Journal of Number Theory **88**, 104–113 (2001)
doi:10.1006/jnth.2000.2613, available online at http://www.idealibrary.com on IDEAL®

Twisted L-Functions and Complex Multiplication

Abdellah Sebbar

Department of Mathematics and Statistics, University of Ottawa, Ottawa, Ontario K1N 6N5, Canada
E-mail: sebbar@mathstat.uottawa.ca

Communicated by H. Darmon

Received November 17, 1999

1. INTRODUCTION

The study of algebraicity of special values of the L-function of a grossencharacter of an imaginary quadratic field K was first initiated by Eisenstein. His work appeared much later in a different formulation in the work of Birch and Swinnerton-Dyer [1] and Damerell [2]. Shimura generalized this aspect to more general CM fields [7]. Using the same language, Goldstein and Schappacher [4] related the work of Eisenstein and Damerell to the conjecture of Birch and Swinnerton-Dyer on elliptic curves and to the Deligne conjecture.

The purpose of this article is to study, with the same methods, the special values of certain twisted L-functions as follows:

Let E be an elliptic curve defined over $\mathbb{Q}$, of conductor N, and let $L(E, s)$ be its usual L-function. Let us write $L(E, s) = \sum_{n \geqslant 1} a_n n^{-s}$ (for $Re(s) > 3/2$).

For every $b \in \mathbb{Z}/N\mathbb{Z}$, we set

$$L(E, b, s) = \sum_{n \geqslant 1} e^{2i\pi bn/N} a_n n^{-s} \qquad (\Re(x) > 3/2). \tag{1.1}$$

It seems reasonable to conjecture that the series $L(E, b, s)$ admit an analytic continuation to every $s \in \mathbb{C}$, and we would like to study the values at $s = 1$ of these Dirichlet series. For $b = 0$, this is essentially one aspect of the Birch and Swinnerton-Dyer conjecture. To be more precise, we need a conjecture of the form

$$L(E, b, 1) = \alpha_b \omega^+ + \beta_b \omega^-, \tag{1.2}$$

with α_b, β_b expressed in terms of the arithmetic of E, and b. The periods ω^+, ω^- are defined in a modular language. For example, they are the so-called $u^{\pm}$ in the article of Shimura [8].

0022-314X/01 $35.00

We study the algebraic properties of $L(E, b, 1)$ in the case when E has complex multiplications by O_K. There exists a finite number (say r) of torsion points x_i of $\mathbb{C}/L$ (for a lattice L corresponding to E over $\mathbb{C}$) and a simple explicit function $C(E, s)$ (depending on the choice of x_i) such that

$$C(E, s)\, L_p(\chi, s) = \sum_{i=1}^{r} K_1(x_i, 0, s, L),$$

where χ is the grossencharacter associated to E, L_p is a certain partial Hecke L-function depending on an integral ideal of O_K relatively prime to $N.O_K$, and K_1 denotes the Kronecker series. The Hecke L-function is recovered by taking the sum of the partial L-series corresponding to integral ideals for which the Artin symbols describe the Galois group of the ray class field modulo N. Because of the complex multiplication, the ray class field is equal to the extension of K generated by the N-torsion points of E/K. Our results concerning $L(E, b, 1)$ follow then from the properties of the Hasse–Weil and Hecke L-functions and the relationship between the Kronecker and the Eisenstein series.

The next two sections deal with the necessary background concerning Hecke characters, L-functions, complex multiplication, and torsions points. The last section presents our study of the values at 1 of the twisted L-function.

2. HECKE CHARACTERS AND L-FUNCTIONS

In this section we recall some important properties which will be used in subsequent sections.

Let F be a number field with O_F its ring of integers, and let $\mathfrak{m}$ be an integral ideal of K and $I_{\mathfrak{m}}$ be the group of fractional ideals relatively prime to $\mathfrak{m}$. Let K be another number field in which F can be embedded. If G is the set of embeddings of F in K, we denote by $\mathbb{Z}[G]$ the free abelian group over G. A Hecke character χ of the field F with values in K^* having conductor $\mathfrak{m}$ and of type at infinity $(n_\sigma)_\sigma$ element of $\mathbb{Z}[G]$ is:

1. A homomorphism from the group of fractional ideals of F relatively prime to $\mathfrak{m}$ into K^*.

2. For every $\alpha \in F^*$ such that the ideal αO_F is relatively prime to $\mathfrak{m}$ and $\alpha = \beta/\gamma$, with $\beta, \gamma \in O_K$ relatively prime to $\mathfrak{m}$ and $\beta \equiv \gamma \bmod \mathfrak{m}$ (we say $\alpha \equiv 1 \bmod \mathfrak{m}$), we have

$$\chi(\alpha O_F) = \prod \sigma(\alpha)^{n_\sigma} \in K^*.$$

If χ_1 is defined over $I_{\mathfrak{m}_1}$ and χ_2 is defined over $I_{\mathfrak{m}_2}$ and if the two give the same map from $I_{\mathfrak{m}_1\mathfrak{m}_2}$ to K^*, we identify them to one character χ. The conductor of χ is the smallest ideal $\mathfrak{m}$ of O_F such that χ is identified with a Hecke character of $I_{\mathfrak{m}}$.

Let E be an elliptic curve defined over a field F with complex multiplications by O_K where K is an imaginary quadratic field. There is an embedding of K into F which we fix, and we look at F as embedded into $\bar{K}$. We define a character attached to these data as a map

$$\chi : I_{\mathfrak{f}} \to K^*,$$

where $\mathfrak{f}$ is an integral ideal which is divisible by all prime ideals at which E has bad reduction, as follows:

If $\mathfrak{p}$ is a prime ideal in $I_{\mathfrak{f}}$, we consider the embedding

$$\mathbb{Q} \otimes_{\mathbb{Z}} End(E) \hookrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} End_{\kappa_{\mathfrak{p}}}(E_{\mathfrak{p}}),$$

where $\kappa_{\mathfrak{p}}$ is the residue field of K at $\mathfrak{p}$. Let

$$\begin{aligned} \mathscr{F} : E_{\mathfrak{p}} &\to E_{\mathfrak{p}} \\ (x, y) &\mapsto (x^{\mathbb{N}\mathfrak{p}}, y^{\mathbb{N}\mathfrak{p}}) \end{aligned}$$

be the Frobenius map at $\mathfrak{p}$. Here $\mathbb{N}\mathfrak{p} = |(O_k/\mathfrak{p}O_K)|$. Then $\mathscr{F}_{\mathfrak{p}}$ is in the center of $End_{\kappa_{\mathfrak{p}}}(E)$ and hence is in the image of the above embedding. Therefore, there exists $\chi(\mathfrak{p})$ in $\mathbb{Q} \otimes End_F(E)$ such that the image of $\chi(\mathfrak{p})$ is $\mathscr{F}_{\mathfrak{p}}$. Since E has complex multiplications, one can show that the map $\chi : I_{\mathfrak{f}} \to K^*$ is a Hecke character (the grossencharacter of E). We talk about $N_{F/K}$ as the type at infinity of χ since it acts on a principal fractional ideal (α) of F^* relatively prime to $\mathfrak{f}$ by

$$\chi(\alpha) = \varepsilon(\alpha)\, N_{F/K}(\alpha), \tag{2.1}$$

where ε is a homomorphism of $(O_K/\mathfrak{f})^*$ into the groups of units of O_K and $N_{F/K}$ is the norm of the extension of F/K. In particular, if $\alpha \equiv 1 \bmod \mathfrak{f}$ then $\chi(\alpha) = N_{F/K}(\alpha)$. Moreover, χ is unramified outside the places of bad reduction of E.

Let $L(\chi, s)$ denote the Hecke L-function attached to χ and $L(E/F, s)$ denote the Hasse–Weil L-function of the elliptic curve E/F. From the properties of χ one can show that

$$L(E/F, s) = L(\chi, s)\, L(\bar{\chi}, s), \tag{2.2}$$

where $\bar{\chi}$ is the complex conjugate of the character χ.

From now on we assume that E is defined over $\mathbb{Q}$ and has complex multiplications by O_K. Using the fact that $\bar{\chi}(\mathfrak{p}) = \chi(\bar{\mathfrak{p}})$ for $\mathfrak{p}$ prime in O_K and comparing the local factors of the L-functions according to whether $\mathfrak{p}$ is ramified, inert, or splits in O_K, the relation (2.2) yields [3]

$$L(E/\mathbb{Q}, s) = L(\chi, s). \tag{2.3}$$

We finish this section by an important relation which will be useful later on. Let N_E be the conductor of E, $\mathfrak{f}$ the conductor of the grossencharacter χ and d_K the discriminant of the field K. Then

$$|d_K| . \mathbb{N}\mathfrak{f} = N_E. \tag{2.4}$$

This follows from the functional equations satisfied by the L-functions and from (2.3).

3. TORSION POINTS OF AN ELLIPTIC CURVE

Let $E/\mathbb{Q}$ be an elliptic curve with complex multiplications by the ring of integers O_K of a quadratic imaginary K. We fix an embedding $K \hookrightarrow \mathbb{C}$. There is a lattice Λ of $\mathbb{C}$ such that we have the Weierstrass isomorphism

$$\xi\colon \frac{\mathbb{C}}{\Lambda} \to E(\mathbb{C})$$

$$(z, \Lambda) \mapsto (\wp(z, \Lambda), \wp'(z, \Lambda)),$$

where $\wp$ is the Weierstrass elliptic function. Since $E/\mathbb{Q}$ has complex multiplications, then $\Lambda = \Omega \cdot O_K$ for $\Omega \in \mathbb{C}^*$. In fact, we can choose Ω to be a nonzero real number.

Let N be a nonnegative integer; the N-torsion group of E over $\mathbb{C}$ is the subgroup of E given by

$$E_N = \{\xi(z, \Lambda), z \in N^{-1}\Lambda\} \subseteq \mathbb{P}_2(\mathbb{C}).$$

If we add the coordinates of all the points in E_N to K, we obtain a finite extension $K(E_N)$ of K. If S denotes the set of places of K at which E has bad reduction, then the extension $K(E_N)/K$ is abelian, nonramified outside the places in K dividing $N . O_K$ and the places in S. Moreover, if $\mathfrak{b}$ is an integral ideal of K relatively prime to N and to S, and if $\rho \in N^{-1}\Lambda/\Lambda$, then the action of the grossencharacter χ on torsion points is given by

$$\xi(\rho, \Lambda)^{(\mathfrak{b}, K(E_N)/K)} = \xi(\chi(\mathfrak{b})\, \rho, \Lambda), \tag{3.1}$$

where $(\mathfrak{b}, K(E_N))$ is the Artin symbol of $\mathfrak{b}$ and χ is the Hecke character attached to E.

Since E is defined over $\mathbb{Q}$ with complex multiplications by O_K, the class number is $h_K = 1$, so that all fractional ideals are principal. Let N be the conductor of $E/\mathbb{Q}$ and $\mathfrak{f}$ the conductor of the grossencharacter of E. A representative αO_K of the principal ray modulo N satisfies $\alpha \equiv 1 \bmod N$, and from (2.4) we deduce that $N . O_K = |d_K| . \bar{\mathfrak{f}} \subseteq \mathfrak{f}$; therefore $\alpha \equiv 1 \bmod \mathfrak{f}$. Hence, any character of $(O_K/\mathfrak{f})^*$ acts trivially on an representative of the principal ray mod N. Using (2.1) and (3.1), we deduce that we have an abelian extension $K(E_N)/K$ nonramified outside N and such that the Artin symbols of ideals in the ray class group modulo N act trivially on $K(E_N)$. From class field theory we have $K(E_N) \subseteq K_N$, where K_N is the ray class field modulo N. On the other hand, using [6, 5.10, p. 124], we have $K_N = K(\Phi_E(P), P \in E_N)$ where Φ is the Weber function $\Phi_E(P) = x(p)^i$, with x being the x-coordinate and $i = \frac{1}{2} |O_K^*|$. Hence $K_N \subseteq K(E_N)$. Therefore, the field $K(E_N)$ is the ray class field mod N.

Let us denote by $\sigma_{\mathfrak{a}}$ the Artin symbol $(\mathfrak{a}, K(E_N)/K)$. Since χ is unramified outside N, then, for $\mathfrak{a} = (\alpha)$ and $\mathfrak{b} = (\beta)$, $\sigma_{\mathfrak{a}} = \sigma_{\mathfrak{b}}$ if and only if for some unit ε in O_K^* we have $\varepsilon\alpha/\beta \equiv 1 \bmod N$. It follows that

$$\text{if} \quad \sigma_{\mathfrak{a}} = \sigma_{\mathfrak{b}} \qquad \text{then} \quad \mathbb{N}\mathfrak{a} \equiv \mathbb{N}\mathfrak{b} \mod N. \tag{3.2}$$

Let μ_N be the group of the Nth root of unity. Consider the Weil pairing $e_N \colon E_N \times E_N \to \mu_N$ (see [9]). The set $\{e_N(S, T), S, T \in E_N\}$ is a subgroup μ_d of μ_N. It follows that for every S and T, $1 = e_N(S, T)^d = e_N([d]\, S, T)$. By the nondegeneracy of the Weil paring, we must have $[d]\, S = 0$; i.e., S is a d-torsion point. Since S was arbitrary, it follows that $d = N$. Moreover, the pairing e_N is equivariant under the Galois action. Then for every $\sigma \in \mathrm{Gal}(\overline{K(E_N)}/K(E_N))$, we have $e_N(S, T)^\sigma = e_N(S^\sigma, T^\sigma) = e_N(S, T)$ since S and T are in $K(E_N)$. It follows that $e_N(S, T) \in K(E_N)$ and therefore $\mu_N \subseteq K(E_N)^*$. Hence, we have the following inclusions

$$\mathbb{Q} \xrightarrow{\;2\;} K \to K(E_N)$$

$$\mathbb{Q} \xrightarrow{\phi(N)} \mathbb{Q}(\mu_N) \to K(E_N).$$

Since the cyclotomic field $\mathbb{Q}(\mu_N)$ is the ray class field modulo N of $\mathbb{Q}$, then if ξ_N is a primitive Nth root of unity, we have

$$\xi_N^{(\mathfrak{a}, K(E_N)/K)} = \xi_N^{\mathbb{N}\mathfrak{a}, \mathbb{Q}(\mu_N)/\mathbb{Q})} = \xi_N^{\mathbb{N}\mathfrak{a}}. \tag{3.3}$$

4. TWISTED L-FUNCTIONS

Let Λ be a lattice in $\mathbb{C}$. If (u, v) is a $\mathbb{Z}$-basis of Λ such that $\Im(u/v) > 0$ then the real number $(\bar{u}v - u\bar{v})/2i\pi$ is positive and independent of the basis; we denote it by $A(\Lambda)$. Let us consider the following homomorphisms of the additive group $\mathbb{C}$ into the unit circle parametrized by the variable z_0:

$$\psi(z, z_0, \Lambda) = \exp\left(\frac{\overline{z_0} z - z_0 \bar{z}}{A(\Lambda)}\right).$$

For $k \geqslant 0$, we define the holomorphic functions on the domain $\Re(s) > 1 + k/2$ by:

$$\mathscr{K}_k(z, z_0, s, \Lambda) = \sum{}' \psi(\omega, z_0, \Lambda) \frac{(\bar{z} + \bar{\omega})^k}{|z + \omega|^{2s}}.$$

The sum is extended to every ω in Λ except $-z$ if $z \in \Lambda$. For i and j integers satisfying $j > i \geqslant 0$ and $z \in \mathbb{C}/\Lambda$ we set

$$E^*_{i,j}(z, \Lambda) = \mathscr{K}_{i+j}(z, 0, j, \Lambda)$$

and

$$E^*_k(z, \Lambda) = E^*_{0,k}(z, \Lambda).$$

The $\mathscr{K}_k$ are the Kronecker double series and the E_{ij} are the Eisenstein series; see [11].

We consider now an elliptic curve $E/\mathbb{Q}$ with complex multiplications by O_K, K being an imaginary quadratic field. Recall that

$$L(E/\mathbb{Q}, s) = L(\chi, s) = \sum_{(\mathfrak{a}, \mathfrak{f}) = 1} \frac{\chi(\mathfrak{a})}{\mathbb{N}\mathfrak{a}^s},$$

where $\mathfrak{f}$ is the conductor of the Hecke character χ. We are interested in nonprimitive L-functions in which the sums in the L-functions are extended over prime ideals which are relatively prime to NO_K where N is the conductor of $E/\mathbb{Q}$. Recall also that $N \in \mathfrak{f}$. For $\sigma \in \mathrm{Gal}(K(E_N)/K)$, we define the partial L-series associated to χ and relative to σ by

$$L(\chi, \sigma, s) = \sum \frac{\chi(\mathfrak{a})}{\mathbb{N}\mathfrak{a}^s},$$

where the sum ranges over integral ideals $\mathfrak{a} \subseteq O_K$ with $\sigma_{\mathfrak{a}} = \sigma$. Each $\sigma \in \mathrm{Gal}(K(E_N)/K)$ corresponds to a $\sigma_{\mathfrak{a}}$ for some integral ideal $\mathfrak{a} \subseteq O_K$; let $\mathscr{A}$ be a complete set of integral ideals in O_K representatives for all elements in Gal$(K(E_N)/K)$. If we denote the series $L(\chi, \sigma_{\mathfrak{a}}, s)$ by $L_{\mathfrak{a}}(\chi, s)$, it is clear that

$$L(\chi, s) = \sum_{\mathfrak{a} \in \mathscr{A}} L_{\mathfrak{a}}(\chi, s).$$

Let Ω be a fixed nonzero real number such that $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$ with $\Lambda = \Omega O_K$.

PROPOSITION 4.1. *Let $\mathfrak{a} = p_0 O_K$ be an ideal relatively prime to $\mathfrak{f}$. Let $\rho \in \Omega K^* \subset \mathbb{C}^*$ such that $\rho \Omega^{-1} O_K = \mathfrak{f}^{-1}$. Then for Re$(s) > 3/2$, we have*:

$$\frac{\chi(p_0 O_K)}{\mathbb{N}(p_0 O_k)} \cdot \frac{\bar{p}_0 \rho}{|p_0 \rho|^{2s}} \cdot L_{\mathfrak{a}}(\bar{\chi}, s) = \sum_{p O_K \in \mathscr{A}} \mathscr{K}_1(\chi(p)\, p_0 \rho, 0, s, \Lambda).$$

Proof. We note first that $p_0 \rho$ is an $\mathfrak{f}$-torsion point. We have $\chi(\mathfrak{a})\, \bar{\chi}(\mathfrak{a}) = \mathbb{N}\mathfrak{a}$ and if $\mathfrak{b}$ is relatively prime to $\mathfrak{f}$, then $\chi(\mathfrak{b})\, O_K = \mathfrak{b}$. Thus

$$\begin{aligned}\sum_{p O_K \in \mathscr{A}} \mathscr{K}_1(\chi(p)\, p_0 \rho, 0, s, \Lambda) &= \sum_{p . O_K} \sum_{\omega} \frac{\overline{\chi(p)\, p_0 \rho + \omega}}{|\chi(p)\, p_0 \rho + \omega|^{2s}} \\ &= \frac{\bar{p}_0 \rho}{|p_0 \rho|^{2s}} \sum_{p . O_K \in \mathscr{A}} \sum_{\alpha \in p_0^{-1} \mathfrak{f}} \frac{\overline{\chi(p) + \alpha}}{|\chi(p)\, \rho + \alpha|^{2s}}.\end{aligned}$$

It remains to show that

$$L_{\mathfrak{a}}(\bar{\chi}, s) = \sum_{p \in \mathscr{A}} \sum_{\alpha \in p_0^{-1} \mathfrak{f}} \frac{\bar{\chi}((\chi(p) + \alpha)\, \mathfrak{a})}{\mathbb{N}((\chi(p) + \alpha)\, \mathfrak{a})}.$$

For this, it is enough to check that for $p . O_K \in \mathscr{A}$ and $\alpha \in p_0^{-1} \mathfrak{f}$, we have $\sigma_{\mathfrak{a}} = \sigma_{(\chi(p) + \alpha)\, \mathfrak{a}}$ $(\alpha \in \mathfrak{a}^{-1} \mathfrak{f})$. This follows from class field theory. ∎

Remark 4.1. There is a similar decomposition in [5]. See also [4]. The series $L_{\mathfrak{a}}$ has an analytic continuation to the whole complex plane in the same way as the Hecke L-series. From the above proposition and the definition of E_1^* we have

COROLLARY 4.2. *With the same notations, we have*:

$$\frac{\chi(\mathfrak{a})}{p_0 \rho} L_{\mathfrak{a}}(\bar{\chi}, 1) = \sum_{p . O_K \in \mathscr{A}} E_1^*(\chi(p . O_K)\, P_0 \rho, \Lambda).$$

PROPOSITION 4.3. *We have*

1. $E_1^*(\chi(p_0 . O_K)\, p_0 \rho, \Lambda) \in K(E_N)$.
2. $E_1^*(\chi(p_0 . O_K)\, p_0 \rho, \Lambda) = E_1^*(\chi(O_K)\, \rho, \Lambda)^{\sigma_{\mathfrak{a}}}$.

This a particular case of [4, Theorem 6.2].

Since $\rho \in \Omega K^*$, the corollary implies that $\Omega^{-1} L_{\mathfrak{a}}(\bar{\chi}, 1)$ and $\sum K_1^*(\chi(p_0)\,\rho, \Lambda)$ are K-proportional. And using the above proposition, we obtain:

PROPOSITION 4.4. *For every* $\mathfrak{a}$, *we have*

1. $L_{\mathfrak{a}}(\bar{\chi}, 1)/\Omega \in K(E_N)$.
2. *If* $\mathfrak{a}$ *is relatively prime to* N, *then*

$$\frac{L_{\mathfrak{a}}(\bar{\chi}, 1)}{\Omega} = \left(\frac{L_{O_K}(\bar{\chi}, 1)}{\Omega}\right)^{\sigma_{\mathfrak{a}}}.$$

PROPOSITION 4.5. *We have*

$$\frac{L(E, 1)}{\Omega} = Trace_{K(E_N)/K}\left(\frac{L_{O_K}(\bar{\chi}, 1)}{\Omega}\right).$$

Proof. Since $\Omega^{-1} L_{O_K}(\bar{\chi}, 1) \in K(E_N)$, the expression makes sense. Since E is defined over $\mathbb{Q}$, for a real number s we have

$$L_{\mathfrak{a}}(\bar{\chi}, s) = L_{\mathfrak{a}}(\chi, s).$$

Since $L(E, s) = \sum_{\mathfrak{a}} L_{\mathfrak{a}}(\chi, s)$, it follows that

$$\frac{L(E, 1)}{\Omega} = \sum_{\mathfrak{a}} \frac{L_{\mathfrak{a}}(\bar{\chi}, 1)}{\Omega} = \sum_{\mathfrak{a}} \left(\frac{L_{O_K}(\bar{\chi}, 1)}{\Omega}\right)^{\sigma_{\mathfrak{a}}},$$

where $\sigma_{\mathfrak{a}}$ describes $\mathrm{Gal}(K(E_N)/K)$. This proves the proposition. ∎

The quantity $L_{O_K}(\bar{\chi}, 1)/\Omega$ is real; indeed, $\bar{\Omega} = \Omega$ and $L_{O_K}(E, s) = \sum \chi(\mathfrak{a})/\mathbb{N}\mathfrak{a}^s$, where the sum is over all $\mathfrak{a} = (\alpha)$ such that $\alpha \equiv 1 (\mathrm{mod}\ N)$ and $\sigma_{\mathfrak{a}} = 1$. Hence

$$\overline{L_{O_K}(E, s)} = \sum_{\bar{\mathfrak{a}} = (\alpha)} \frac{\chi(\bar{\mathfrak{a}})}{\mathbb{N}\bar{\mathfrak{a}}^s} = L_{O_K}(E, s). \tag{4.1}$$

We now introduce the twisted L-function associated with $E/\mathbb{Q}$ which still has complex multiplications by O_K and N is its conductor.

If the Hasse–Weil L-function of E is given by

$$L(E, s) = \sum_{n \geqslant 1} a_n n^{-s} \qquad \text{for} \quad \Re(s) > 3/2,$$

then, for every $b \in \mathbb{Z}/N\mathbb{Z}$, we set

$$L(E, b, s) = \sum_{n \geqslant 1} e^{2i\pi bn/N} a_n n^{-s}.$$

PROPOSITION 4.6. *For* $\Re(s) > 3/2$, *we have*

$$L(E, b, s) = \sum_{\mathfrak{a} \in \mathscr{A}} e^{2i\pi n \mathbb{N}\mathfrak{a}/N} L_{\mathfrak{a}}(\bar{\chi}, s).$$

Proof. This follows from the definition and from (3.2) which says that if $\sigma_{\mathfrak{a}} = \sigma_{\mathfrak{b}}$, then $\mathbb{N}\mathfrak{a} \equiv \mathbb{N}\mathfrak{b} \bmod N$. ∎

Let $\xi_N = e^{2i\pi/N}$ be a primitive Nth root of unity. From (4.1) we have

$$\xi_N^{\sigma_{\mathfrak{a}}} = \xi_N^{(\mathbb{N}\mathfrak{a},\, \mathbb{Q}(\mu_N)/\mathbb{Q})} = \xi_N^{\mathbb{N}\mathfrak{a}}.$$

Therefore, using (4.4), we have

$$\begin{aligned} \frac{L(E, b, 1)}{\Omega} &= \sum_{\mathfrak{a}} (\xi_N^b)^{\mathbb{N}\mathfrak{a}} \frac{L_{\mathfrak{a}}(\bar{\chi}, 1)}{\Omega} \\ &= \sum_{\mathfrak{a}} (\xi_N^b)^{\mathbb{N}\mathfrak{a}} \left(\frac{L_{O_K}(\bar{\chi}, 1)}{\Omega}\right)^{\sigma_{\mathfrak{a}}} \\ &= \sum_{\mathfrak{a}} \left(\xi_N^b \frac{L_{O_K}(\bar{\chi}, 1)}{\Omega}\right)^{\sigma_{\mathfrak{a}}}. \end{aligned}$$

The above sums are over ideals $\mathfrak{a}$ such that $\sigma_{\mathfrak{a}}$ runs over the Galois group $\mathrm{Gal}(K(E_N)/K)$. Therefore, we have

THEOREM 4.7. *The value* $L(E, b, 1)$ *is a K-rational multiple of the period* Ω; *more precisely*:

$$\frac{L(E, b, 1)}{\Omega} = \mathit{Trace}_{K(E_N)/K}\left(\xi_N^b \frac{L_{O_K}(\bar{\chi}, 1)}{\Omega}\right).$$

Remark 4.2. We should note that Ω is well determined up to a scalar in K^*. It also has a homological interpretation; see [4]. The formula in the theorem gives the expression (1.2). It remains to express these quantities in terms of the arithmetic of E.

ACKNOWLEDGMENT

I thank Norbert Schappacher, who taught me the subject and inspired this work.

REFERENCES

1. B. J. Birch and H. P. F. Swinnerton-Dyer, Notes on elliptic curves. II, *J. Reine Angew. Math.* **218** (1965), 79–108.
2. R. Damerell, L-functions of elliptic curves with complex multiplication, I, *Acta Arith.* **17** (1970), 287–301.
3. M. Deuring, Die Zetafunktion einer algebraischen Kurve vom Geschlechte eins, I, II, III, IV, *Gott. Nach.* (1953, 1955, 1956, 1957).
4. C. Goldstein and N. Schappacher, Séries d'Eisenstein et fonctions L de courbes élliptiques à multiplication complexe, *J. Reine Angew. Math.* **327** (1981), 184–218.
5. D. Rohrlich, Elliptic curves and values of L-functions, *Can. Math. Soc. Conf. Proc.* **7** (1987), 371–387.
6. G. Shimura, "Introduction to the Arithmetic Theory of Automorphic Functions," Princeton Univ. Press, Princeton, NJ, 1971.
7. G. Shimura, On some arithmetic properties of modular forms of one and several variables, *Ann. of Math.* **102** (1975), 491–515.
8. G. Shimura, On the periods of modular forms, *Ann. Math.* **229** (1977), 211–221.
9. J. Silvermann, "Arithmetic of Elliptic Curves," Springer-Verlag, Berlin/New York, 1986.
10. J. Silvermann, "Advanced Topics in the Arithmetic of Elliptic Curves," Springer-Verlag, Berlin/New York, 1994.
11. A. Weil, "Elliptic Functions According to Eisenstein and Kronecker," Springer-Verlag, Berlin/Heidelberg/New York, 1976.