_0) = z_0$ or $h(z_0) = \bar{z}_0$ and so it is not real. We can choose a small open disc D around $h(z_0)$ such that $V = h^{-1}(D)$ is an open neighborhood of z_0 not containing any elliptic fixed point, and such that no two points of V are Γ-equivalent. The disc D meets $\mathbb{R}$ in an open interval I. Meanwhile, since Γ has finite index in $\mathrm{SL}_2(\mathbb{Z})$, each orbit of a cusp for Γ is dense in $\mathbb{R}$ and hence in the interval I. In particular, there are infinitely many rational points that are Γ-equivalent to the cusp ∞. Let $r \in I$ be such a rational number with $r = h(z)$, $z \in V$, and let $\gamma \in \Gamma$ such that $\gamma \cdot r = \infty$. Then $\gamma \cdot z$ is a pole of h. Indeed,

$$h(\gamma \cdot z) = \gamma \cdot h(z) = \gamma \cdot r = \infty.$$

In fact, the point $\gamma \cdot z$ is a zero of $f'(z)$. Furthermore, no two zeros $\gamma_1 \cdot z_1$ and $\gamma_2 \cdot z_2$ of $f'(z)$ thus constructed are Γ-equivalent because this would imply that z_1 and z_2, which are both in V, are Γ-equivalent and this is not the case by construction of V. Therefore, $f'(z)$ has infinitely many zeros in $\mathbb{H}$ that are inequivalent relative to Γ. $\square$

Remark 4.2. Since $f'(z)$ is invariant under the translations that are in Γ, it is clear that $f'(z)$ has infinitely many inequivalent zeros in the vertical strip $-l/2 < \operatorname{Re} z \leqslant l/2$ where l is the cusp width at infinity defined to be the smallest positive integer n such that $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$.

As for the multiplicity of the zeros of f', we have:

Proposition 4.3. *If f is a nonzero modular form for a modular subgroup Γ, then f' has only a finite number of inequivalent multiple zeros.*

Proof. If f has weight k, then one can easily show that $F(z) = kff'' - (k+1)f'^2$ is a modular form of weight $2k+4$. Now, suppose that z_0 is a multiple zero of f', then $f''(z_0) = 0$ and hence $F(z_0) = 0$. However, $F(z)$, being a modular form, has only finitely many inequivalent zeros. The proposition follows. $\square$

Remark 4.4. The form $F(z)$ introduced in the above proof is an example of the so-called Rankin–Cohen bracket [10]. In fact $(k+1)F(z) = [f, f]_2$, the bracket of order 2. For a modular form f of weight k and a modular form of weight l, the Rankin–Cohen bracket of order $n \geqslant 0$ is defined by

$$[f, g]_n = \sum_{r=0}^{n} (-1)^r \binom{k+n-1}{n-r} \binom{l+n-1}{r} f^{(r)} g^{(n-r)}.$$

It defines a modular form of weight $k+l+2n$.

We end this section with an interesting application. If f is a modular form of weight k for a modular subgroup Γ and if l is the cusp width at ∞, then f has the following q-expansion, where $q = \exp 2\pi iz/l$

$$f(z) = \sum_{n \geqslant n_0} a_n q^n, \quad a_n \in \mathbb{C}.$$

As a corollary of the results of this section, we have the following:

Theorem 4.5. *If f is a nonzero modular form for a modular subgroup Γ, then infinitely many of the q-coefficients a_n are nonzero.*

Proof. If f is polynomial in q, so is f'. Moreover, there are infinitely many inequivalent zeros to f' in the vertical strip $-l/2 < \operatorname{Re} z \leqslant l/2$ yielding infinitely many different values of q that are roots of $f'(q)$ which is a contradiction. Therefore, f cannot be polynomial in q. $\square$

In fact, this is a classical theorem for which a proof can be found in [6] using the Rankin–Selberg zeta function attached to f or using the theory of vector-valued modular forms.

5. Quasimodular forms

In this section we establish a similar result for quasimodular forms. They were introduced by M. Kaneko and D. Zagier in [5] as a generalization of modular forms. A holomorphic function f on $\mathbb{H}$ is called a quasimodular form of weight k and depth p for a modular subgroup Γ if it satisfies

$$(cz+d)^{-k} f(\gamma \cdot z) = \sum_{i=0}^{p} f_i(z) \left(\frac{c}{cz+d} \right)^i, \quad \gamma \in \Gamma, \ z \in \mathbb{H}, \tag{5.1}$$

where each f_i is a holomorphic function of $\mathbb{H}$. The quasimodular polynomial attached to f is defined by

$$F_k^p(z, X) := \sum_{i=0}^{p} f_i(z) X^i. \tag{5.2}$$

Setting $\gamma = I_2$, the identity matrix, in (5.1) shows that $f_0 = f$.

We focus on the quasimodular forms of weight 2 and depth 1, and in particular on those whose quasimodular polynomial is monomial, that is, of the form

$$F_2^1(z, X) = f_0(z) + X. \tag{5.3}$$

Example of such quasimodular form is given by

$$\phi_0(z) = \frac{\pi i}{6} E_2(z),$$

where E_2 is the Eisenstein series given by:

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n$$

where $\sigma_1(n)$ is the sum of the positive divisors of n. In fact ϕ_0 is a quasimodular form for $\mathrm{SL}_2(\mathbb{Z})$ as well as for any of its subgroups. We have

Proposition 5.1. *(See [2].) Let f be a quasimodular form for a modular subgroup Γ with a quasimodular polynomial given by* (5.3)*. Then the function*

$$h(z) = z + \frac{1}{f(z)}$$

is equivariant for Γ.

Proof. This follows from a straightforward calculations. □

As in the previous section we deduce

Corollary 5.2. *Every quasimodular form for a modular subgroup Γ with a quasimodular polynomial as in* (5.3) *has infinitely many zeros that are inequivalent relative to Γ.*

There is a different way to look at the above result. As it was mentioned above, according to the valence formula for modular forms, the number of inequivalent zeros of f counted with multiplicity is $2\mu/12$ where μ is the index of Γ in $\mathrm{SL}_2(\mathbb{Z})$. However if we perform a perturbation to f by a fixed quasimodular form as above, *e.g.* by ϕ_0, then we have the following:

Proposition 5.3. *Let $\phi_0 = (\pi i/6)E_2$ and let f be any modular form of weight* 2 *for a modular subgroup Γ, then $f + \phi_0$ has infinitely many zeros that are inequivalent relative to Γ.*

Proof. If f is a weight 2 modular form for Γ, and if ϕ is a quasimodular form for Γ with a quasimodular polynomial as in (5.3), then $f + \phi$ is also a quasimodular form for Γ with the quasimodular polynomial having the same shape as in (5.3). The proposition follows from Corollary 5.2. □

Remark 5.4. The results of Sections 4 and 5 are still valid for meromorphic modular forms and meromorphic quasimodular forms and also for modular forms with a multiplier system since the equivariance of (2.1) remains valid in these cases [2]. Moreover, the weight of the modular forms under consideration can be any positive real number, and not necessary an integer.

6. Examples

In this section we provide few examples involving the Eisenstein series and the Jacobi theta functions. The main reference for the formulas below is [7]. We recall the following:

$$E_2(z) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n,$$

$$E_4(z) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n,$$

$$E_6(z) = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n.$$

The Eisenstein series E_4 and E_6 are modular forms for $\mathrm{SL}_2(\mathbb{Z})$ of weight 4 and 6 respectively, and E_2 is a quasimodular form of weight 2 and depth 1. Moreover, E_2 satisfies

$$E_2(z) = \frac{1}{2\pi i}\frac{\Delta'(z)}{\Delta(z)}, \tag{6.1}$$

where Δ is the weight 12 cusp form for $\mathrm{SL}_2(\mathbb{Z})$ given by

$$\Delta(z) = q\prod_{n\geqslant 1}\left(1-q^n\right)^{24}.$$

The Eisenstein series satisfy the Ramanujan relations

$$\frac{6}{\pi i}E_2' = E_2^2 - E_4, \tag{6.2}$$

$$\frac{3}{2\pi i}E_4' = E_4E_2 - E_6, \tag{6.3}$$

$$\frac{1}{\pi i}E_6' = E_6E_2 - E_4^2. \tag{6.4}$$

The Jacobi theta functions are given by

$$\theta_2(z) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2},$$

$$\theta_3(z) = \sum_{n=-\infty}^{\infty} q^{n^2},$$

$$\theta_4(z) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}.$$

These theta functions are modular forms of weight 1/2 with multiplier systems for conjugates of the level 2 congruence subgroup $\Gamma(2)$.

As a consequence of Corollary 5.2, the Eisenstein series E_2 (and hence Δ') has infinitely many zeros in the strip $-1/2 < \mathrm{Re}(z) \leqslant /12$. Moreover, all these zeros are simple since E_2 and E_2' cannot vanish at the same time because of (6.2) and the fact that E_4 vanishes only at the orbit of the cubic root of unity ρ. Similarly, the zeros of E_4' are also simple because of (6.3) and the fact that E_6 vanishes only at the orbit of i. Using the same argument and (6.4), the zeros of E_6' are all simple. We also note that E_2 is real on the axis $\mathrm{Re}(z) = 0$ and $\mathrm{Re}(z) = 1/2$, and one can show that it has a unique zero on each axis given approximately by $i0.5235217000$ and $1/2 + i0.1309190304$. Also, E_4' and E_6' are purely imaginary on both axes and have (unique) zeros only on $\mathrm{Re}(z) = 1/2$ given respectively by $1/2 + i0.4086818600$ and $1/2 + i0.6341269863$.

As for the theta functions, we have the following relations:

$$\vartheta_2(z) = 2\frac{\eta(4z)^2}{\eta(2z)},$$

$$\vartheta_3(z) = \frac{\eta(2z)^5}{\eta(z)^2\eta(4z)^2},$$

$$\vartheta_4(z) = \frac{\eta(z)^2}{\eta(2z)},$$

where the Dedekind eta function is given by $\eta(z) = \Delta(z)^{1/24}$. One can see that the three theta functions are infinite products and thus do not vanish in $\mathbb{H}$. Using (6.1), we have the following formulas for the derivative of the theta functions:

$$\frac{1}{4\pi i}\frac{\theta_2'(z)}{\theta_2(z)} = 4E_2(4z) - E_2(2z), \tag{6.5}$$

$$\frac{24}{\pi i}\frac{\theta_3'(z)}{\theta_3(z)} = 5E_2(2z) - E_2(z) - 4E_2(4z), \tag{6.6}$$

$$\frac{1}{2\pi i}\frac{\theta_4'(z)}{\theta_4(z)} = E_2(z) - E_2(2z). \tag{6.7}$$

It is interesting to notice that each of the above combinations with E_2, E_4 and E_6 vanishes infinitely many times at inequivalent points in a vertical strip in $\mathbb{H}$.

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