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Quantum Screening Operators and Canonical q-de Rham Cocycles

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Abstract: In this paper a close connection is established between certain cohomology spaces of representations of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$, and a twisted q-de Rham (Jackson–Aomoto) cohomology of configuration spaces using the quantum screening operators.

1. Introduction

The representation theory of Virasoro algebras, established by Feigin and Fuchs [7], gives a way to obtain intertwining operators between the Fock space representations, out of the top homology classes of certain one dimensional local systems over configuration spaces. A similar construction exists for affine Kac-Moody algebras. These intertwining operators are built up from the so-called *screening operators*. In [10], Ginzburg and Schechtman made the remark that in fact these screening operators contain more information. Namely, they provide canonical cocycles of the Virasoro and affine Kac-Moody Lie algebras, with coefficients in the de Rham complex of an operator-valued local system on the configuration space. This makes it possible to obtain canonical morphisms from higher homology groups of the above local system to appropriate Ext-groups between the Fock modules.

The purpose of this paper is to investigate this connection between the geometry of configuration spaces and representation theory in the case of the quantum affine algebras $U_q(\widehat{\mathfrak{sl}_2})$. The quantized universal enveloping algebras of semisimple Lie algebras were studied in this context in [17], and the representations in question were the Verma modules. In this paper we consider the q-analog of the Wakimoto modules [19] via *bosonization* of the quantum affine algebras, using three scalar boson fields. From the same boson fields, we construct the screening operator $S(z)$. We introduce a certain bracket $\langle \cdot, \cdot \rangle$ which is a pairing between elements of $U_q(\widehat{\mathfrak{sl}_2})$ and operators between the q-Wakimoto modules. This bracket is defined using the Hopf algebra structure of $U_q(\widehat{\mathfrak{sl}_2})$. Then we solve the difference equation

$$\langle x, S(z)\rangle = \partial_q S(x,z) \quad x \in U_q(\widehat{\mathfrak{sl}_2}).$$

It turns out that the solution $S(z)$ and $S(x,z)$ have very nice homological properties, in particular, $S(x,z)$ satisfies a cocycle condition of the form:

$$S(xy,z) = x \cdot S(y,z) + \varepsilon(y) S(x,z),$$

where the dot means the action of x on the Hom space between the modules, and ε is the counit map of the Hopf algebra $U_q(\widehat{\mathfrak{sl}_2})$. The above difference equation is built by means of q-difference operators of Jackson type, which generate a flat connection in a one dimensional vector bundle over the n-dimensional torus. We then consider a "q-de Rham" complex of the space of formal algebraic q-differential forms over the n-torus, whose "differentials" are the above q-difference operators. The homology groups of this complex can be regarded as the homology groups of the n-torus with coefficients in a local system with stalk $\mathbb{C}$. From these data, we construct the canonical cocycles, which in fact live in the total space of a double complex produced by mixing the above q-de Rham complex and a Hochshild complex which will be introduced below. These cocycles yield canonical maps between the homology groups of the local systems and the Ext-spaces between the representations in question.

I should mention the emphasis made on the Hopf algebra structure of $U_q(\widehat{\mathfrak{sl}_2})$, in fact, we make sense of the above notions only by the use of this structure. Also, techniques such as the vertex operators calculus and the normally ordered products were very helpful.

2. Hopf Algebras and Their Actions

In this section we follow closely the treatment in [17] with less details. We present some constructions on Hopf algebras and their representations, and establish some results related to them. All the algebraic structures will be over the field of complex numbers.

2.1. Composition of maps. Let H be a Hopf algebra, and let Δ, A, and ε denote respectively the comultiplication, the antipode and the counit maps.

If $\mathcal{M}$ and $\mathcal{N}$ are two H-modules, one can define a structure of left module on both $\mathcal{M} \otimes \mathcal{N}$ and $\mathrm{Hom}(\mathcal{M},\mathcal{N})$ by:

$$x \cdot (m \otimes n) = \sum_{(x)} (x'm) \otimes (x''n) \quad (m \in \mathcal{M}, n \in \mathcal{N},\ x \in H),$$

$$(x \cdot f)(m) = \sum_{(x)} x' f(A(x'')m) \quad (m \in \mathcal{M},\ f \in \mathrm{Hom}(\mathcal{M},\mathcal{N}),\ x \in H),$$

where we have used the Sweedler notation for the comultiplication:

$$\Delta(x) = \sum_{(x)} x' \otimes x'' \qquad (x \in H).$$

(If more than one superscript $'$ or $''$ is involved, we will use numerical subscripts.)

Each vector space carries a structure of an H-module through the map ε. Therefore, the dual space $\mathcal{M}^* = \mathrm{Hom}(\mathcal{M}, \mathbb{C})$ is an H-module with the action of H given by:

$$(x \cdot \phi)(m) = \phi(A(x)m) \quad x \in H,\ \phi \in \mathcal{M}^*,\ m \in \mathcal{M}.$$

Lemma 2.1. *For every $x \in H$, the following relation holds in $H \otimes H \otimes H$:*

$$\sum_{(x)} x_1 \otimes 1 \otimes x_2 = \sum_{(x)} x_{1,1} \otimes A(x_{1,2})x_{2,1} \otimes x_{2,2}. \tag{2.1}$$

Proof. We prove the identity by applying the coassociativity of the map Δ several times. Let $x \in H$, by coassociativity we have:

$$x_1 \otimes x_{2,1} \otimes x_{2,2} = x_{1,1} \otimes x_{1,2} \otimes x_2.$$

Now, applying $1 \otimes \Delta \otimes 1$ to the left side, and $1 \otimes 1 \otimes \Delta$ to the right side, we obtain by coassociativity:

$$x_1 \otimes x_{2,1,1} \otimes x_{2,1,2} \otimes x_{2,2} = x_{1,1} \otimes x_{1,2} \otimes x_{2,1} \otimes x_{2,2}.$$

Therefore,

$$\begin{aligned} x_{1,1} \otimes A(x_{1,2})x_{2,1} \otimes x_{2,2} &= x_1 \otimes A(x_{2,1,1})x_{2,1,2} \otimes x_{2,2} \\ &= x_1 \otimes \varepsilon(x_{2,1}) \otimes x_{2,2} \\ &= x_1 \otimes 1 \otimes x_2, \end{aligned}$$

using the axioms of the counit. This proves the lemma. □

Proposition 2.2 (*Composition lemma*). *If $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{P}$ are three H-modules, then for every $f \in \mathrm{Hom}(\mathcal{N}, \mathcal{P})$ and for every $g \in \mathrm{Hom}(\mathcal{M}, \mathcal{N})$ and $x \in H$, we have:*

$$x \cdot (f \circ g) = \sum_{(x)} (x' \cdot f) \circ (x'' \cdot g).$$

Proof. The relation can be written as:

$$x_1 f(g(A(x_2)m)) = x_{1,1} f(A(x_{1,2})x_{2,1} g(A(x_{2,2})m))) \quad (m \in \mathcal{M}),$$

which follows from the above lemma after applying $1 \otimes 1 \otimes A$ to its two sides. □

The composition lemma seems to be just a consequence of the axiomatic definition of the Hopf algebra, especially from the coassociativity. Let us consider the composition map $(f, g) \longrightarrow f \circ g$. It is a bilinear map and therefore induces a linear map

$$\mathrm{Hom}(\mathcal{N}, \mathcal{P}) \otimes \mathrm{Hom}(\mathcal{M}, \mathcal{N}) \xrightarrow{\circ} \mathrm{Hom}(\mathcal{M}, \mathcal{P}).$$

Using the action of H on the tensor product and on the Hom space, we can restate the composition lemma as the following result

Corollary 2.3. *The composition map is H-linear.*

Remark 2.1. The linearity of the composition map is known and proved in the literature only when $\mathcal{N}$ (or both $\mathcal{M}$ and $\mathcal{P}$) is finite dimensional, in which case $\mathrm{Hom}(\mathcal{M}, \mathcal{N})$ is isomorphic to $\mathcal{M}^* \otimes \mathcal{N}$, see [13]. Here, we established it for the general case, for all the representations we will be considering are infinite dimensional.

2.2. A bracket and a cochain complex. Let H be a Hopf algebra over $\mathbb{C}$ with the associated maps as above. We define a bilinear map $\langle \cdot, \cdot \rangle$ on $H \otimes H$ by

$$\langle x, y \rangle = \sum_{(x)} x' y A(x'') - \varepsilon(x) y \quad (x, y \in H). \tag{2.2}$$

This bracket satisfies the following relations:

Proposition 2.4. *For all x, y, z in H, we have :*

(1) $\langle xy, z \rangle = \langle x, \langle y, z \rangle \rangle + \varepsilon(x) \langle y, z \rangle + \varepsilon(y) \langle x, z \rangle$,
(2) $\varepsilon(\langle x, y \rangle) = 0$,
(3) $A^2(\langle x, y \rangle) = \left\langle A^2(x), A^2(y) \right\rangle$.

Proof. The first relation follows from that fact that Δ is an algebra homomorphism and that A is an algebra anti-homomorphism. The second relation follows from the antipode axiom:

$$\sum_{(x)} x' A(x'') = \sum_{(x)} A(x') x'' = \varepsilon(x) \cdot \mathbf{1},$$

and from the fact that ε is an algebra homomorphism. We will prove the third relation: From the identity

$$\Delta(A(x)) = \sum_{(x)} A(x'') \otimes A(x')$$

we obtain

$$\Delta(A^2(x)) = \sum A^2(x') \otimes A^2(x'').$$

Using this relation and the fact that $\varepsilon(A(x)) = \varepsilon(x)$ we obtain:

$$\left\langle A^2(x), A^2(y) \right\rangle = A^2(\langle x, y \rangle). \qquad \square$$

Note that $\langle 1, x \rangle = \langle x, 1 \rangle = 0$ for every $x \in H$. And if H is commutative then $\langle x, y \rangle = 0$ for every $x, y \in H$, while if H is cocommutative, then

$$\langle x, A(y) \rangle = A \langle x, y \rangle .$$

If $\mathcal{M}$ is a left H-module, then $m \in \mathcal{M}$ is called an *H-invariant* if $xm = \varepsilon(x)m$ for every $x \in H$. If $\mathcal{N}$ is another H-module, we set

$$\langle x, \phi \rangle = x \cdot \phi - \varepsilon(x)\phi \quad x \in H,\ \phi \in \mathrm{Hom}(\mathcal{M}, \mathcal{N}). \tag{2.3}$$

Notice the analogy with the definition of $\langle x, y \rangle$. We say that ϕ is an *invariant* if $\langle x, \phi \rangle = 0$ for every $x \in H$.

Proposition 2.5. *For all x, y in H and for all ϕ, ψ in* $\mathrm{Hom}(\mathcal{M}, \mathcal{N})$ *we have:*

$$\langle xy, \phi \rangle = x \cdot \langle y, \phi \rangle + \varepsilon(y) \langle x, \phi \rangle , \tag{2.4}$$

$$\langle x, \phi\psi \rangle = \sum_{(x)} \langle x', \phi \rangle \langle x'', \psi \rangle + \langle x, \phi \rangle \psi + \phi \langle x, \psi \rangle . \tag{2.5}$$

Proof. The first relation is the same as the relation (1) in Proposition 2.4 when we substitute z by ϕ. The second relation is a consequence of the composition lemma and the fact that $\varepsilon(x) = \sum_{(x)} \varepsilon(x')\varepsilon(x'')$. $\square$

The importance of the bracket $\langle\cdot,\cdot\rangle$ will appear throughout this work. The expression for $\langle x,\phi\rangle$ is simply the difference of two actions of x on ϕ. In the case of the Universal enveloping algebras of Lie algebras, this bracket coincides with the Lie bracket, and for the quantized version of these algebras the appearance of the trivial action in this bracket will emphasize the role of the group-like elements as will be seen in the next sections. Another importance of this bracket will appear below in defining the differential for the cochain complex.

Let $\mathcal{M}$ be an H-module, and let us consider the following sequence:

$$C^\bullet = C(H^{\otimes\bullet},\mathcal{M}) : 0 \longrightarrow \mathcal{M} \longrightarrow \mathrm{Hom}(H,\mathcal{M}) \longrightarrow \cdots \longrightarrow \mathrm{Hom}(H^{\otimes n},\mathcal{M}) \longrightarrow \cdots,$$

and the linear map

$$d : \mathrm{Hom}(H^{\otimes n-1},\mathcal{M}) \longrightarrow \mathrm{Hom}(H^{\otimes n},\mathcal{M})$$

defined as follows:

If $\phi \in \mathrm{Hom}(H^{\otimes n-1},\mathcal{M})$ and $x_1 \otimes x_2 \otimes \ldots \otimes x_n \in H^{\otimes n}$, then

$$\begin{aligned} d\phi(x_1,x_2,\ldots,x_n) = {} & x_1 \cdot \phi(x_2,\ldots,x_n) \\ & + \sum_{i=1}^{n-1}(-1)^i \phi(x_1,\ldots,x_{i-1},x_i x_{i+1},x_{i+2},\ldots,x_n) \\ & + (-1)^n \phi(x_1,x_2,\ldots,x_{n-1})\varepsilon(x_n). \end{aligned}$$

One can look at this sequence as a Hochshild complex of the associative algebra H with a left and a right action which commute on the space $\mathcal{M}$, the right action is given by the trivial action, i.e. by ε, [3]. It follows that $(C^\bullet, d)$ is a cochain complex, i.e. $d^2 = 0$.

One can choose the coefficients of the cochains in $\mathrm{Hom}(\mathcal{M},\mathcal{N})$, where $\mathcal{M}$ and $\mathcal{N}$ are two H-modules. Thus we obtain a complex

$$C^\bullet(H,\mathcal{M},\mathcal{N}) = \mathrm{Hom}(H^{\otimes\bullet},\mathrm{Hom}(\mathcal{M},\mathcal{N})).$$

If $\phi \in \mathrm{Hom}(\mathcal{M},\mathcal{N})$, then $d\phi(x) = \langle x,\phi\rangle$. Hence, $d\phi = 0$ implies that $\langle x,\phi\rangle = 0$ for all $x \in H$. It follows that the 0^{th} cohomology space is the space of H-invariants. We will see that in the case of quantum groups, the space of invariants coincides with the space of intertwiners. More generally, the cohomology spaces are the Ext-spaces $\mathrm{Ext}_H^\bullet(\mathcal{M},\mathcal{N})$.

3. The Quantum Affine Algebra $U_q(\widehat{\mathfrak{sl}_2})$

3.1. The affine algebra $\widehat{\mathfrak{sl}_2}$. We recall the definition of the affine algebra $\widehat{\mathfrak{sl}_2}$ and we fix some notations. Let E, F, H be the standard generators of the Lie algebra $\mathfrak{sl}_2$. For X, Y in $\mathfrak{sl}_2$, we set $(X,Y) = \mathrm{tr}(XY)$, this defines an invariant bilinear form on $\mathfrak{sl}_2$. Thus $(E,F) = (F,E) = 1$, $(H,H) = 2$. We fix a complex number k and we set $B(X,Y) = k(X,Y)$. The corresponding affine algebra $\widehat{\mathfrak{sl}_2}$ is defined by the generators X_n ($X \in \mathfrak{sl}_2,\ n \in \mathbb{Z}$) and $\mathbf{1}$, and the relations

$$\text{(a)}\quad [X_n,Y_m] = [X,Y]_{m+n} + nB(X,Y)\delta_{m+n,0}.\mathbf{1} \quad (X,Y \in \mathfrak{sl}_2,\ m,n \in \mathbb{Z}).$$

This algebra is realized as a central extension of the loop algebra $\mathfrak{sl}_2 \otimes \mathbb{C}[z,z^{-1}]$ by $\mathbb{C}.\mathbf{1}$, and we identify X_n with $X \otimes z^n$. The affine algebra $\widehat{\mathfrak{sl}_2}$ is isomorphic to the Kac-Moody algebra $\mathfrak{sl}_2{}^{(1)}$ corresponding to the Cartan matrix of affine type $A_1^{(1)}$. To be more precise, we have to add a derivation to $\widehat{\mathfrak{sl}_2}$, that is an element D satisfying: $[D, X_n] = nX_n$ and $[D, \mathbf{1}] = 0$. And we have the identification

$$\mathfrak{sl}_2{}^{(1)} = \mathfrak{sl}_2 \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}.\mathbf{1} \oplus \mathbb{C}.D.$$

Let $P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda \oplus \mathbb{Z}\delta$ be the weight lattice and let $Q = \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1$ be the root lattice endowed with a symmetric bilinear form $(.,.)$ defined by $(\Lambda_0, \lambda_0) = 0$, $(\Lambda_0, \alpha_1) = 0$, $(\Lambda_0, \delta) = 1$, $(\alpha_0, \alpha_1) = 2$, $(\alpha_1, \delta) = 0$, $(\delta, \delta) = 0$, where $\Lambda_1 = \Lambda_0 + \frac{\alpha_1}{2}$, $\delta = \alpha_0 + \alpha_1$.

We define $P^* = \mathbb{Z}H_0 \oplus \mathbb{Z}H_1 \oplus \mathbb{Z}D$ as the dual space of P. The dual pairing is defined by

$$\langle H_i, \lambda\rangle = (\alpha_i, \lambda) \qquad (i = 0, 1) \quad \text{for } \lambda \in P.$$

If k is a nonnegative integer we denote by $P_k = \{(k - i)\Lambda_0 + i\Lambda_1,\ i = 0, 1, \dots, k\}$ the set of dominant integral weights of level k, and we set $\lambda_i = (k - i)\Lambda_0 + i\Lambda_1$.

3.2. Deformation of the affine algebra. The quantum affine algebra $U_q(\mathfrak{sl}_2{}^{(1)})$ is an associative algebra over $\mathbb{Q}(q)$ with $\mathbf{1}$, where q is a transcendental complex number, generated by e_i, f_i, $i = 0, 1$, and q^h ($h \in P^*$). The defining relations are as follows:

$$q^h q^{h'} = q^{h+h'}, \quad q^0 = 1\ ,$$

$$q^h e_i q^{-h} = q^{\langle h,\alpha_i\rangle} e_i\ , \qquad q^h f_i q^{-h} = q^{-\langle h,\alpha_i\rangle} f_i\ ,$$

$$[e_i, f_j] = \delta_{i,j} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}},$$

$$e_i^3 e_j - [3]e_i^2 e_j e_i + [3]e_i e_j e_i^2 - e_j e_i^3 = 0 \qquad (i \neq j),$$

$$f_i^3 f_j - [3]f_i^2 f_j f_i + [3]f_i f_j f_i^2 - f_j f_i^3 = 0 \qquad (i \neq j).$$

Here $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$.

We consider the algebra $U_q(\widehat{\mathfrak{sl}_2})$ to be the subalgebra of $U_q(\mathfrak{sl}_2{}^{(1)})$ generated by e_i, f_i, $t_i = q^{h_i}$ $(i = 0, 1)$. The algebra $U_q(\mathfrak{sl}_2{}^{(1)})$ has a Hopf algebra structure. The comultiplication is given by

$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i\ ,$$

$$\Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i \quad (i = 0, 1)\ ,$$

$$\Delta(q^h) = q^h \otimes q^h \qquad (h \in P^*).$$

The antipode A is given by

$$A(q^h) = q^{-h}\ , \quad A(e_i) = -t_i^{-1} e_i\ , \quad A(f_i) = -f_i t_i \quad (i = 0, 1)\ .$$

The counit ε is given by

$$\varepsilon(e_i) = \varepsilon(f_i) = 0 \quad (i = 0, 1)\ , \quad \varepsilon(q^h) = 1.$$

This definition of the quantum affine algebra by Chevalley generators is due to Drinfeld and Jimbo [5, 11]. Later, Drinfeld gave another realization for $U_q(\widehat{\mathfrak{sl}_2})$ [6], which is the loop algebra version of the above algebra. In this realization $U_q(\widehat{\mathfrak{sl}_2})$ is an associative algebra generated by $\{E_n,\ n \in \mathbb{Z}\}$, $\{F_n,\ n \in \mathbb{Z}\}$, $\{H_n,\ n \in \mathbb{Z} - \{0\}\}$ and invertible K and $q^{\frac{k}{2}}$ satisfying the following relations:

$$[H_n, K] = 0\ , \quad [H_n, H_m] = \delta_{n+m,0}\frac{1}{n}[2n][k],$$

$$KE_nK^{-1} = q^2E_n\ , \quad KF_nK^{-1} = q^{-2}F_n,$$

$$[H_n, E_m] = \frac{[2n]}{n}q^{\frac{-|n|k}{2}}E_{n+m}\ , \quad [H_n, F_m] = -\frac{[2n]}{n}q^{\frac{|n|k}{2}}F_{n+m}\ ,$$

$$E_{n+1}E_m - q^2E_mE_{n+1} = q^2E_nE_{m+1} - E_{m+1}E_n\ ,$$

$$F_{n+1}F_m - q^{-2}F_mF_{n+1} = q^{-2}F_nF_{m+1} - F_{m+1}F_n\ ,$$

$$[E_n, F_m] = \frac{1}{q - q^{-1}}\left(q^{\frac{k(n-m)}{2}}\psi_{n+m} - q^{\frac{k(m-n)}{2}}\phi_{n+m}\right),$$

where ψ_n and ϕ_n are related to H_l by

$$\sum_{n\in\mathbb{Z}}\psi_n z^{-n} = K\exp\left((q - q^{-1})\sum_{l=1}^{\infty}H_l z^{-l}\right), \tag{3.1}$$

$$\sum_{n\in\mathbb{Z}}\phi_n z^{-n} = K^{-1}\exp\left(-(q - q^{-1})\sum_{l=1}^{\infty}H_{-l} z^{l}\right)\ . \tag{3.2}$$

Here, $\phi_n = \psi_{-n} = 0$ for $n \geq 0$. We define H_0 by the formula

$$K = \exp\left((q - q^{-1})\frac{H_0}{2}\right).$$

The standard Chevalley generators $\{e_i, f_i, t_i\}$ are given by the identification

$$t_0 = q^kK^{-1}\ ,\ t_1 = K\ ,\ e_1 = E_0\ ,\ f_1 = F_0\ ,\ e_0t_1 = F_1\ ,\ t_1^{-1}f_0 = E_{-1}. \tag{3.3}$$

This identification leads to an algebra isomorphism between the above realizations.

Equivalently, the Drinfeld realization can be obtained using the generators E_n, F_n ($n \in \mathbb{Z}$), ϕ_{-n}, ψ_n ($n \in \mathbb{N}$) and $q^{\pm\frac{k}{2}}$. And if we consider the currents

$$E(z) = \sum_{n\in\mathbb{Z}}E_n z^{-n},\ \ F(z) = \sum_{n\in\mathbb{Z}}F_n z^{-n},\ \ \phi(z) = \sum_{n=0}^{\infty}\phi_{-n}z^{n},\ \ \psi(z) = \sum_{n=0}^{\infty}\psi_n z^{-n}\ ,$$

then the defining relations can be written as

$$\phi_0\psi_0 = \psi_0\phi_0 = 1, \tag{3.4}$$

$$\phi(z)\phi(w) = \phi(w)\phi(z)\ , \quad \psi(z)\psi(w) = \psi(w)\psi(z), \tag{3.5}$$

$$\phi(z)\psi(w) = \frac{(zq^{k-2} - w)(zq^{-k+2} - w)}{(zq^{k+2} - w)(zq^{-k-2} - w)}\psi(w)\phi(z), \tag{3.6}$$

$$\phi(z)E(w) = \frac{zq^{-\frac{k}{2}+2} - w}{zq^{-\frac{k}{2}} - wq^2} E(w)\phi(z), \tag{3.7}$$

$$\phi(z)F(w) = \frac{zq^{\frac{k}{2}-2} - w}{zq^{\frac{k}{2}} - wq^{-2}} F(w)\phi(z), \tag{3.8}$$

$$\psi(z)E(w) = \frac{wq^{-\frac{k}{2}} - zq^2}{wq^{-\frac{k}{2}+2} - z} E(w)\psi(z), \tag{3.9}$$

$$\psi(z)F(w) = \frac{wq^{\frac{k}{2}} - q^{-2}z}{wq^{\frac{k}{2}-2} - z} F(w)\psi(z), \tag{3.10}$$

$$[E(z), F(w)] = \frac{1}{q - q^{-1}} \left(\delta(\frac{w}{z} q^k)\psi(wq^{\frac{k}{2}}) - \delta(\frac{w}{z} q^{-k})\phi(wq^{\frac{-k}{2}}) \right), \tag{3.11}$$

$$(z - q^2 w)E(z)E(w) = (q^2 z - w)E(w)E(z), \tag{3.12}$$

$$(z - q^{-2} w)F(z)F(w) = (q^{-2} z - w)F(w)F(z), \tag{3.13}$$

where

$$\delta(z) = \sum_{n \in \mathbb{Z}} z^n.$$

These relations are understood to be between formal power series.

Since the Drinfeld currents are suitable for bosonization, we will use this realization for the rest of this chapter. Drinfeld also gave the Hopf algebra structure for this current realization.

Proposition 3.1. *[4] The algebra $U_q(\widehat{\mathfrak{sl}_2})$ has a Hopf algebra structure given by*

- *Comultiplication* Δ

$$\Delta(q^k) = q^k \otimes q^k,$$

$$\Delta(E(z)) = E(z) \otimes 1 + \phi(zq^{\frac{k}{2}\otimes 1}) \otimes E(zq^{k\otimes 1}),$$

$$\Delta(F(z)) = 1 \otimes F(z) + F(zq^{1\otimes k}) \otimes \psi(q^{1\otimes \frac{k}{2}}),$$

$$\Delta(\phi(z)) = \phi(zq^{-1\otimes \frac{k}{2}}) \otimes \phi(zq^{\frac{k}{2}\otimes 1}),$$

$$\Delta(\psi(z)) = \psi(zq^{1\otimes \frac{k}{2}}) \otimes \psi(zq^{-\frac{k}{2}\otimes 1}).$$

- *Counit* ε

$$\varepsilon(q^k) = \varepsilon(\psi(z)) = \varepsilon(\phi(z)) = 1,$$

$$\varepsilon(E(z)) = \varepsilon(F(z)) = 0.$$

• *Antipode* A

$$
\begin{aligned}
A(q^k) &= q^{-k},\\
A(E(z)) &= -\phi(zq^{-\frac{k}{2}})^{-1}E(zq^{-k}),\\
A(F(z)) &= -F(zq^{-k})\psi(zq^{-\frac{k}{2}})^{-1},\\
A(\phi(z)) &= \phi(z)^{-1},\\
A(\psi(z)) &= \psi(z)^{-1}.
\end{aligned}
$$

If $\phi(z) = \sum_{n=0}^{\infty} \phi_{-n} z^n$ then the formula for $\Delta(\phi(z))$ means that

$$\Delta(\phi(z)) = \sum_{m,n} (q^{\frac{mk}{2}} \phi_{-n} z^n) \otimes (q^{-\frac{nk}{2}} \phi_{-m} z^m).$$

4. Bosonization and q-Analog of Wakimoto Modules

4.1. Free boson realization of $U_q(\widehat{\mathfrak{sl}_2})$*.* Here we follow [1, 9], with some modifications, we introduce the Heisenberg algebra generated by three free boson fields a, b and c. We construct a homomorphism from the quantum affine algebra to the Heisenberg algebra which will enable us to express the Drinfeld generators in terms of the Heisenberg generators.

The generators of the quantum Heisenberg algebra $\mathcal{H}_q(\mathfrak{sl}_2)$ are a_n, b_n, c_n, $n \in \mathbb{Z}$, p_a, p_b, and p_c. The relations are

$$[a_n, a_m] = \frac{1}{n}[(k+2)n][2n]\delta_{n+m,0}\ , \quad [a_0, p_a] = \frac{4h}{q-q^{-1}}(k+2), \tag{4.1}$$

$$[b_n, b_m] = -\frac{1}{n}[n]^2\delta_{m+n,0}\ , \quad [b_0, p_b] = \frac{-2h}{q-q^{-1}}, \tag{4.2}$$

$$[c_n, c_m] = \frac{1}{n}[n]^2\delta_{m+n,0}\ , \quad [c_0, p_c] = \frac{2h}{q-q^{-1}}, \tag{4.3}$$

where $q = e^h$. The remaining commutators vanish.

We define the completion $\widetilde{\mathcal{H}}_q(\mathfrak{sl}_2)$ of $\mathcal{H}_q(\mathfrak{sl}_2)$ as follows:

$$\widetilde{\mathcal{H}}_q(\mathfrak{sl}_2) = \varprojlim \mathcal{H}_q/I_n \quad (n > 0),$$

where I_n is the left ideal of $\mathcal{H}_q(\mathfrak{sl}_2)$ generated by all the polynomials in a_m, b_m, c_m, $m > 0$, of degrees greater than or equal to n (we set $\deg a_m = \deg b_m = \deg c_m = m$).

We form the generating functions:

$$a_\pm(z) = \pm(q-q^{-1})\left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_{\pm n} z^{\mp n}\right),$$

$$b_\pm(z) = \pm(q-q^{-1})\left(\frac{b_0}{2} + \sum_{n=1}^{\infty} b_{\pm n} z^{\mp n}\right),$$

$$b(z) = -\sum_{n\neq 0} \frac{b_n}{[n]} z^{-n} + \frac{q-q^{-1}}{2h} b_0 \log z + p_b,$$

$$c_\pm(z) = \pm(q-q^{-1}) \left(\frac{c_0}{2} + \sum_{n=1}^{\infty} c_{\pm n} z^{\mp n} \right),$$

$$c(z) = -\sum_{n\neq 0} \frac{c_n}{[n]} z^{-n} + \frac{q-q^{-1}}{2h} c_0 \log z + p_c.$$

For a real number α we define:

$$a(z;\alpha) = -\sum_{n\neq 0} \frac{a_n}{[(k+2)n]} q^{-\alpha|n|} z^{-n} + \frac{1}{k+2} \left(\frac{q-q^{-1}}{2h} a_0 \log z + p_a \right).$$

Let : : denote the normal ordering of a product of operators defined by moving the creation operators to the left and moving the annihilation operators to the right. In our case the annihilation operators are $\{a_n,\ b_n,\ c_n\ n \geq 0\}$ and the creation operators are $\{a_n,\ b_n,\ c_n,\ p_a,\ p_b,\ p_c,\ n < 0\}$. For example

$$:\exp(b(z)) := \exp\left(-\sum_{n<0} \frac{b_n}{[n]} z^{-n} \right) \exp\left(-\sum_{n>0} \frac{b_n}{[n]} z^{-n} \right) e^{p_b} z^{\frac{q-q^{-1}}{2h} b_0}.$$

Proposition 4.1 ([1, 9]). *There is a homomorphism ω from $U_q(\widehat{\mathfrak{sl}_2})$ to $\widetilde{\mathcal{H}}_q(\mathfrak{sl}_2)$ which is defined on generators as follows:*

$$\begin{aligned}
\omega[E'(z)] &= -:e^{b_+(z)-(b+c)(zq)}: + :e^{b_-(z)-(b+c)(zq^{-1})}:,\\
\omega[F'(z)] &= e^{a_+(zq^{\frac{k}{2}+1})} :e^{b_+(zq^{k+2})+(b+c)(zq^{k+1})}:\\
&\quad - e^{a_-(zq^{-\frac{k}{2}-1})} :e^{b_-(zq^{-k-2})+(b+c)(zq^{-k-1})}:,\\
\omega[\psi(z)] &= e^{a_+(zq)} e^{b_+(zq^{\frac{k}{2}})+b_+(zq^{\frac{k}{2}+2})},\\
\omega[\phi(z)] &= e^{a_-(zq^{-1})} e^{b_-(zq^{-\frac{k}{2}})+b_-(zq^{-\frac{k}{2}-2})},
\end{aligned}$$

where $E'(z) = (q-q^{-1})E(z)$ and $F'(z) = (q-q^{-1})F(z)$.

In order to prove this proposition, we need the following two lemmas, which will be used later on too.

Lemma 4.2 ([8]). *Let X and Y be two operators such that $[X,Y]$ commutes with X and with Y, then*

$$[X, e^Y] = [X,Y]e^Y \quad \textit{and} \quad e^X e^Y = e^Y e^X e^{[X,Y]}.$$

The following lemma can be proved by direct computation of the operator product expansions. It will be used later on without any mention of a specific relation.

Lemma 4.3. *We have the following commutation relations:*

$$e^{a_+(z)}e^{a_-(w)} = \frac{(w - zq^{k+4})(w - zq^{-k-4})}{(w - zq^k)(w - zq^{-k})} e^{a_-(w)}e^{a_+(z)},$$

$$e^{b_+(z)}e^{b_-(w)} = \frac{(z - w)^2}{(z - wq^2)(z - wq^{-2})} e^{b_-(w)}e^{b_+(z)},$$

$$e^{c_+(z)}e^{c_-(w)} = \frac{(z - wq^2)(z - wq^{-2})}{(z - w)^2} e^{c_-(w)}e^{c_+(z)},$$

$$e^{b_+(z)} : e^{b(w)} := \frac{z - wq}{zq - w} : e^{b(w)} : e^{b_+(z)}, \quad e^{c_+(z)} : e^{c(w)} := \frac{zq - w}{z - wq} : e^{c(w)} : e^{c_+(z)},$$

$$e^{b_-(z)} : e^{b(w)} := \frac{wq - z}{w - zq} : e^{b(w)} : e^{b_-(z)}, \quad e^{c_-(z)} : e^{c(w)} := \frac{w - zq}{wq - z} : e^{c(w)} : e^{c_-(z)},$$

$$: e^{(b+c)(z)} :: e^{(b+c)(w)} :=: e^{(b+c)(w)} :: e^{(b+c)(z)} : .$$

For simplicity we will use the same notation for the elements of $U_q(\widehat{\mathfrak{sl}_2})$ and their images in $\widetilde{\mathcal{H}}(\mathfrak{sl}_2)$.

Using the above lemma and the fact that

$$\phi_0 = e^{-\frac{q-q^{-1}}{2}(a_0+b_0+c_0)} \quad \text{and} \quad \psi_0 = e^{\frac{q-q^{-1}}{2}(a_0+b_0+c_0)},$$

one can easily prove that $E(z)$, $F(z)$, $\phi(z)$ and $\psi(z)$ satisfy the defining relations of the algebra $U_q(\widehat{\mathfrak{sl}_2})$, except for the relation involving $[E(z), F(w)]$ which needs an explanation. We look at z and w as complex variables and we set $E'(z) = -E'_+(z) + E'_-(z)$ and $F'(z) = F'_+(z) - F'_-(z)$ in the obvious way. Then we have

$$E'_+(z)F'_+(w) = \frac{zq^{-1} - wq^{k+1}}{z - wq^k} \; : E'_+(z)F'_+(w) : \quad (|z| > |wq^k|),$$

$$E'_-(z)F'_-(w) = \frac{zq - wk^{-k-1}}{z - wq^{-k}} \; : E'_-(z)F'_-(w) : \quad (|z| > |wq^{-k}|),$$

$$F'_+(w)E'_+(z) = \frac{wq^{k+1} - zq^{-1}}{wq^k - z} \; : F'_+(w)E'_+(z) : \quad (|z| < |wq^k|),$$

$$F'_-(w)E'_-(z) = \frac{wq^{-k-1} - zq}{wq^{-k} - z} \; : F'_-(w)E'_-(z) : \quad (|z| < |wq^{-k}|).$$

The other products have no poles, more precisely:

$$E'_+(z)F'_-(w) = F'_-(w)E'_+(z) =: E'_+(z)F'_-(w) : ,$$

$$E'_-(z)F'_+(w) = F'_+(w)E'_-(z) =: E'_-(z)F'_+(w) : .$$

For $|z| \gg |w|$, it follows that:

$$E'(z)F'(w) = -\frac{zq^{-1} - wq^{k+1}}{z - wq^k} : E'_+(z)F'_+(w) : - \frac{zq - wq^{-k-1}}{z - wq^{-k}} : E'_-(z)F'_-(w) :$$
$$+ : E'_+(z)F'_-(w) : + : E'_-(z)F'_+(w) :,$$

and for $|z| \ll |w|$ we have:

$$F'(w)F'(z) = -\frac{zq^{-1} - wq^{k+1}}{z - wq^k} : F'_+(w)E'_+(z) : - \frac{zq - wq^{-k-1}}{z - wq^{-k}} : F'_-(w)E'_-(z) : \\ + : F'_+(w)E'_-(z) : + : F'_-(w)E'_+(z) : .$$

Since the normally ordered product does not depend on the order of the factors, we conclude that $E'(z)F'(w)$ and $F'(w)E'(z)$ have the same analytic continuation. The coefficient of z^{-n-1} in the Laurent expansion of $E'(z)F'(w) - F'(w)E'(z)$ is

$$\frac{1}{2\pi i}\int_{C_R} E'(z)F'(w)z^n dz - \frac{1}{2\pi i}\int_{C_r} F'(w)E'(z)z^n dz,$$

where C_R and C_r are circles on the z-plane of radii $R \gg |w|$ and $r \ll |w|$ respectively, which is equal to the sum of the residues of the common analytic continuation. The latter is equal to

$$(wq^k)^{n+1}(q - q^{-1}) : E'_+(wq^k)F'_+(w) : -(wq^{-k})^{n+1}(q - q^{-1}) : E'_-(wq^{-k})F'_-(w) : .$$

Moreover,

$$: E'_+(wq^k)F'_+(w) := \psi(wq^{\frac{k}{2}})$$

and

$$: E'_-(wq^{-k})F'_-(w) := \phi(wq^{-\frac{k}{2}}).$$

Hence

$$[E'(z), F'(w)] = (q - q^{-1})\left(\delta(\frac{w}{z}q^k)\psi(wq^{\frac{k}{2}}) - \delta(\frac{w}{z}q^{-k})\phi(wq^{-\frac{k}{2}})\right),$$

which provides the right formula for $[E(z), F(w)]$.

4.2. Representations. The infinite dimensional representations of $U_q(\widehat{\mathfrak{sl}_2})$ are created from the Fock module of the Heisenberg algebra via the homomorphism ω. We start by considering the vacuum state Ω of the boson Fock space which satisfies

$$a_n.\Omega = b_n.\Omega = c_n.\Omega = 0 \quad (n \geq 0).$$

We define the vector $\Omega_{r,s}$ by:

$$\Omega_{r,s} = \exp\left(r\frac{p_a}{2(k+2)} + s(p_b + p_c)\right)\Omega \quad (r, s \in \mathbb{Z}).$$

Let $\mathcal{F}$ be the free $\mathbb{Q}(q)$-algebra generated by $\{a_n, b_n, c_n,\ n < 0\}$ and let $\mathcal{F}_{r,s}$ be the Fock module defined by

$$\mathcal{F}_{r,s} := \mathcal{F}.\Omega_{r,s}.$$

It is clear that $\phi(z)$ and $\psi(z)$ map $\mathcal{F}_{r,s}$ to $\mathcal{F}_{r,s} \otimes \mathbb{C}((z))$ and from the simple observation that

$$e^{\pm(p_b+p_c)}\Omega_{r,s} = e^{r\frac{p_a}{2(k+2)}+(s\pm1)(p_a+p_b)}\Omega,$$

we deduce that $E(z)$ maps $\mathcal{F}_{r,s}$ to $\mathcal{F}_{r,s-1} \otimes \mathbb{C}((z^{-1}))$ and $F(z)$ maps $\mathcal{F}_{r,s}$ to $\mathcal{F}_{r,s+1} \otimes \mathbb{C}((z^{-1}))$.

We denote by $V(\lambda)$ the Verma module over $U_q(\widehat{\mathfrak{sl}_2})$ of highest weight λ and generated by the highest weight vector v_λ. Thus

$$e_0 v_\lambda = e_1 v_\lambda = 0\ , \quad t_0 v_\lambda = q^\alpha v_\lambda\ , \quad t_1 v_\lambda = q^\beta v_\lambda, \tag{4.4}$$

where $\alpha\Lambda_0 + \beta\Lambda_1$ is the classical part of λ; Λ_0 and Λ_1 are the fundamental weights of $\mathfrak{sl}_2$. We refer to (4.4) as the highest weight condition.

Proposition 4.4. *The vector $\Omega_{r,0}$ satisfies:*

$$\begin{aligned} E_n\Omega_{r,0} &= F_n\Omega_{r,0} = H_n\Omega_{r,0} = 0 \quad \textit{if } n > 0,\\ E_0\Omega_{r,0} &= 0,\\ K\Omega_{r,0} &= q^r\Omega_{r,0}. \end{aligned}$$

Proof. We have

$$\begin{aligned} E(z)\Omega_{r,0} &= \exp\left(\sum_{n<0}\frac{b_n + c_n}{[n]}(zq)^{-n}\right)\Omega_{r,-1}\\ &\quad + \exp\left((q-q^{-1})\sum_{n<0} b_n z^{-n} + \sum_{n<0}\frac{b_n + c_n}{[n]}(zq^{-1})^{-n}\right)\ \Omega_{r,-1}. \end{aligned}$$

Hence, for $n \geq 0$, $E_n\Omega_{r,0} = 0$. We have a similar formula for $F(z)\Omega_{r,0}$ except that, due to the presence of $a_\pm(z)$, the first term comes with the factor q^r and the second term comes with the factor q^{-r}, which implies that $F_n\Omega_{r,0} = 0$ for $n < 0$ only. Since $\psi(z)\Omega_{r,0} = q^r\Omega_{r,0}$ by (3.1), we deduce that $H_n\Omega_{r,0} = 0$ for $n \geq 0$ and $K\Omega_{r,0} = q^r\Omega_{r,0}$. □

Corollary 4.5. *The vector $\Omega_{r,0}$ satisfies the highest weight condition (4.4) and can be identified with the highest vector v_{λ_r}, where*

$$\lambda_r = (k-r)\Lambda_0 + r\Lambda_1.$$

Proof. This follows from the proposition and the identification (3.3). □

The following proposition follows from the identification of v_{λ_r} with $\Omega_{r,0}$ and the action of the generators of $U_q(\widehat{\mathfrak{sl}_2})$ on $\Omega_{r,0}$:

Proposition 4.6. *There is an embedding of the highest weight module in a direct sum of Fock modules:*

$$V(\lambda_r) \hookrightarrow \bigoplus_{s\in\mathbb{Z}} \mathcal{F}_{r,s}. \tag{4.5}$$

Remark 4.1. The module $V(\lambda_r)$ is reducible as is known in Conformal Field Theory. To obtain irreducible modules, one has to use the q-analog of the Felder resolution, see [14].

We set

$$\mathcal{W}_r := \bigoplus_{s\in\mathbb{Z}} \mathcal{F}_{r,s}. \tag{4.6}$$

This Fock space carries a $U_q(\widehat{\mathfrak{sl}_2})$-module structure defined by the bosonization formulae of Proposition 4.1. These are the q-analogs of the *Wakimoto* modules [19].

5. Screening Operators

5.1. The operator $S(z)$. The so-called screening operator $S(z)$ is used to investigate the irreducible representations of $U_q(\widehat{\mathfrak{sl}_2})$, and to compute correlation functions in Conformal Field Theory. It is an element of $\widetilde{\mathcal{H}}(\mathfrak{sl}_2)$ which acts then on Fock space representations.

Definition 5.1 ([1]). The screening operator is given by

$$S(z) = \frac{1}{(q-q^{-1})z} : e^{-a(z,\frac{k+2}{2})} : \left(: e^{-b_-(z)-(b+c)(qz)} : - : e^{-b_+(z)-(b+c)(q^{-1}z)} : \right).$$

For simplicity we write $z(q-q^{-1})S(z) = S_1(z) - S_2(z)$.

Proposition 5.1. *The operator $S(z)$ sends $\mathcal{F}_{r,s}$ to $\mathcal{F}_{r-1,s-1}$, and therefore sends the module $\mathcal{W}_r$ to the module $\mathcal{W}_{r-2} \otimes z^{-\frac{r}{k+2}}\mathbb{C}((z^{-1}))$.*

Proof. $\exp\left(\frac{-p_a}{k+2}\right)$ sends $\Omega_{r,s}$ to $\Omega_{r-1,s}$, $\exp\left(-\frac{p_b+p_c}{2}\right)$ sends $\Omega_{r,s}$ to $\Omega_{r,s-1}$, and

$$\begin{aligned} e^{\frac{-p_a}{k+2}} e^{-\frac{q-q^{-1}}{2h(k+2)}a_0 \log z}\, \Omega_{r,s} &= e^{\frac{-p_a}{k+2}} e^{[-\frac{q-q^{-1}}{2h(k+2)}a_0 \log z, \frac{rp_a}{2(k+2)}]}\, \Omega_{r,s} \\ &= z^{-\frac{r}{k+2}} \Omega_{r-2,s}. \qquad \square \end{aligned}$$

Now we compute the pairing between $E(z)$, $F(z)$, $\phi(z)$, $\psi(z)$ and $S(w)$.

Proposition 5.2. *The operator $S(w)$ satisfies:*

$$\langle E(z), S(w)\rangle = E(z)S(w) - \phi(zq^{\frac{k}{2}})S(w)\phi(zq^{\frac{k}{2}})^{-1}E(z), \tag{5.1}$$

$$\langle F(z), S(w)\rangle = F(zq^{-k})S(w)\psi(zq^{-\frac{k}{2}})^{-1} - S(w)F(zq^{-k})\psi(zq^{-\frac{k}{2}})^{-1}, \tag{5.2}$$

$$\langle \phi(z), S(w)\rangle = \phi(zq^{\frac{k}{2}})S(w)\phi(zq^{\frac{k}{2}})^{-1} - S(w), \tag{5.3}$$

$$\langle \psi(z), S(w)\rangle = \psi(zq^{-\frac{k}{2}})S(w)\psi(zq^{-\frac{k}{2}})^{-1} - S(w), \tag{5.4}$$

where these relations are understood to be between formal power series.

Proof. We have

$$\begin{aligned} \Delta(E(z)) &= E(z) \otimes 1 + \phi(zq^{\frac{k}{2}\otimes 1}) \otimes E(zq^{k\otimes 1}) \\ &= E(z) \otimes 1 + \sum_{m\geq 0,n} \phi_{-m} q^{\frac{k}{2}m - km} z^m \otimes E_n z^{-n}, \end{aligned}$$

hence

$$(\mathrm{id} \otimes A)\Delta(E(z)) = E(z) \otimes 1 + \sum_{m,n} \phi_{-m} q^{\frac{k}{2}m-kn} z^m \otimes A(E_n) z^{-n},$$

therefore

$$\begin{aligned}\langle E(z), S(w)\rangle &= E(z)S(w) + \sum_{m,n} \phi_{-m} q^{\frac{k}{2}m-kn} z^m S(w)A(E_n)z^{-n} \\ &= E(z)S(w) + \sum_{m,n} \phi_{-m} q^{\frac{k}{2}} z^m S(w)A(E_n)(q^k z)^{-n} \\ &= E(z)S(w) - \phi(zq^{\frac{k}{2}})S(w)\phi(zq^{\frac{k}{2}})^{-1}E(z).\end{aligned}$$

This proves (5.1). The proof of the other formulas is similar, i.e. by expanding the currents and applying A to the second component of the tensor product, and using the fact that $q^{\frac{k}{2}}$ is central. □

Proposition 5.3. *We have the following relations between formal power series:*

$$\phi(z)S(w) = S(w)\phi(z) =: \phi(z)S(w) :, \tag{5.5}$$
$$\psi(z)S(w) = S(w)\psi(z) =: \psi(z)S(w) :, \tag{5.6}$$
$$E(z)S(w) = S(w)E(z) \quad \textit{and the products have no poles}. \tag{5.7}$$

Proof. The relations (5.5) and (5.6) follow from direct computation. Equation (5.7) needs some explanation. Set

$$E_1(z) =: \exp(b_+(z) - (b+c)(zq)) : \quad \text{and} \quad E_2(z) =: \exp(b_-(z) - (b+c)(zq^{-1})) : .$$

Then we have

$$E_1(z)S_1(w) = S_1(w)E_1(z) = q : E_1(z)S_1(w) :,$$

and

$$E_2(z)S_2(w) = S_2(w)E_2(z) = q^{-1} : E_2(z)S_2(w) : .$$

The other products have a pole at $z = w$:

$$\begin{aligned}E_1(z)S_2(w) = S_2(w)E_1(z) &= q\frac{z - wq^{-2}}{z - w} : E_1(z)S_2(w) : \\ &= qw\frac{1 - q^{-2}}{z - w} : E_1(w)S_2(w) : + \text{regular part at } z = w\end{aligned}$$

and

$$\begin{aligned}E_2(z)S_1(w) = S_1(w)E_2(z) &= q^{-1}\frac{z - wq^{2}}{z - w} : E_2(z)S_1(w) : \\ &= q^{-1}w\frac{1 - q^{2}}{z - w} : E_2(w)S_1(w) : + \text{regular part at } z = w.\end{aligned}$$

It follows that

$$\begin{aligned}E_1(z)S_2(w) + E_2(z)S_1(w) &= S_2(w)E_1(z) + S_1(w)E_2(z) \\ &= w\frac{q - q^{-1}}{z - w}(: E_1(w)S_2(w) : - : E_2(w)S_1(w) :) + \cdots .\end{aligned}$$

It is easy to see that $: E_1(w)S_2(w) :=: E_2(w)S_1(w) :$ and (5.7) follows. □

Before investigating how the Fourier coefficients behave with $S(z)$, we introduce the notion of q-difference operator:

For a function $f(z)$ and for $\alpha \in \mathbb{C}$, we define

$$\frac{\mathcal{D}_\alpha}{d_q z} f(z) := \frac{f(zq^\alpha) - f(zq^{-\alpha})}{z(q - q^{-1})}.$$

We can also write

$$\mathcal{D}_\alpha f(z) = \frac{f(zq^\alpha) - f(zq^{-\alpha})}{z(q - q^{-1})} d_q z. \tag{5.8}$$

In particular

$$\mathcal{D}_\alpha z^n = [\alpha n] z^{n-1} d_q z.$$

If f and g are two functions, the q-difference of the products is given by:

$$\begin{aligned} \mathcal{D}_\alpha(f(z)g(z)) &= f(zq^\alpha)\mathcal{D}_\alpha(g(z)) + \mathcal{D}_\alpha(g(z))f(zq^{-\alpha}) \\ &= f(zq^{-\alpha})\mathcal{D}_\alpha(g(z)) + \mathcal{D}_\alpha(f(z))g(zq^\alpha). \end{aligned}$$

For $p, s \in \mathbb{C} - \{0\}$ with $|p| < 1$ and for a function $f(u)$, we define

$$\int_0^{s\infty} f(u) d_p u = s(1-p) \sum_{n=-\infty}^{\infty} f(sp^n) p^n$$

whenever it is convergent. This is a q-difference analog of the ordinary integration, and it is called the Jackson integral along a q-interval $[0, s\infty]$. This integral has the following property:

$$\int_0^{s\infty} \frac{\mathcal{D}_\alpha}{d_p u} f(u) d_p u = 0.$$

All these notions extend to multi-variable functions.

Theorem 5.4. *For every integer n, we have:*

$$\langle E_n, S(w)\rangle = 0, \tag{5.9}$$

$$\langle \phi_{-n}, S(w)\rangle = \langle \psi_n, S(w)\rangle = 0 \quad (n \geq 0), \tag{5.10}$$

$$\langle F_n, S(w)\rangle = \frac{\mathcal{D}_{k+2}}{d_q w} \left((wq)^n : e^{-a(w, -\frac{k+2}{2})} \psi(wq^{\frac{k}{2}})^{-1} : \right). \tag{5.11}$$

Proof. The relations (5.9) and (5.10) follow immediately from Proposition 5.2 and Proposition 5.3.

Set $(q - q^{-1})F(zq^{-k}) = F_1(zq^{-k}) - F_2(z^{-k})$, where

$$F_1(zq^{-k}) = e^{a_+(zq^{-\frac{k}{2}+1})} : \exp(b_+(zq^2) + (b+c)(zq)) :$$

and

$$F_2(zq^{-k}) = e^{a_-(zq^{-\frac{3k}{2}-1})} : \exp(b_-(zq^{-2k-2}) + (b+c)(zq^{-2k-1})) : .$$

We look at z and w as complex variables. To normally order $F_1(zq^{-k})S_1(w)$, we need the following brackets:

$$\left[a_+(zq^{-\frac{k}{2}+1}), -a\left(w, \frac{k+2}{2}\right)\right]$$
$$= \left[\frac{q-q^{-1}}{2}a_0, \frac{-1}{k+2}p_a\right] - \sum_{n=1}^{\infty}(q-q^{-1})\frac{q^{-2n}}{[(k+2)n]}[a_n, a_{-n}]\left(\frac{w}{z}\right)^n$$
$$= -2h + \sum_{n=1}^{\infty}\frac{1}{n}(q^{-4n}-1)\left(\frac{w}{z}\right)^n.$$

And $[b_+(zq^2), -b_-(w) - b(qw)] + [b(zq) - p_b, -b_-(w)]$ gives

$$h + \sum_{n=1}^{\infty}\frac{1}{n}(1-q^{-4n})\left(\frac{w}{z}\right)^n.$$

Therefore

$$F_1(zq^{-k})S_1(w) = q^{-1} : F_1(zq^{-k})S_1(w) : . \tag{5.12}$$

Meanwhile

$$S_1(w)F_1(zq^{-k}) = e^{[\frac{q-q^{-1}}{2}b_0, p_b]} : S_1(w)F_1(zq^{-k}) := q^{-1} : F_1(zq^{-k})S_1(w) : . \tag{5.13}$$

Similarly, we have:

$$F_2(zq^{-k})S_2(w) = S_2(w)F_2(zq^{-k}) = q : F_2(zq^{-k})S_2(w) : . \tag{5.14}$$

The other products have poles:

$$F_1(zq^{-k})S_2(w) = q^{-1}\frac{z-w}{z-wq^{-2}} : F_1(zq^{-k})S_2(w) : \quad (|w| \ll |z|), \tag{5.15}$$

$$S_2(w)F_1(zq^{-k}) = q\frac{w-z}{z-wq^2} : S_2(w)F_1(zq^{-k}) : \quad (|z| \ll |w|). \tag{5.16}$$

Notice that the expressions at the right are the same. And

$$F_2(zq^{-k})S_1(w) = q\frac{z-wq^{2k}}{z-wq^{2k+2}} : F_2(zq^{-k})S_1(w) : \quad (|w| \ll |z|), \tag{5.17}$$

$$S_1(w)F_2(zq^{-k}) = q^{-1}\frac{w-zq^{-2}}{w-zq^{-2k-2}} : S_1(w)F_2(zq^{-k}) : \quad (|z| \ll |w|). \tag{5.18}$$

Since the normally ordered products do not depend on the order of the factors, we deduce from (5.12)–(5.18) that $F(zq^{-k})S(w)\psi(zq^{-\frac{k}{2}})^{-1}$ and $S(w)F(zq^{-k})\psi(zq^{-\frac{k}{2}})^{-1}$ have the same analytic continuation ($\psi(zq^{-\frac{k}{2}})^{-1}$ contains only positive modes, so it stays to the right when the products are normally ordered).

Therefore, in view of Proposition 5.2, we obtain:

$$\langle F_n, S(w)\rangle = \frac{1}{2\pi i}\int_{C_R} F(zq^{-k})S(w)\psi(zq^{-\frac{k}{2}})^{-1}z^{n-1}dz$$
$$-\frac{1}{2\pi i}\int_{C_r} S(w)F(zq^{-k})\psi(zq^{-\frac{k}{2}})^{-1}z^{n-1}dz,$$

where C_R and C_r are circles of radii $R \gg |w|$ and $r \ll |w|$ respectively. It follows that $\langle F_n, S(w)\rangle$ is equal the sum of the residues of

$$\frac{z^{n-1}}{(q-q^{-1})^2 w}\left(\frac{q^{-1}(z-w)}{z-wq^{-2}} : F_1(zq^{-k})S_2(w) : \right.$$
$$\left. + \frac{q(z-wq^{2k})}{z-wq^{2k+2}} : F_2(zq^{-k})S_1(w) : \right)\psi(zq^{-\frac{k}{2}})^{-1},$$

therefore

$$\langle F_n, S(w)\rangle = \frac{1}{(q-q^{-1})w}\left(-(wq^{-2})^n : F_1(wq^{-k-2})S_2(w)\psi(wq^{-\frac{k}{2}-2})^{-1} : \right.$$
$$\left. + (wq^{2k+2})^n : F_2(wq^{k+2})S_1(w)\psi(wq^{\frac{3k}{2}+2})^{-1} : \right).$$

Meanwhile

$$: F_1(wq^{-k-2})S_2(w) : =: e^{a_+(wq^{-\frac{k}{2}-1}) - a(w, \frac{k+2}{2})} :$$
$$=: e^{-a(wq^{-k-2}, -\frac{k+2}{2})} : .$$

Notice the change of sign in the argument of a. Similarly

$$: F_2(wq^{k+2})S_1(w) :=: e^{-a(wq^{k+2}, -\frac{k+2}{2})} : .$$

It follows that

$$\langle F_n, S(w)\rangle = \frac{\mathcal{D}_{k+2}}{d_q w}\left((wq^k)^n : e^{-a(w, -\frac{k+2}{2})}\psi(wq^{\frac{k}{2}})^{-1} : \right),$$

which proves (5.11). □

As a consequence, the bracket of the Jackson integral of $S(w)$ for $p = q^{k+2}$ with the generators of $U_q(\widehat{\mathfrak{sl}_2})$ vanishes. Using(5.11) we have

Corollary 5.5. *The Jackson integral,* $\int_0^{s\infty} S(w)d_p w \quad (p = q^{k+2})$, *of the screening operator is an invariant for* $U_q(\widehat{\mathfrak{sl}_2})$.

5.2. Difference equations. We will state and prove two results concerning the screening operators, which deal with the following difference equation:

$$\langle x, S(w)\rangle = \frac{\mathcal{D}_{k+2}}{d_q w} S(x, w), \tag{5.19}$$

for every $x \in U_q(\widehat{\mathfrak{sl}_2})$.

Theorem 5.6. *There exists a well defined operator $S(x,w)$, for each $x \in U_q(\widehat{\mathfrak{sl}_2})$ such that:*

(1) *$S(x,w)$ vanishes on the Fourier coefficients of $E(z)$, $\phi(z)$, $\psi(z)$, and on $q^{\pm\frac{k}{2}}$ and*

$$S(F_m,w) = (wq)^m : e^{-a(w,-\frac{k+2}{2})}\psi(wq^{\frac{k}{2}})^{-1} : \quad (m \in \mathbb{Z}).$$

(2) *For every $x,y \in U_q(\widehat{\mathfrak{sl}_2})$ we have:*

$$S(xy,w) = x \cdot S(y,w) + \varepsilon(y)S(x,w). \tag{5.20}$$

Proof. We consider $S(x,w)$ to be linearly dependent on x and defined on the generators as in (1), and we extend it to the free algebra generated by E_m, F_m, ϕ_{-n}, ψ_n ($m \in \mathbb{Z}$, $n \in \mathbb{N}$) and $q^{\pm\frac{k}{2}}$. In order to have $S(x,w)$ well defined for every $x \in U_q(\widehat{\mathfrak{sl}_2})$, we need to show that the relation (5.20) is compatible with the defining relations of $U_q(\widehat{\mathfrak{sl}_2})$. By linearity of $S(x,w)$, we have:

$$S(E(z),w) = S(\phi(z),w) = S(\psi(z),w) = 0, \tag{5.21}$$

and

$$S(F(z),w) = \delta\left(\frac{wq^k}{z}\right) : e^{-a(w,-\frac{k+2}{2})}\psi(zq^{\frac{k}{2}})^{-1} : . \tag{5.22}$$

It is clear that the relation (5.20) is compatible with the relations (3.4)–(3.7), (3.9) and (3.12) since (5.20) vanishes on both sides of these relations (they do not involve $F(z)$). We will prove the compatibility with (3.8) and (3.11).

From (3.8) we have

$$\phi(z_1)F(z_2) = g_k(z)F(z_2)\phi(z_1), \quad \text{with } g_k(z) = \frac{z_1q^{\frac{k}{2}-2} - z_2}{z_1q^{\frac{k}{2}} - z_2q^{-2}}.$$

And

$$\begin{aligned} S(\phi(z_1)F(z_2),w) &= \phi(z_1)\cdot S(F(z_2),w) + \varepsilon(F_2(z_2))S(\phi(z_1),w) \\ &= \phi(z_1q^{\frac{k}{2}})\delta\left(\frac{wq^k}{z_2}\right) : e^{-a(w,-\frac{k+2}{2})}\psi(wq^{\frac{k}{2}})^{-1} : \phi(z_1q^{\frac{k}{2}})^{-1}. \end{aligned}$$

Meanwhile,

$$\psi(wq^{\frac{k}{2}})^{-1}\phi(z_1q^{\frac{k}{2}})^{-1} = \frac{(z_1^{k+2}-w)(z_1q^{-k-2}-w)}{(z_1q^{k-2}-w)(z_1q^{-k+2}-w)}\phi(z_1q^{\frac{k}{2}})^{-1}\psi(wq^{\frac{k}{2}})^{-1}. \tag{5.23}$$

And since

$$\left[-a\left(w,-\frac{k+2}{2}\right), -a(z_1q^{\frac{k}{2}-1})\right] = 2h + \sum_{n=1}^{\infty}\left(q^{(k+2)n} - q^{(k-2)n}\right)\left(\frac{z_1}{w}\right)^n,$$

we obtain

$$e^{-a(w,-\frac{k+2}{2})}\phi(z_1q^{\frac{k}{2}})^{-1} = \frac{wq^2 - z_1q^k}{w - z_1q^{k+2}}\phi(zq^{\frac{k}{2}})^{-1}e^{-a(w,-\frac{k+2}{2})}. \tag{5.24}$$

From (5.23) and (5.24) we obtain:

$$S(\phi(z_1)F(z_2), w) = \frac{wq^2 - z_1 q^{-k}}{w - z_1 q^{-k+2}} \delta\left(\frac{wq^k}{z_2}\right) : e^{-a(w, -\frac{k+2}{2})} \psi(wq^{\frac{k}{2}})^{-1} : . \quad (5.25)$$

We need to compare this expression with $S(g_k(z)F(z_2)\phi(z_1), w)$.
Since $S(g_k(z)\phi(z_1), w) = 0$, we have:

$$\begin{aligned} S(g_k(z)F(z_2)\phi(z_1), w) &= \varepsilon(g_k(z)\phi(z_1))S(F(z_2), w) \\ &= g_0(z)S(F(z_2), w). \end{aligned}$$

Now we use techniques from Vertex Operator Calculus [8], and we have:

$$g_0(z)\delta\left(\frac{wq^k}{z_2}\right) = \frac{z_2 - q^{-2}}{q^{-2}z_2 - z_1}\delta\left(\frac{wq^k}{z_2}\right) = \frac{wq^k - z_1 q^{-2}}{q^{-2}z_2 - z_1}\delta\left(\frac{wq^k}{z_2}\right).$$

This follows from the identity:

$$z^{-n}\delta\left(\frac{w}{z}\right) = w^{-n}\delta\left(\frac{w}{z}\right).$$

Therefore

$$S(g_k(z)F(z_2)\phi(z_1), w) = \frac{q^2 w - q^{-k} z_1}{w - q^{-k+2} z_1} S(F(z_2), w),$$

which is equal to (5.25). The case of (3.10) is similar.

We will use a different approach to prove the compatibility with

$$[E(z), F(w)] = \frac{1}{q - q^{-1}}\left(\delta(\frac{w}{z}q^k)\psi(wq^{\frac{k}{2}}) - \delta(\frac{w}{z}q^{-k})\phi(wq^{\frac{-k}{2}})\right).$$

Since $S(x, w)$ vanishes if x is the right side, we need to show that $S(E(z)F_m, w)$ and $S(F_m E(z))$ $(m \in \mathbb{Z})$, given by (5.20), are equal.

On one hand

$$\begin{aligned} S(F_m E(z), w) &= F_m \cdot S(E(z), w) + \varepsilon(E(z))S(F_m, w) \\ &= 0. \end{aligned}$$

On the other hand

$$S(E(z)F_m, w) = E(z) \cdot S(F_m, w) + 0$$
$$= E(z)S(F_m, w) - (wq^k)^n \phi(zq^{\frac{k}{2}}) : e^{-a(w,-\frac{k+2}{2})}\psi(wq^{\frac{k}{2}})^{-1} : \phi(zq^{\frac{k}{2}})^{-1}E(z).$$

Using (3.9), we have

$$\psi(wq^{\frac{k}{2}})^{-1}E(z) = \frac{w - zq^{-k+2}}{wq^2 - zq^{-k}}E(z)\psi(wq^{\frac{k}{2}})^{-1}.$$

Keeping in mind the relations (5.23) and(5.24), we see that the second term in the expression of $S(E(z)F_m, w)$ is $E(z)S(F_m, w)$ times

$$\frac{(w - q^{k+2}z)(w - q^{-k-2})}{(w - q^{-k+2})(w - q^{k-2}z)} \cdot \frac{wq^2 - zq^k}{w - q^{k+2}} \cdot \frac{wq^{-2} - q^{-k}z}{w - q^{-k-2}z}.$$

This product is easily seen to be equal to 1. Therefore $S(E(z)F_m, w) = 0$. The case of (3.13) is similar. □

Theorem 5.7. *The screening operator $S(z)$ satisfies:*

$$\langle x, S(z)\rangle = \frac{\mathcal{D}_{k+2}}{d_q z} S(x, z), \tag{5.26}$$

for every $x \in U_q(\widehat{\mathfrak{sl}_2})$.

Proof. This is already satisfied by the generators of $U_q(\widehat{\mathfrak{sl}_2})$ by Theorem 5.4. Since both sides are well defined for every $x \in U_q(\widehat{\mathfrak{sl}_2})$, assume that the equality holds for x and y, then, by Theorem 5.6, we have

$$\begin{aligned}\langle xy, S(z)\rangle &= x \cdot \langle y, S(z)\rangle + \varepsilon(y)\,\langle x, S(z)\rangle \\ &= \frac{\mathcal{D}_{k+2}}{d_q z}(x \cdot S(y, z) + \varepsilon(y)S(x, z)) \\ &= \frac{\mathcal{D}_{k+2}}{d_q z} S(xy, z),\end{aligned}$$

which proves the theorem. □

6. Canonical q-de Rham Cocycles

6.1. The simple case. From now on, we set $\alpha = k + 2$ and we suppose $\alpha \neq 0$ (the level is noncritical), and let r be a complex number. Recall that the operators $z^{\frac{r}{\alpha}} S(z)$ and $z^{\frac{r}{\alpha}} S(x, z)$ ($x \in U_q(\widehat{\mathfrak{sl}_2})$) send $\mathcal{W}_r$ to $\mathcal{W}_{r-2} \otimes \mathbb{C}((z^{-1}))$. We consider the following complex

$$\mathrm{Hom}\left(U_q(\widehat{\mathfrak{sl}_2})^{\otimes\bullet}, \mathrm{Hom}(\mathcal{W}_r, \mathcal{W}_{r-2})\right). \tag{6.1}$$

The differential d' of this complex was introduced in Sect. 2 and is given by

$$d'\phi(x_1,\dots,x_n) = x_1\cdot\phi(x_2,\dots,x_n) + \sum_{i=1}^{n-1}(-1)^i\phi(x_1,\dots,x_{i-1},x_ix_{i+1},x_{i+2}\dots,x_n) + (-1)^n\varepsilon(x_n)\phi(x_1,\dots,x_{n-1}).$$

The cohomology groups are $\mathrm{Ext}^{\bullet}_{U_q(\widehat{\mathfrak{sl}_2})}(\mathcal{W}_r,\mathcal{W}_{r-2})$. And the 0-cohomolgy group is exactly the space of $U_q(\widehat{\mathfrak{sl}_2})$-invariants.

Let $\Omega^0 = \mathbb{C}((z^{-1}))$ and $\Omega^1 = \mathbb{C}((z^{-1}))d_qz$ (the space of formal algebraic q-differentials 1-forms). We consider the following complex:

$$\Omega^{\bullet}\ :\ 0\longrightarrow\Omega^0\longrightarrow\Omega^1\longrightarrow 0. \tag{6.2}$$

The differential of this complex is given by

$$d''(f(z)) = \mathcal{D}'_\alpha(f(z)) - [r]\frac{q^{r-\alpha}f(z)d_qz}{z},$$

where $\mathcal{D}'_\alpha(f(z)) = \mathcal{D}_\alpha f(zq^\alpha)$ (the last derivative should be read as $f'(zq^\alpha)$ and not as $(f(zq^\alpha))'$).

We consider the double complex

$$C^{\bullet\bullet} = \mathrm{Hom}\left(U_q(\widehat{\mathfrak{sl}_2})^{\otimes\bullet}, \mathrm{Hom}(\mathcal{W}_r,\mathcal{W}_{r-2}\otimes\Omega^\bullet)\right) = \mathrm{Hom}\left(U_q(\widehat{\mathfrak{sl}_2})^{\otimes\bullet}\otimes\mathcal{W}_r,\mathcal{W}_{r-2}\otimes\Omega^\bullet\right).$$

Let $C^\bullet$ be the simple complex associated with this double complex. Set

$$S^{01} = q^{2r}z^{\frac{r}{\alpha}}S(zq^\alpha)d_qz,$$

$$S^{10}(x) = z^{\frac{r}{\alpha}}S(x,z)\quad (x\in U_q(\widehat{\mathfrak{sl}_2})).$$

Then (S^{01},S^{10}) is a one-cochain in C^1, and we have:

Theorem 6.1. *The cochain* (S^{01},S^{10}) *is a one-cocycle in* C^1.

Proof. We need to prove that

$$d''S^{01} = 0, \tag{6.3}$$
$$d'S^{10}(z) = 0, \tag{6.4}$$
$$d'S^{01} = d''S^{10}. \tag{6.5}$$

Equation (6.3) derives from the fact that the complex $\Omega^\bullet$ has length 1, and (6.4) is a consequence of Theorem 5.6. The relation (6.5) also follows from Theorem 5.6 in the following way:

$$\mathcal{D}_\alpha(z^{\frac{r}{\alpha}}S(x,z)) = \mathcal{D}_\alpha(z^{\frac{r}{\alpha}})S(x,zq^{-\alpha}) + (zq^\alpha)^{\frac{r}{\alpha}}\mathcal{D}_\alpha S(x,z) = [r]z^{\frac{r}{\alpha}-1}S(x,zq^{-\alpha}) + q^rz^{\frac{r}{\alpha}}\mathcal{D}_\alpha S(x,z),$$

therefore

$$\begin{aligned}\mathcal{D}'_\alpha(z^{\frac{r}{\alpha}}S(x,z)) &= [r]q^{r-\alpha}z^{\frac{r}{\alpha}-1}S(x,z) + q^{2r}z^{\frac{r}{\alpha}}\mathcal{D}_\alpha S(x.zq^\alpha)\\ &= [r]q^{r-\alpha}z^{\frac{r}{\alpha}-1}S(x,z) + q^{2r}z^{\frac{r}{\alpha}}\left\langle x, S(zq^\alpha)\right\rangle,\end{aligned}$$

by Theorem 5.7. We deduce that

$$\begin{aligned}d''(z^{\frac{r}{\alpha}}S(x,z)) &= \left\langle x, q^{2r}z^{\frac{r}{\alpha}}S(zq^\alpha)\right\rangle\\ &= d'S^{01}. \qquad \square\end{aligned}$$

6.2. Compositions. Let p be a positive integer, and let us consider the ring

$$A_p = \mathbb{C}[[z_1,\dots,z_p]][\prod_{i=1}^p z_i^{-1}];$$

we look at A_p as the ring of functions on the p^{th} power of the formal punctured disk X_p.

Let Ω^a $(1 \le a \le p)$ denote the space of algebraic q-differential a-forms on the formal variety X_p. Thus, Ω^0 is just A_p, and elements of Ω^a have the form

$$\sum f(z_1,\dots,z_p)d_q z_{i_1}\wedge\dots\wedge d_q z_{i_a} \quad (f(z_1,\dots,z_p)\in A_p).$$

We consider the following complex

$$\Omega^\bullet \;:\; 0\longrightarrow\Omega^0\longrightarrow\dots\longrightarrow\Omega^p\longrightarrow 0, \tag{6.6}$$

its differential is given by

$$d'' = \mathcal{D}'_\alpha - \sum_{i=1}^p q^{r-\alpha-2(i-1)}[r-2(i-1)]\frac{d_q z_i}{z_i},$$

where, in several variables, $\mathcal{D}'_\alpha$ is defined by:

$$\mathcal{D}'_\alpha(f) = \sum_{i=1}^p \frac{\mathcal{D}'_\alpha(f)}{d_q z_i}\wedge d_q z_i.$$

For $i = 1,\dots,p$, we set:

$$S'(z_i) = q^{2(r-i)}\,z_i^{\frac{r-2(i-1)}{\alpha}}S(z_i q^\alpha),$$

$$S'(x,z_i) = z_i^{\frac{r-2(i-1)}{\alpha}}S(x,z_i).$$

The operators $S'(z_i)$ and $S'(x,z_i)$ are elements of

$$\mathrm{Hom}\left(\mathcal{W}_{r-2(i-1)}, \mathcal{W}_{r-2i}\otimes A_p\right).$$

We consider the following double complex:

$$\begin{aligned}C^{\bullet\bullet} &= \mathrm{Hom}\left(U_q(\widehat{\mathfrak{sl}_2})^{\otimes\bullet}, \mathrm{Hom}(\mathcal{W}_r, \mathcal{W}_{r-2p}\otimes\Omega^\bullet)\right)\\ &= \mathrm{Hom}\left(U_q(\widehat{\mathfrak{sl}_2})^{\otimes\bullet}\otimes\mathcal{W}_r, \mathcal{W}_{r-2p}\otimes\Omega^\bullet\right).\end{aligned}$$

And for each $m = 0,\dots,p$, we define the operators

$$S^{m,p-m} \in \operatorname{Hom}\left(U_q(\widehat{\mathfrak{sl}_2})^{\otimes m}, \operatorname{Hom}(\mathcal{W}_r, \mathcal{W}_{r-2p} \otimes \Omega^{p-m})\right)$$

as follows:

$$\begin{aligned}&S^{m,p-m}(x_1, \dots, x_m)\\ &= (-1)^{\frac{m(m+1)}{2}} \sum_{1\le i_1\le\dots\le i_m\le p} (-1)^{i_1+\dots+i_m} \mathcal{S}(x_1, \dots, x_m; i_1, \dots, i_m),\end{aligned}$$

and $\mathcal{S}$ is given by

$$\begin{aligned}&\sum_{(x_1,\dots,x_n)} S'(z_1) \cdot \dots \cdot S'(x_1', z_{i_1})x_1''\\ &\cdot \left(S'(z_{i_1+1}) \cdot \dots \cdot S'(x_m', z_{i_m})x_m'' \cdot (S'(z_{i_m+1}) \cdot \dots \cdot S'(z_p))\right)\\ &\qquad d_q z_1 \wedge \dots \wedge \widehat{d_q} z_{i_1} \wedge \dots \wedge \widehat{d_q} z_{i_m} \wedge \dots \wedge d_q z_p.\end{aligned}$$

In this expression, we consider initially the composition $S'(z_1) \cdot \dots \cdot S'(z_p)$, and we substitute each $S'(z_{i_k})$ by $S'(x_k', z_{i_k})x_k'' \cdot \dots$ and x_k'' is acting on all the remaining factors on the right if there are any; if not, it is just $S'(x_k, z_{i_k})$ (which happens only when $i_k = p$) . And the sum is taken over all the terms involved in the Sweedler notation for the comultiplication of $x_1, \dots, x_m$.

The element $\mathcal{S} = (\mathcal{S}^{0,p}, \dots, \mathcal{S}^{p,0})$ is p-cochain in the simple complex associated with the double complex $C^{\bullet\bullet}$.

Theorem 6.2. *The cochain $\mathcal{S}$ is a p-cocycle.*

We need to prove that
$$d''\mathcal{S}^{0,p} = d'\mathcal{S}^{p,0} = 0,$$
and for $k = 0, \dots, p-1$,
$$d'\mathcal{S}^{k,p-k} = (-1)^k d''\mathcal{S}^{k+1,p-k-1}.$$

These relations follow from Theorem 5.6 and Theorem 5.7 and some lemmas on Hopf algebras. We give the proof in the case $p = 2$ in order to illustrate the techniques used, the general case is proved exactly in the same way but with lengthy formulas.

Let us assume $p = 2$, we have:

$$\begin{aligned}\mathcal{S}^{0,2} &= S'(z_1)S'(z_2)d_q z_1 \wedge d_q z_2,\\ \mathcal{S}^{1,1}(x) &= \sum_{(x)} \left(\mathcal{S}'(x', z_1)x'' \cdot S'(z_2)d_q z_2 - S'(z_1)S'(x, z_2)d_q z_1\right),\\ \mathcal{S}^{2,0}(x, y) &= \sum_{(x)} S'(x', z_1)x'' \cdot S'(y, z_2).\end{aligned}$$

We will use the counit axiom and the composition lemmas several times without mentioning them. We also will drop the σ in the Sweedler notation to simplify the expressions.

Since $\Omega^\bullet$ is of length 1, it is clear that $d''S^{0,2} = 0$. Let us prove that $d'S^{2,0} = 0$, we have:

$$\begin{aligned}d'S^{2,0}(x, y, z) = x \cdot \left(S'(y', z_1)y'' \cdot S'(z, z_2)\right) - S'(x'y', z_1)(x''y'') \cdot S'(z, z_2)\\ + S'(x', z_1)x'' \cdot S'(yz, z_2) - \varepsilon(z)S'(x', z_1)x'' \cdot S'(y, z_2),\end{aligned}$$

using Theorem 5.6, we get:

$$\begin{aligned} x\cdot\big(S'(y',z_1)y''\cdot S'(z,z_2)\big) &= x'\cdot S'(y',z_1)(x''y'')\cdot S'(z,z_2),\\ S'(x'y',z_1)(x''y'')\cdot S'(z,z_2) &= x'\cdot S'(y',z_1)(x''y'')\cdot S'(z,z_2)\\ &\quad + S'(x',z_1)(x''y)\cdot S'(z,z_2),\\ S'(x',z_1)x''\cdot S'(yz,z_2) &= S'(x',z_1)(x''y)\cdot S'(z,z_2)+\varepsilon(z)S'(x',z_1)x''\cdot S'(y,z_2). \end{aligned}$$

Adding up, we get $d'S^{2,0}=0$.

Now we prove that $d'S^{1,1}=-d''S^{2,0}$:

$$\begin{aligned} d'S^{1,1}(x,y) =& x\cdot(S'(y',z_1)y''\cdot S'(z_2))d_qz_2 - x\cdot(S'(z_1)S'(y,z_2))d_qz_1\\ &- S'(x'y',z_1)(x''y'')\cdot S'(z_2)d_qz_2 + S'(z_1)S'(xy,z_2)d_qz_1\\ &+\varepsilon(y)S'(x',z_1)x''\cdot S'(z_2)d_qz_2 - \varepsilon(y)S'(z_1)S'(x'z_2)d_qz_1. \end{aligned}$$

Using Theorem 5.6, we get:

$$d'S^{1,1}(x,y) = -\langle x',S(z_1)\rangle x''\cdot S'(y,z_2)d_qz_1 - S'(x',z_1)x''\cdot\langle y,S'(z_2)\rangle d_qz_2.$$

On the other hand:

$$\begin{aligned} \mathcal{D}'_\alpha S^{2,0}(x,y) =& [r]q^{r-\alpha}z_1^{\frac{r}{\alpha}-1}z_2^{\frac{r-2}{\alpha}}S(x',z_1)x''\cdot S(y,z_2)d_qz_1\\ &+q^{2r}z_1^{\frac{r}{\alpha}}z_2^{\frac{r-2}{\alpha}}\langle x',S(zq^\alpha)\rangle x''\cdot S(y,z_2)d_qz_1\\ &+[r-2]q^{r-2-\alpha}z_1^{\frac{r}{\alpha}}z_2^{\frac{r-2}{\alpha}-1}S(x',z_1)x''\cdot S(y,z_2)d_qz_2\\ &+q^{2(r-2)}z_1^{\frac{r}{\alpha}}z_2^{\frac{r-2}{\alpha}}S(x',z_1)x''\cdot\langle x,S(z_2q^\alpha)\rangle d_qz_2, \end{aligned}$$

where we have used Theorem 5.7. Therefore

$$\begin{aligned} d''S^{2,0}(x,y) =& q^{2r}z_1^{\frac{r}{\alpha}}z_2^{\frac{r-2}{\alpha}}\langle x',S(zq^\alpha)\rangle x''\cdot S(y,z_2)d_qz_1\\ &+q^{2(r-2)}z_1^{\frac{r}{\alpha}}z_2^{\frac{r-2}{\alpha}}S(x',z_1)x''\cdot\langle x,S(z_2q^\alpha)\rangle d_qz_2\\ &=\langle x',S(z_1)\rangle x''\cdot S'(y,z_2)d_qz_1 + S'(x',z_1)x''\cdot\langle y,S'(z_2)\rangle d_qz_2, \end{aligned}$$

which show that $d'S^{1,1}=-d''S^{2,0}$.

Finally, we need to prove that $d'S^{0,2}=d''S^{1,1}$:

$$\begin{aligned} d'S^{0,2}(x) =& x\cdot(S'(z_1)S'(z_2))d_qz_1\wedge d_qz_2 - \varepsilon(x)S'(z_1)S'(z_2)d_qz_1\wedge d_qz_2\\ =& x'\cdot S'(z_1)x''\cdot S'(z_2)d_qz_1\wedge d_qz_2 - \varepsilon(x)S'(z_1)S'(z_2)d_qz_1\wedge d_qz_2\\ =& \langle x',S'(z_1)\rangle x''\cdot S'(z_2)d_qz_1\wedge d_qz_2 + \varepsilon(x')S'(z_1)x''\cdot S'(z_2)d_qz_1\wedge d_qz_2\\ &-\varepsilon(x)S'(z_1)S'(z_2)d_qz_1\wedge d_qz_2\\ =& \langle x',S'(z_1)\rangle x''\cdot S'(z_2)d_qz_1\wedge d_qz_2 + S'(z_1)\langle x,S'(z_2)\rangle d_qz_1\wedge d_qz_2. \end{aligned}$$

Meanwhile,

$$\begin{aligned}\mathcal{D}'_\alpha(S^{1,1}(x)) =& q^{r-\alpha}[r]q^{2(r-2)}z_1^{\frac{r}{\alpha}-1}z_2^{\frac{r-2}{\alpha}}S(x',z_1)x''\cdot S(z_2)d_qz_1\wedge d_qz_2\\ &+q^{2r}q^{2(r-2)}z_1^{\frac{r}{\alpha}}z_2^{\frac{r-2}{\alpha}}\langle x',S(z_1q^\alpha)\rangle x''\cdot S(z_2)d_qz_1\wedge d_qz_2\\ &+q^{2r}[r-2]q^{r-2-\alpha}z_1^{\frac{r}{\alpha}}z_2^{\frac{r-2}{\alpha}-1}S(z_1)S(x,z_2)d_qz_1\wedge d_qz_2\\ &+q^{2r}q^{2(r-2)}z_1^{\frac{r}{\alpha}}z_2^{\frac{r-2}{\alpha}}S(z_1)\langle x,S(z_2q^\alpha)\rangle d_qz_1\wedge d_qz_2.\end{aligned}$$

Therefore

$$\begin{aligned}d''S^{1,1}(x) =& q^{2r}q^{2(r-2)}z_1^{\frac{r}{\alpha}}z_2^{\frac{r-2}{\alpha}}\langle x',S(z_1q^\alpha)\rangle x''\cdot S(z_2)d_qz_1\wedge d_qz_2\\ &+q^{2r}q^{2(r-2)}z_1^{\frac{r}{\alpha}}z_2^{\frac{r-2}{\alpha}}S(z_1)\langle x,S(z_2q^\alpha)\rangle d_qz_1\wedge d_qz_2\\ =&\langle x',S'(z_1)\rangle x''\cdot S'(z_2)d_qz_1\wedge d_qz_2+S'(z_1)\langle x,S'(z_2)\rangle d_qz_1\wedge d_qz_2,\end{aligned}$$

which proves that $d'S^{0,2}=d''S^{1,1}$. □

Finally, by homological considerations we deduce:

Corollary 6.3. *The cocycle $\mathcal{S}$ induces canonical linear maps*

$$s_m:\mathcal{H}^m(\Omega^\bullet)^*\longrightarrow \mathrm{Ext}^{p-m}_{U_q(\widehat{\mathfrak{sl}_2})}\left(\mathcal{W}_r,\mathcal{W}_{r-2p}\right)\quad 0\le m\le p.$$

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