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Quantized Universal Enveloping Algebras and q-de Rham Cocycles

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1. INTRODUCTION

In this work a close connection is established between certain cohomology spaces of quantized universal enveloping algebras and a twisted q-de Rham (Jackson–Aomoto) cohomology of configuration spaces.

In the Lie algebra case, the idea of such a connection belongs to V. Ginzburg and V. Schechtman. In [4] and [5], these authors have constructed canonical morphisms between the de Rham homology of certain local systems over configuration spaces and Ext-spaces between Fock-type modules over Kac–Moody and Virasoro Lie algebras. This construction is, in turn, a generalization of the classical Feigin–Fuchs construction. In their study of representation theory of Virasoro algebras, Feigin and Fuchs have discovered a way of obtaining intertwiners between Fock modules over the Virasoro algebra from the top homology of certain one-dimensional local systems.

We investigate this connection between the geometry of configuration spaces and the representation theory in the case of quantum groups. The representations considered here are Verma modules over the quantized enveloping algebras of semisimple Lie algebras. The existence and the uniqueness of these modules were established by Lusztig [9]. We consider a family of operators between the Verma modules that satisfy certain difference equations and certain cocycle conditions. These equations are built using a family of q-difference operators that generate a flat connec-

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tion in a one-dimensional vector bundle over the n-dimensional torus. In fact, these difference operators are the "differentials" of a "q-de Rham" complex of the space of formal algebraic q-differential forms over the n-torus. The homology groups of this complex can be regarded as the homology groups of the n-torus with coefficients in a local system with stalk $\mathbb{C}$. From these data, we construct the canonical "q-de Rham" cocycles, and consequently we obtain the canonical maps between the homology of the local systems and the Ext-spaces between the Verma modules.

This paper is organized as follows. Section 2 is concerned with Hopf algebras; we make some constructions and prove some results that we need later. Two key ingredients are introduced, namely a bracket that will have a major role in all of the work, and a cochain complex of Hochschild type that will lead to the Ext-spaces. In Section 3, we treat the simple case of the algebra $\mathscr{U}_q(\mathfrak{sl}_2)$. And in Section 4 we treat the general case of a semisimple Lie algebra; we define a sequence of certain *vertex operators* that, taken together, define a cocycle in a double complex. This double complex is a mix of the difference de Rham complex and the Hochschild cochain complex with coefficients in a Hom-space between two Verma modules. In the course of the work, we found some nice features of the q-deformed picture. We mention one of them: the appearance of the Kashiwara operators ∂_i and ${}_i\partial$ of Lusztig in the solution of the main difference equations. I should mention also that the same construction has been made in [13] for the quantum affine algebra $\mathscr{U}_q(\widehat{\mathfrak{sl}_2})$, with different techniques. The operators are the so-called screening operators, and the representations are the q-analog of the Wakimoto modules.

2. HOPF ALGEBRAS AND THEIR ACTIONS

In this section we present some constructions concerning Hopf algebras and their representations and establish some results related to them. All of the algebraic structures will be over the field of complex numbers.

Let H be a Hopf algebra, and let Δ, A, and ε denote, respectively, the comultiplication, the antipode, and the counit maps. These maps satisfy the following axioms, in which we use the Sweedler notation for the comultiplication:

$$\Delta(x) = \sum_{(x)} x' \otimes x'' \qquad (x \in H).$$

$$\sum_{(x)} x' \otimes (x'')' \otimes (x'')'' = \sum_{(x)} (x')' \otimes (x')'' \otimes x'' \qquad \text{(Coassociativity axiom)}. \quad (2.1)$$

$$\sum_{(x)} x'\varepsilon(x'') = \sum_{(x)} \varepsilon(x')x'' = x \qquad \text{(Counit axiom)}. \tag{2.2}$$

$$\sum_{(x)} x'A(x'') = \sum_{(x)} A(x')x'' = \varepsilon(x).\mathbf{1} \qquad \text{(Antipode axiom)}. \tag{2.3}$$

Moreover, A is an antihomomorphism, and $\varepsilon \circ A = \varepsilon$.

The modules considered in this section are algebra left-modules. If $\mathscr{M}$ and $\mathscr{N}$ are two H-modules, one can define a structure of left module on both $\mathscr{M} \otimes \mathscr{N}$ and $\operatorname{Hom}(\mathscr{M}, \mathscr{N})$ by

$$x \cdot (m \otimes n) = \sum_{(x)} (x'm) \otimes (x''n) \qquad (m \in \mathscr{M}, n \in \mathscr{N}, x \in H),$$

$$(x \cdot f)(m) = \sum_{(x)} x'f(A(x'')m) \qquad (m \in \mathscr{M}, f \in \operatorname{Hom}(\mathscr{M}, \mathscr{N}), x \in H).$$

Each vector space carries a structure of H-module through the map ε. Therefore the dual space $\mathscr{M}^* = \operatorname{Hom}(\mathscr{M}, \mathbb{C})$ is an H-module, where the action of H is given by

$$\begin{aligned}(x \cdot \phi)(m) &= \sum_{(x)} \varepsilon(x')\phi(A(x'')m) \\ &= \sum_{(x)} \phi(A(\varepsilon(x')x'')m) \\ &= \phi(A(x)m) \qquad \text{using (2.2)}.\end{aligned}$$

2.1. *Composition of Maps*

If $\mathscr{M}, \mathscr{N}$, and $\mathscr{P}$ are three H-modules, we wish to factorize the action of elements of H on maps in $\operatorname{Hom}(\mathscr{M}, \mathscr{P})$ that are compositions of maps from $\operatorname{Hom}(\mathscr{M}, \mathscr{N})$ and $\operatorname{Hom}(\mathscr{N}, \mathscr{P})$. To simplify the notations, we change the superscripts $'$ and $''$ to numerical subscripts when more than one is involved. The proof of the following lemma was outlined to me by S. Montgomery.

LEMMA 2.1. *For every $x \in H$, the following relation holds in $H \otimes H \otimes H$:*

$$\sum_{(x)} x_1 \otimes 1 \otimes x_2 = \sum_{(x)} x_{1,1} \otimes A(x_{1,2})x_{2,1} \otimes x_{2,2}. \tag{2.4}$$

Proof. We prove the identity by applying the coassociativity of the map Δ several times. By coassociativity we have for $x \in H$

$$x_1 \otimes x_{2,1} \otimes x_{2,2} = x_{1,1} \otimes x_{1,2} \otimes x_2.$$

Now, applying $1 \otimes \Delta \otimes 1$ to the left side, and $1 \otimes 1 \otimes \Delta$ to the right side, we obtain by coassociativity

$$x_1 \otimes x_{2,1,1} \otimes x_{2,1,2} \otimes x_{2,2} = x_{1,1} \otimes x_{1,2} \otimes x_{2,1} \otimes x_{2,2}.$$

Hence

$$\begin{aligned} x_{1,1} \otimes A(x_{1,2})x_{2,1} \otimes x_{2,2} &= x_1 \otimes A(x_{2,1,1})x_{2,1,2} \otimes x_{2,2} \\ &= x_1 \otimes \varepsilon(x_{2,1}) \otimes x_{2,2} \\ &= x_1 \otimes 1 \otimes x_2, \end{aligned}$$

using (2.2) twice. This proves the lemma. ∎

PROPOSITION 2.2 (Composition lemma). *If $\mathscr{M}$, $\mathscr{N}$, and $\mathscr{P}$ are three H-modules, then for every $f \in \mathrm{Hom}(\mathscr{N}, \mathscr{P})$ and for every $g \in \mathrm{Hom}(\mathscr{M}, \mathscr{N})$ and $x \in H$, we have*

$$x \cdot (f \circ g) = \sum_{(x)} (x' \cdot f) \circ (x'' \cdot g).$$

Proof. The relation can be written as

$$x_1 f\big(g(A(x_2)m)\big) = x_{1,1} f\big(A(x_{1,2})x_{2,1} g(A(x_{2,2}m))\big) \qquad (m \in \mathscr{M}),$$

which follows from (2.4) in the above lemma after applying $1 \otimes 1 \otimes A$ to both sides. ∎

The composition lemma seems to be just a consequence of the axiomatic definition of the Hopf algebra, especially from the coassociativity. Let us consider the composition map $(f, g) \to f \circ g$. It is a bilinear map and therefore induces a linear map

$$\mathrm{Hom}(\mathscr{N}, \mathscr{P}) \otimes \mathrm{Hom}(\mathscr{M}, \mathscr{N}) \overset{\circ}{\to} \mathrm{Hom}(\mathscr{M}, \mathscr{P}).$$

Using the action of H on the tensor product and on the Hom space, we can restate the composition lemma as the following result

COROLLARY 2.3. *The composition map is H-linear.*

Remark 2.1. The linearity of the composition map is known and proved in the literature only when $\mathscr{N}$ (or both $\mathscr{M}$ and $\mathscr{P}$) is finite-dimensional, in which case $\mathrm{Hom}(\mathscr{M}, \mathscr{N})$ is isomorphic to $\mathscr{M}^* \otimes \mathscr{N}$ (see [8]). Here we established it for the general case, because all of the representations we will be considering are infinite-dimensional.

2.2. *A Bracket and a Cochain Complex*

Let H be a Hopf algebra over $\mathbb{C}$ with the associated maps as above. We define a bilinear map $\langle \cdot , \cdot \rangle$ on $H \otimes H$ by

$$\langle x, y\rangle = \sum_{(x)} x'yA(x'') - \varepsilon(x)y \qquad (x, y \in H). \tag{2.5}$$

This bracket satisfies the following relations:

PROPOSITION 2.4. *For all x, y, z in H, we have*

(1) $\langle xy, z\rangle = \langle x, \langle y, z\rangle\rangle + \epsilon(x)\langle y, z\rangle + \varepsilon(y)\langle x, z\rangle$.
(2) $\varepsilon(\langle x, y\rangle) = 0$.
(3) $A^2(\langle x, y\rangle) = \langle A^2(x), A^2(y)\rangle$.

Proof. The first relation follows since Δ is an algebra homomorphism and A is an algebra antihomomorphism. The second relation follows from (1.3) and from ε being an algebra homomorphism. We will prove the third relation: we use the identity $\Delta(A(x)) = \sum_{(x)} A(x'') \otimes A(x')$, the fact that A is an antihomomorphism, and that $\varepsilon(A(x)) = \varepsilon(x)$. We have

$$\begin{aligned}
\Delta\big(A^2(x)\big) &= \sum A^2(x)' \otimes A^2(x)'' \\
&= \sum A(A(x)'') \otimes A(A(x)') \\
&= (A \otimes A)\Big(\sum A(x)'' \otimes A(x)'\Big) \\
&= (A^2 \otimes A^2)\Big(\sum x' \otimes x''\Big) \\
&= \sum A^2(x') \otimes A^2(x'').
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle A^2(x), A^2(y)\rangle &= \sum A^2(x')A^2(y)A\big(A^2(x'')\big) - \varepsilon\big(A^2(x)\big)A^2(y) \\
&= A^2\Big(\sum x'yA(x'')\Big) - \varepsilon(x)A^2(y) \\
&= A^2(\langle x, y\rangle).
\end{aligned}$$

∎

Note that $\langle 1, x\rangle = \langle x, 1\rangle = 0$ for every $x \in H$. And if H is commutative, then $\langle x, y\rangle = 0$ for every $x, y \in H$, whereas if H is cocommutative, then

$$\langle x, A(y)\rangle = A\langle x, y\rangle.$$

Let $\mathscr{M}$ be a left H-module. Following [11], we call $m \in \mathscr{M}$ an H-*invariant* if $xm = \varepsilon(x)m$ for every $x \in H$. If $\mathscr{N}$ is another H-module, we set

$$\langle x, \phi \rangle = x \cdot \phi - \varepsilon(x)\phi \qquad x \in H, \quad \phi \in \operatorname{Hom}(\mathscr{M}, \mathscr{N}). \tag{2.6}$$

Notice the analogy with the definition of $\langle x, y \rangle$. We say that ϕ is an *invariant* if $\langle x, \phi \rangle = 0$ for every $x \in H$.

PROPOSITION 2.5. *For all x, y in H and for all ϕ, ψ in* $\operatorname{Hom}(\mathscr{M}, \mathscr{N})$ *we have*

$$\langle xy, \phi \rangle = x \cdot \langle y, \phi \rangle + \varepsilon(y) \langle x, \phi \rangle. \tag{2.7}$$

$$\langle x, \phi\psi \rangle = \sum_{(x)} \langle x', \phi \rangle \langle x'', \psi \rangle + \langle x, \phi \rangle \psi + \phi \langle x, \psi \rangle. \tag{2.8}$$

Proof. The first relation is the same as the relation (1) in the above proposition when we substitute ϕ for z. The second relation is a consequence of the composition lemma and the fact that $\varepsilon(x) = \sum_{(x)} \varepsilon(x')\varepsilon(x'')$, which follows from (2.2). ∎

Remark 2.2. The importance of the bracket $\langle \cdot, \cdot \rangle$ will appear throughout this work. For Lie algebras, the expression $[x, \phi]$ defined by $[x, \phi](m) = x\phi(m) - \phi(xm)$, where ϕ is a homomorphism between two modules, defines a left action of the Lie algebra on the Hom-space. This action is enough to define homological sequences, e.g., Koszul complexes. For our case, we are dealing with associative algebras that need two actions (left and right) to define homological sequences, e.g., Hochshild complexes (see below). This explains for the moment the choice of the bracket $\langle x, \phi \rangle$, which we define as the difference of two actions of x on ϕ. In the case of the universal enveloping algebra, this bracket coincides with the Lie bracket, and for the quantized versions of these algebras, the appearance of the trivial action will emphasize the role of the group-like elements, as will be seen in the next sections.

Let $\mathscr{M}$ be a H-module, and let us consider the following sequence:

$$\begin{aligned} C^{\bullet} = C(H^{\otimes \bullet}, \mathscr{M}) : 0 \to \mathscr{M} \to \operatorname{Hom}(H, \mathscr{M}) \to \cdots \\ \to \operatorname{Hom}(H^{\otimes n}, \mathscr{M}) \to \cdots, \end{aligned}$$

and the linear map

$$d : \operatorname{Hom}(H^{\otimes n-1}, \mathscr{M}) \to \operatorname{Hom}(H^{\otimes n}, \mathscr{M}),$$

defined as follows: If $\phi \in \operatorname{Hom}(H^{\otimes n-1}, \mathscr{M})$ and $x_1 \otimes x_2 \otimes \cdots \otimes x_n \in H^{\otimes n}$, then

$$\begin{aligned} d\phi(x_1, x_2, \ldots, x_n) &= x_1 \cdot \phi(x_2, \ldots, x_n) \\ &\quad + \sum_{i=1}^{n-1} (-1)^i \phi(x_1, \ldots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \ldots, x_n) \\ &\quad + (-1)^n \phi(x_1, x_2, \ldots, x_{n-1}) \varepsilon(x_n). \end{aligned}$$

One can look at this sequence as a Hochshild complex of the associative algebra H with a left and a right action that commute on the space $\mathscr{M}$. The right action is given by the trivial action, i.e., by ε [1]. It follows that $(C^\bullet, d)$ is a cochain complex, i.e., $d^2 = 0$.

One can choose the coefficients of the cochains in $\operatorname{Hom}(\mathscr{M},\mathscr{N})$, where $\mathscr{M}$ and $\mathscr{N}$ are two H-modules. Thus we obtain a complex,

$$C^\bullet(H, \mathscr{M},\mathscr{N}) = \operatorname{Hom}\big(H^{\otimes \bullet}, \operatorname{Hom}(\mathscr{M},\mathscr{N})\big).$$

If $\phi \in \operatorname{Hom}(\mathscr{M},\mathscr{N})$, then $d\phi(x) = \langle x, \phi \rangle$. Hence, $d\phi = 0$ implies that $\langle x,\phi \rangle = 0$ for all $x \in H$. It follows that the 0th cohomology space is the space of H-invariants. We will see that in the case of quantum groups, the space of invariants coincides with the space of intertwiners. More generally, the cohomology spaces are the Ext-spaces $\operatorname{Ext}^\bullet_H(\mathscr{M},\mathscr{N})$.

3. THE ALGEBRA $\mathscr{U}_q(\mathfrak{sl}_2)$, INTERTWINERS, AND COCYCLES

3.1. *The Main Constructions*

Let q be a nonzero complex parameter that is not a root of unity. The quantum group $\mathscr{U}_q = \mathscr{U}_q(\mathfrak{sl}_2)$ is the associative algebra generated by four variables E, F, $K^{\pm 1}$ and the relations

$$KK^{-1} = K^{-1}K = 1, \tag{3.1}$$

$$KE = q^2 EK, \tag{3.2}$$

$$KF = q^{-2} FK, \tag{3.3}$$

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}. \tag{3.4}$$

The associative algebra $\mathscr{U}_q$ has the structure of a Hopf algebra. Comultiplication, counit, and antipode are given by

$$\Delta(E) = 1 \otimes E + E \otimes K,$$

$$\Delta(F) = K^{-1} \otimes F + F \otimes 1,$$

$$\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1},$$

$$\varepsilon(E) = \varepsilon(F) = 0 \quad \text{and} \quad \varepsilon(K^{\pm 1}) = 1$$

and

$$A(E) = -EK^{-1}, \quad A(F) = -KF, \quad A(K^{\pm 1}) = K^{\mp 1}.$$

If λ is a nonzero complex number, $\mathscr{M}(\lambda)$ will denote the Verma module over $\mathscr{U}_q$ with highest weight λ. This module is generated by a nonzero vector v_λ satisfying

$$Ev_\lambda = 0, \quad Kv_\lambda = q^\lambda v_\lambda.$$

The action of the generators of $\mathscr{U}_q$ on the Verma module $\mathscr{M}(\lambda)$ is summarized in the following proposition, which one can easily prove by induction.

PROPOSITION 3.1. *Let v_λ be the highest weight vector of $\mathscr{M}(\lambda)$. Then*

$$K^n F^a v_\lambda = q^{n(\lambda - 2a)} F^a v_\lambda \qquad (a \in \mathbb{N}, n \in \mathbb{Z}),$$

$$E^n F^a v_\lambda = \prod_{k=0}^{n-1} [a-k][\lambda - a + k + 1] F^{a-n} v_\lambda \qquad (a \in \mathbb{N}, n \in \mathbb{N}),$$

where

$$[a] = \frac{a^q - a^{-q}}{q - q^{-1}}.$$

With the convention that $F^a v_\lambda = 0$ if $a < 0$.

Let λ and $\lambda' \in \mathbb{C}$, and let us examine the $\mathbb{C}$-linear map

$$V_n \colon \mathscr{M}(\lambda' - 1) \to \mathscr{M}(\lambda - 1) \qquad (n \in \mathbb{N}),$$

given by

$$V_n(F^a v_{\lambda' - 1}) = F^{a+n} v_{\lambda - 1} \qquad (a \in \mathbb{N}, n \in \mathbb{N}).$$

By direct computation, we obtain

PROPOSITION 3.2. *The pairing of V_n with the generators of $\mathscr{U}_q$ is given by*

(i) $\langle E, V_n \rangle (F^a v_{\lambda' - 1}) = q^{-\lambda' + 2a + 1}([a+n][\lambda - a - n] - [a][\lambda' - a]) F^{a+n-1} v_{\lambda - 1}$,

(ii) $\langle F, V_n \rangle (F^a v_{\lambda'-1}) = (1 - q^{\lambda' - \lambda + 2n}) F^{a+n+1} v_{\lambda - 1}$,
(iii) $\langle K^m, V_n \rangle (F^a v_{\lambda'-1}) = (-1 + q^{m(\lambda - \lambda' - 2n)}) F^{a+n} v_{\lambda - 1}$,

for every $n, a \in \mathbb{N}$ *and* $m \in \mathbb{Z}$.

If $\alpha \in \mathbb{C}$, we define a twisted differential $d_\alpha \colon \mathbb{C}[[z]] \to \mathbb{C}[[z]]dz/z$ linearly by

$$d_\alpha(z^n) = [n + \alpha] z^n \frac{dz}{z},$$

where z is a formal variable. Let $\mathscr{M}(\lambda - 1)[[z^{-1}]]$ be the module of Laurent series in z^{-1} with coefficients in $\mathscr{M}(\lambda - 1)$ and consider the operator

$$V(z) = \sum_{n \geq 0} V_n z^{-n-1} dz : \mathscr{M}(\lambda' - 1) \to \mathscr{M}(\lambda - 1)\big[[z^{-1}]\big] \frac{dz}{z}.$$

We would like to find a number $\alpha \in \mathbb{C}$ and an operator

$$V(E, z) = \sum_{n \geq 0} V_n(E) z^{-n} : \mathscr{M}(\lambda' - 1) \to \mathscr{M}(\lambda - 1)\big[[z^{-1}]\big]$$

such that

$$\langle E, V(z) \rangle = d_\alpha V(E, z). \tag{3.5}$$

This equation is equivalent to

$$\langle E, V_n \rangle = [-n + \alpha] V_n(E) \qquad (n \in \mathbb{N}).$$

Applying this to $F^a v_{\lambda'-1}$ for a nonnegative integer a, we obtain

$$\begin{aligned}[-n + \alpha] & V_n(F^a v_{\lambda'-1}) \\ &= q^{-\lambda' + 2a + 1}([a + n][\lambda - a - n] - [a][\lambda' - a]) F^{a+n-1} v_{\lambda - 1}.\end{aligned}$$

We look for a number a' depending on a such that, for every n, we have

$$[a + n][\lambda - a - n] - [a][\lambda' - a] = [-n + \alpha][n + a'].$$

After multiplication by $(q - q^{-1})^2$, the right side gives

$$-q^{2n - \alpha + a'} - q^{-2n + \alpha - a'} + q^{\alpha + a'} + q^{-\alpha - a'},$$

and the left side gives

$$q^{2n + 2a - \lambda} - q^{-2n - 2a + \lambda} + q^{2a - \lambda'} + q^{-2a + \lambda'} + q^{\lambda} + q^{-\lambda} - q^{\lambda'} - q^{-\lambda'}.$$

Identifying the powers containing $2n$ (q is not a root of unity), we obtain $-\alpha + a' = 2a - \lambda$; hence $a' = 2a + \alpha - \lambda$. Now identifying the powers containing $2a$, we get $\alpha + a' = 2a - \lambda'$. This gives $2\alpha = \lambda - \lambda'$. We must also have $q^{\lambda} + q^{-\lambda} = q^{\lambda'} + q^{-\lambda'}$, which gives $q^{\lambda} = q^{\pm\lambda'}$. If $q^{\lambda} = q^{\lambda'}$, then we are dealing with the same Verma module, since it is q^{λ} that is involved in the highest weight condition. Therefore without loss of generality, we can assume that $\lambda = \pm\lambda'$.

From now on, we suppose $\lambda = -\lambda'$; hence $a' = 2a$ and $\alpha = \lambda$. Thus

$$\langle E, V_n \rangle = [-n + \lambda] V_n(E),$$

where

$$V_n(E)(F^a v_{-\lambda-1}) = q^{\lambda+2a+1}[n + 2a] F^{a+n-1} v_{\lambda-1}. \tag{3.6}$$

By doing the same for the other generators F, K, K^{-1}, and using Proposition 3.2, we have

PROPOSITION 3.3. *For $X = E,\ F,\ K^{\pm1}$ and for every $n \in \mathbb{N}$, one can define operators*

$$V_n(X)\colon \mathscr{M}(-\lambda - 1) \to \mathscr{M}(\lambda - 1)$$

given by

$$V_n(E)(F^a v_{-\lambda-1}) = q^{\lambda+2a+1}[n + 2a] F^{a+n-1} v_{\lambda-1},$$

$$V_n(F)(F^a v_{-\lambda-1}) = q^{n-\lambda}(q - q^{-1}) F^{a+n+1} v_{\lambda-1},$$

$$V_n(K^{\pm1})(F^a v_{-\lambda-1}) = -q^{\pm(-n+\lambda)}(q - q^{-1}) F^{a+n} v_{\lambda-1},$$

and which satisfy

$$\langle X, V_n \rangle = [-n + \lambda] V_n(X).$$

Remark 3.1. If we omit the term with $\varepsilon(x)$ in the definition of the pairing $\langle \cdot , \cdot \rangle$, $V_n(K^{\pm1})$ cannot be defined; at the same time, $K^{\pm1}$ are the only generators for which ε is not zero.

Next, we need to define the operator $V_n(x)$ for every $x \in \mathscr{U}_q$.

PROPOSITION 3.4. *Let $\mathfrak{f}$ be the free associative algebra generated by E, F, and $K^{\pm1}$. Then for every $x \in \mathfrak{f}$ and $n \in \mathbb{N}$, there exists an operator $V_n(x)$: $\mathscr{M}(-\lambda - 1) \to \mathscr{M}(\lambda - 1)$, which satisfies*

$$\langle x, V_n \rangle = [-n + \lambda] V_n(x). \tag{3.7}$$

Proof. Assume that for x and y in $\mathfrak{f}$, and for every $n \in \mathbb{N}$, one can define $V_n(x)$ and $V_n(y)$ satisfying (3.7). Then for every $n \in \mathbb{N}$ one has

$$\begin{aligned} \langle xy, V_n \rangle &= x \cdot \langle y, V_n \rangle + \varepsilon(y) \langle x, Vn \rangle \qquad \text{using (2.7)} \\ &= [-n + \lambda]\big(x \cdot V_n(y) + \varepsilon(y) V_n(x)\big). \end{aligned}$$

We set $V_n(xy) = x \cdot V_n(y) + \varepsilon(y)V_n(x)$. Then we have $\langle xy, V_n\rangle = [-n + \lambda]V_n(xy)$. Since, for x, a generator of $\mathfrak{f}$, $V_n(x)$ exists and satisfies (3.7), the proposition follows. ∎

Now we extend the definition of $V_n(x)$ to $\mathscr{U}_q$.

PROPOSITION 3.5. *For every x in $\mathscr{U}_q$ and for every $n \in \mathbb{N}$, there exists an operator $V_n(x)$: $\mathscr{M}(-\lambda - 1) \to \mathscr{M}(\lambda - 1)$ satisfying the relation* (3.7).

Proof. Recall that on $\mathfrak{f}$ we have

$$V_n(xy) = x \cdot V_n(y) + \varepsilon(y)V_n(x). \tag{3.8}$$

In view of the above proposition, we need to prove that this relation is compatible with the defining relations of the algebra $\mathscr{U}_q$. For the relation (3.1), we have

$$\begin{aligned} V_n(KK^{-1}) &= K \cdot V_n(K^{-1}) + \varepsilon(K^{-1})V_n(K) \\ &= KV_n(K^{-1})K^{-1} + V_n(K). \end{aligned}$$

Hence

$$\begin{aligned} &V_n(KK^{-1})(F^a v_{-\lambda-1}) \\ &\quad = KV_n(K^{-1})\left(q^{2a+\lambda+1}F^a v_{-\lambda-1}\right) + q^{-n+\lambda}(q - q^{-1})F^{a+n}v_{\lambda-1} \\ &\quad = \left(-q^{2a+\lambda+1}q^{n-\lambda}(q - q^{-1})K + q^{-n+\lambda}(q - q^{-1})\right)F^{a+n}v_{\lambda-1} \\ &\quad = 0. \end{aligned}$$

On the other hand, it is clear that $V_n(1) = 0$. Similar calculations hold for $K^{-1}K = 1$.

For the relation (3.2), we have

$$\begin{aligned} V_n(KE)(F^a v_{-\lambda-1}) &= K \cdot V_n(E)(F^a v_{-\lambda-1}) + \varepsilon(E)V_n(K)(F^a v_{-\lambda-1}) \\ &= KV_n(E)K^{-1}(F^a v_{-\lambda-1}) + 0 \\ &= q^{2a+3\lambda-2n+3}[n + 2a]F^{a+n-1}v_{\lambda-1} \end{aligned}$$

and

$$\begin{aligned} &V_n(EK)(F^a v_{-\lambda-1}) \\ &\quad = (E \cdot V_n(K) + V_n(E))(F^a v_{-\lambda-1}) \\ &\quad = -V_n(K)EK^{-1}F^a v_{-\lambda-1} + V_n(K)K^{-1}F^a v_{-\lambda-1} + V_n(E)F^a v_{-\lambda-1}. \end{aligned}$$

The coefficient of $F^{a+n-1}v_{-\lambda-1}$ is

$$\begin{aligned} & q^{2a+\lambda+1}\big((q-q^{-1})q^{-n+\lambda}[a][\lambda+a] \\ & \qquad +(q-q^{-1})q^{-n+\lambda}[a+n][\lambda-a-n]+[n+2a]\big) \\ & \quad = q^{2a+\lambda+1}\frac{1}{q-q^{-1}}\left(q^{-n+2\lambda+2a}-q^{-3n+2\lambda-2a}\right) \\ & \quad = q^{2a+3\lambda-2n+1}[n+2a]. \end{aligned}$$

Therefore

$$\begin{aligned} & V_n(q^2EK)(F^a v_{-\lambda-1}) \\ & \quad = q^{2a+3\lambda-2n+3}[n+2a]F^{a+n-1}v_{\lambda-1} = V_n(KE)(F^a v_{-\lambda-1}). \end{aligned}$$

The compatibility with (3.3) is checked in the same way. Finally,

$$\begin{aligned} & V_n\left(\frac{K-K^{-1}}{q-q^{-1}}\right)(F^a v_{-\lambda-1}) \\ & \quad = \frac{1}{q-q^{-1}}\left(q^{-n+\lambda}(q-q^{-1})+q^{n-\lambda}(q-q^{-1})\right)F^{a+n}v_{\lambda-1} \\ & \quad = \left(q^{n-\lambda}+q^{-n+\lambda}\right)F^{a+n}v_{\lambda-1}. \end{aligned}$$

On the other hand:

$$V_n([E,F]) = -V_n(F)EK^{-1}+EV_n(F)K^{-1}+K^{-1}V_n(E)KF-FV_n(E).$$

Applying this to $(q-q^{-1})F^a v_{-\lambda-1}$, we obtain successively the following factors as coefficients of $F^{a+n}v_{\lambda-1}$:

$$\begin{gathered} \left(q^{3a+n+1}-q^{a+n+1}\right)\left(q^{\lambda+a}-q^{-\lambda-a}\right), \\ \left(q^{3a+2n+2}-q^{a}\right)\left(q^{\lambda-(a+n+1)}-q^{-\lambda+(a+n+1)}\right), \\ q^{-\lambda+2a+2n+1}\left(q^{n+2a+2}-q^{-n-2a-2}\right), \quad \text{and} \quad q^{\lambda+2a+1}\left(q^{n+2a}-q^{-n-2a}\right). \end{gathered}$$

Summing these expressions, we get

$$\begin{aligned} & (q-q^{-1})V_n([E,F])F^a v_{-\lambda-1} \\ & \quad = \left(q^{-\lambda+n}(q-q^{-1})+q^{\lambda-n}(q-q^{-1})\right)F^{a+n}v_{\lambda-1}, \end{aligned}$$

which gives the same expression as $V_n(K-K^{-1})$. ∎

Recall that we have set

$$V(z) = \sum_{n\geq 0} V_n z^{-n-1}dz\colon \mathscr{M}(-\lambda-1) \to \mathscr{M}(\lambda-1)\big[[z^{-1}]\big]\frac{dz}{z},$$

$$V(x,z) = \sum_{n\geq 0} V_n(x) z^{-n}\colon \mathscr{M}(-\lambda-1) \to \mathscr{M}(\lambda-1)\big[[z^{-1}]\big] \qquad (x \in \mathscr{U}_q).$$

Using the definition of the twisted differential d_λ and the previous propositions, we have

THEOREM 3.6. *For every $x \in \mathscr{U}_q$, there exists an operator*

$$V(x,z) = \sum_{n\geq 0} V_n(x) z^{-n}\colon \mathscr{M}(-\lambda-1) \to \mathscr{M}(\lambda-1)\big[[z^{-1}]\big]$$

linearly dependent on x such that

$$\langle x, V(z)\rangle = d_\lambda V(x,z). \tag{3.9}$$

Moreover, for x, y in $\mathscr{U}_q$, we have

$$V(xy,z) = x \cdot V(y,z) + \varepsilon(y) V(x,z). \tag{3.10}$$

3.2. *Intertwiners and Cocycles*

We consider the complex of length one:

$$\Omega^\bullet : 0 \to \Omega^0 \overset{d_\lambda}{\to} \Omega^1 \to 0,$$

where

$$\Omega^0 = \mathbb{C}\big[[z^{-1}]\big], \quad \Omega^1 = \mathbb{C}\big[[z^{-1}]\big]\frac{dz}{z}.$$

Recall that d_λ is defined linearly by

$$d_\lambda(z^{-n}) = [\lambda - n] z^{-n}\frac{dz}{z}.$$

The length one is due to the fact that $\mathfrak{sl}_2$ has one simple root. From this complex and the cochain complex $C^\bullet$ introduced in the first section, we construct the following bigraded space:

$$\begin{aligned} C^{ij} &= C^{ij}\big(\mathscr{U}_q, \mathscr{M}(-\lambda-1), \mathscr{M}(\lambda-1)\big) \\ &= \operatorname{Hom}\big(\mathscr{U}_q^{\otimes i}, \operatorname{Hom}\big(\mathscr{M}(-\lambda-1), \mathscr{M}(\lambda-1)\otimes\Omega^j\big)\big) \end{aligned}$$

for $i, j > 0$, where $\Omega^j = 0$ for $j \geq 2$.

Note that C^{ij} is isomorphic to $\mathrm{Hom}(\mathscr{U}_q^{\otimes i} \otimes \mathscr{M}(-\lambda-1), \mathscr{M}(\lambda-1) \otimes \Omega^j)$. The bigraded space C^{ij} has a natural structure of a bicomplex. The first differential $d'\colon C^{ij} \to C^{i+1,j}$ is induced by the differential d introduced in the first section. The second differential $d''\colon C^{ij} \to C^{i,j+1}$ is induced by the differential d_λ of $\Omega^\bullet$.

The operator $V(z)$ is an element of C^{01}, and we denote it by $V^{01}(z)$. The operator $V(x,z) (x \in \mathscr{U}_q)$ defines an element $V^{10}(z)$ of the space C^{10} given by

$$V^{10}(z)(x) = V(x,z).$$

PROPOSITION 3.7. *The elements $V^{01}(z)$ and $V^{10}(z)$ satisfy*

$$d'V^{01}(z) = d''V^{10}(z), \tag{3.11}$$

$$d'V^{10}(z) = 0. \tag{3.12}$$

Proof. For $x \in \mathscr{U}_q$ one has, by the definition of d',

$$\begin{aligned} d'\big(V^{01}(z)\big)(x) &= x \cdot V^{01}(z) - \varepsilon(x) V^{01}(z) \\ &= \langle x, V^{01}(z)\rangle, \end{aligned}$$

and $d''V^{10}(z)(x) = d_\lambda V(x,z)$. Thus, the first relation is simply a consequence of the relation (3.9) of Theorem 3.6. And for $x, y \in \mathscr{U}_q$, one has

$$\begin{aligned} d'\big(V^{10}(z)\big)(x \otimes y) &= x \cdot V^{10}(z)(y) - V^{10}(z)(xy) + \varepsilon(y) V^{10}(z)(x) \\ &= x \cdot V(y,z) - V(xy,z) + \varepsilon(y) V(x,z) \\ &= 0, \end{aligned}$$

using the relation (3.10) of Theorem 3.6. ∎

Let $\mathscr{C}^\bullet = \mathscr{C}^\bullet(\mathscr{U}_q, \mathscr{M}(-\lambda-1), \mathscr{M}(\lambda-1))$ denote the simple complex associated with the double complex $C^{\bullet\bullet}$, that is,

$$\mathscr{C}^n = \bigoplus_{a+b=n} C^{ab} \quad (n \in \mathbb{Z}).$$

Its differential $\mathfrak{d}$ is defined by [1]:

$$\mathfrak{d}\,|_{C^{ab}} = d' + (-1)^a d''.$$

THEOREM 3.8. *The element $(V^{01}(z), V^{10}(z))$ is a 1-cocycle of the complex $\mathscr{C}^\bullet$.*

Proof. The element $(V^{01}(z), V^{10}(z))$ is in $\mathscr{C}^1(\mathscr{U}_q, \mathrm{Hom}(\mathscr{M}(-\lambda-1), \mathscr{M}(\lambda-1)))$. Applying $\mathfrak{d}$, we get

$$\mathfrak{d}\big(V^{01}(z) + V^{10}(z)\big) = d'V^{01}(z) + d''V^{01}(z) + d'V^{10}(z) - d''V^{10}(z).$$

And we have $d''V^{01}(z) = 0$, since $\Omega^\bullet$ is of length one. Using Proposition 3.7, we have $d'V^{10}(z) = 0$ and $d'V^{01}(z) = d''V^{10}(z)$. Hence

$$\mathfrak{d}\big(V^{01}(z) + V^{10}(z)\big) = 0.$$

∎

Let us consider again the complex $\Omega^\bullet$, a monomial $z^{-n} \in \operatorname{Ker} d_\lambda$ if and only if $[-n+\lambda] = 0$. Since q is not a root of unity, this is equivalent to $\lambda = n$. Thus the complex $\Omega^\bullet$ is acyclic if λ is not an integer. We assume that λ is a nonnegative integer for the rest of this section. Thus the space $\mathscr{H}^0(\Omega^\bullet)$ is a one-dimensional space generated by the function $z^{-\lambda}$. The space $\mathscr{H}^1(\Omega^\bullet)$ is generated by the class of the form $z^{-\lambda}dz/z$.

If we consider the homology spaces $\mathscr{H}_i = \mathscr{H}^{i*}$, then $\mathscr{H}_1$ is a one-dimensional space generated by the linear form

$$\begin{aligned} \Omega^1 &\to \mathbb{C} \\ \omega &\mapsto \operatorname{Res}_{z=0}(\omega z^\lambda). \end{aligned}$$

The space $\mathscr{H}^0$ is generated by the linear form

$$\begin{aligned} \Omega^0 &\to \mathbb{C} \\ f(z) &\mapsto \operatorname{Res}_{z=0}\left(f(z)z^\lambda \frac{dz}{z}\right). \end{aligned}$$

PROPOSITION 3.9. *The operator* $\operatorname{Res}_{z=0}(V^{01}(z^\lambda))$ *in* $\operatorname{Hom}_{\mathbb{C}}(\mathscr{M}(-\lambda-1), \mathscr{M}(\lambda-1))$ *is an intertwiner.*

Proof. From the first section, we know that C^0 is a space of $\mathscr{U}_q$-invariants. We need to show that in fact it coincides with the space of intertwiners in our case. Let $\phi \in C^0$, i.e., $d\phi = 0$. We need to show that ϕ intertwines with the generators E, F, and $K^{\pm 1}$. Let $m \in \mathscr{M}(-\lambda-1)$; since $\langle E, \phi\rangle(m) = 0$, we have $-\phi(EK^{-1}m) + E\phi(K^{-1}m) = 0$. Therefore $\phi(EK^{-1}m) = E\phi(K^{-1}m)$. This shows that E intertwines with ϕ (note that K^{-1} is invertible). Now, if $\langle K^{-1}, \phi\rangle(m) = 0$, then $K^{-1}\phi(Km) - \phi(m) = 0$, which shows that K intertwines with ϕ. The intertwining property for K^{-1} follows from the invariance of ϕ with K. Finally, if $\langle F, \phi\rangle(m) = 0$, then $-K^{-1}\phi(KFm) + F\phi(m) = 0$; since K intertwines with ϕ, we see that F does too. ∎

Remark 3.2. The operator $\operatorname{Res}_{z=0}(V^{01}(z^\lambda))$: $\mathscr{M}(-\lambda-1) \to \mathscr{M}(\lambda-1)$ is the unique $\mathscr{U}_q$-homomorphism sending $v_{-\lambda-1}$ to $F^\lambda v_{\lambda-1}$. Hence it is nontrivial (notice that $F^\lambda v_{-\lambda-1}$ is also a singular vector).

From the discussion preceding the proposition, we have the following:

COROLLARY 3.10. *We have*

$$\mathscr{H}_1(\Omega^\bullet) \cong \mathscr{H}^0\big(\mathscr{U}_q, \mathrm{Hom}(\mathscr{M}(-\lambda-1), \mathscr{M}(\lambda))\big).$$

PROPOSITION 3.11. *The operator* $\mathrm{Res}_{z=0}(V^{10}(z)dz/z)$ *is a nontrivial element of the space* $\mathrm{Ext}^1_{\mathscr{U}_q}(\mathscr{M}(-\lambda-1), \mathscr{M}(\lambda-1))$.

Proof. The operator $\mathrm{Res}_{z=0}(V^{01}z^\lambda dz/z)$ is in the space $\mathrm{Hom}(\mathscr{U}_q, \mathrm{Hom}(\mathscr{M}(-\lambda-1), \mathscr{M}(\lambda)))$. By Theorem 3.8 it is a 1-cocycle of the algebra $\mathscr{U}_q$ with coefficients in the $\mathscr{U}_q$-module $\mathrm{Hom}_{\mathbb{C}}(\mathscr{M}(-\lambda-1), \mathscr{M}(\lambda-1))$; hence it defines an element of $\mathrm{Ext}^1_{\mathscr{U}_q}(\mathscr{M}(-\lambda-1), \mathscr{M}(\lambda-1))$. It is a nontrivial element; indeed,

$$V^{10}(z)z^\lambda\frac{dz}{z}(E) = \sum_{n\geq 0} q^{\lambda+1}[n]z^{-n+\lambda}\frac{dz}{z}.F^{n-1}v_{\lambda-1}.$$

(Recall that $V_n(E)(v_{-\lambda-1}) = q^{\lambda+1}[n]F^{n-1}v_{\lambda-1}v_{\lambda-1}$). Therefore

$$\mathrm{Res}_{z=0}\left(V^{10}(z)z^\lambda\frac{dz}{z}\right)(E)(v_{-\lambda-1}) = q^{\lambda+1}[\lambda]F^{\lambda-1}v_{\lambda-1}.$$

The right side is not zero since λ is a nonzero integer and q is not a root of unity. ∎

Remark 3.3. The case when q is a root of unity does not present a significant difference with the generic case, except for the fact that the homology spaces are not one-dimensional, and all of the above constructions can be carried out, obtaining an infinite family of linearly independent cocycles [12].

4. GENERALIZATION TO THE DEFORMATION OF A SEMISIMPLE LIE ALGEBRA

4.1. *The Quantum Group* $\mathscr{U}_q(\mathfrak{g})$

Root Systems

Let $(a_{ij})_{1\leq i,j\leq N}$ be an $n\times n$ indecomposable matrix with integer entries such that $a_{ii}=2$ and $a_{ij}\leq 0$ for $i\neq j$, and let $(d_1,\ldots,d_N)$ be a vector with relatively prime entries such that the matrix (d_ia_{ij}) is symmetric and positive definite. Notice that (a_{ij}) is a Cartan matrix of a simple finite-dimensional Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra, $\Pi = \{\alpha_i,$

$1 \le i \le N\}$ the corresponding root system, and $\Pi^\vee = \{\alpha_i^\vee, 1 \le i \le N\}$ the corresponding coroot system. $(\mathfrak{h}, \Pi, \Pi^\vee)$ is called a realization of $\mathfrak{g}$ [7]. There exists a nondegenerate symmetric bilinear $\mathbb{C}$-valued form $(.,.)$ on $\mathfrak{h}$ satisfying

$$\left(\alpha_i^\vee, h\right) = \langle h, \alpha_i\rangle d_i^{-1} \qquad \text{for } h \in \mathfrak{h}, \quad i = 1, \dots, N.$$

Here $\langle .,. \rangle$ is the natural pairing between $\mathfrak{h}^*$ and $\mathfrak{h}$. Since $(.,.)$ is nondegenerate, there is an isomorphism μ: $\mathfrak{h} \to \mathfrak{h}^*$ defined by

$$\langle h_1, \mu(h)\rangle = (h_1, h) \qquad (h_1, h \in \mathfrak{h}).$$

This isomorphism induces a symmetric (nondegenerate) bilinear form $(.,.)$ on $\mathfrak{h}^*$. Thus we have

$$\mu\left(\alpha_i^\vee\right) = d_i^{-1}\alpha_i, \qquad i = 1, \dots, N,$$

and

$$(\alpha_i, \alpha_j) = d_i a_{ij} = d_j a_{ji}.$$

Let ρ be the element of $\mathfrak{h}^*$ defined by

$$\langle \alpha_i^\vee, \rho\rangle = 1, \qquad i = 1, \dots, N.$$

Then ρ satisfies $(\rho, \alpha_i) = d_i$. We define also the fundamental reflexions r_i, $1 \le i \le n$, of $\mathfrak{h}^*$ by

$$r_i(\lambda) = \lambda - \langle \alpha_i^\vee, \lambda\rangle \qquad (\lambda \in \mathfrak{h}^*).$$

In particular, $r_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$.

Gaussian Binomial Coefficients

Let q be an indeterminate. For $n \in \mathbb{Z}$, $d \in \mathbb{N}$, we define the q-integer $[n]_d$ by

$$[n]_d = \frac{q^{dn} - q^{-dn}}{q^d - q^{-d}}.$$

If $d = 1$, we denote it simply by $[n]$, and we have $[n]_d = [nd]/[d]$. We also set

$$[n]_d! = \prod_{s=1}^{n} [s]_d.$$

And we define the q-binomial coefficients

$$\begin{bmatrix} n \\ j \end{bmatrix}_d = \frac{[d]_d \cdots [n-j+1]_d}{[j]_d!} \qquad \text{for } j \in \mathbb{Z}, \quad j \le n,$$

and

$$\begin{bmatrix} n \\ j \end{bmatrix}_d = 0 \qquad \text{if } j > n.$$

We have

$$\begin{bmatrix} n \\ j \end{bmatrix}_d = (-1)^j \begin{bmatrix} -n+j-1 \\ j \end{bmatrix}_d$$

and

$$\prod_{s=0}^{n-1} \left(1 + q^{2sd} z\right) = \sum_{j=0}^{n} q^{dj(n-1)} \begin{bmatrix} n \\ j \end{bmatrix}_d z^j \qquad (n \geq 0), \tag{4.1}$$

where z is another indeterminate. It follows that $[{n \atop j}]_d \in \mathbb{Z}[q, q^{-1}]$.

If m and n are in $\mathbb{Z}$ and $j \in \mathbb{N}$, then

$$\begin{bmatrix} m+n \\ j \end{bmatrix}_d = \sum_{k+l=j} q^{d(ml-nk)} \begin{bmatrix} m \\ k \end{bmatrix}_d \begin{bmatrix} n \\ l \end{bmatrix}_d.$$

By putting $z = -1$ in (4.1) and using $[n]_d = [n]_{-d}$ for integer n, we obtain

$$\sum_{j=0}^{n} q^{dj(1-n)} \begin{bmatrix} n \\ j \end{bmatrix}_d = 0. \tag{4.2}$$

The Drinfeld–Jimbo Algebra $\mathscr{U}_q$ [2], [6]

We assume that q is a generic complex number, and we set $q_i = q^{d_i}$. We consider the algebra $\mathscr{U}_q$ defined by the generators E_i, F_i, $K_i^{\pm 1}$ $(1 \leq i \leq n)$ and the relations

$$K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \tag{4.3}$$

$$K_i E_j = q_i^{a_{ij}} E_j K_i, \quad K_i F_j = q_i^{-a_{ij}} F_j K_i, \tag{4.4}$$

$$\left[E_i, F_j\right] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \tag{4.5}$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} E_i^{1-a_{ij}-s} E_j E_i^s = 0 \qquad \text{if } i \neq j, \tag{4.6}$$

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} F_i^{1-a_{ij}-s} F_j F_i^s = 0 \qquad \text{if } i \neq j. \tag{4.7}$$

The last two relations are referred to as the Serre relations. There is a unique algebra involution $\omega: \mathscr{U}_q \to \mathscr{U}_q$ such that $\omega(E_i) = F_i$, $\omega(F_i) = E_i$, $\omega(K_i) = K_i^{-1}$.

The associative algebra $\mathscr{U}_q$ has a Hopf algebra structure given by [9]:

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i,$$

$$\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,$$

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta\left(K_i^{-1}\right) = K_i^{-1} \otimes K_i^{-1},$$

$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = \varepsilon\left(K_i^{-1}\right) = 1,$$

$$A(E_i) = -K_i^{-1}E_i, \quad A(F_i) = -F_iK_i, \quad A\left(K_i^{\pm 1}\right) = K_i^{\mp 1}.$$

For $i = 1 \dots N$.

4.2. *Differentials and Operators*

The Maps ∂_i and ${}_i\partial$

Following [9], we let $\mathfrak{f}'$ be the free algebra with $\mathbf{1}$ generated by the F_i's. Let $\mathbb{Z}[\Pi]$ be the root lattice and $\mathbb{N}[\Pi]$ be the submonoid of $\mathbb{Z}[\Pi]$ of all linear combinations of elements of Π with coefficients in $\mathbb{N}$. For any $\alpha = \sum a_i \alpha_i$ in $\mathbb{N}[\Pi]$, we denote by $\mathfrak{f}'_\alpha$ the subalgebra of $\mathfrak{f}'$ spanned by monomials $F_{i_1}F_{i_2} \cdots F_{i_r}$ such that for any i, the number of occurrences of i in the sequence $i_1, i_2, \dots, i_r$ is equal to a_i. Each $\mathfrak{f}'_\alpha$ is a finite-dimensional vector space, and we have a direct sum decomposition $\mathfrak{f}' = \bigoplus_\alpha \mathfrak{f}'_\alpha$, where α runs over $\mathbb{N}[\Pi]$. We also have $\mathfrak{f}'_\alpha \mathfrak{f}'_{\alpha'} \subset \mathfrak{f}'_{\alpha+\alpha'}$, $\mathbf{1} \in \mathfrak{f}'_0$, and $F_i \in \mathfrak{f}'_{\alpha_i}$. An element $x \in \mathfrak{f}'$ is said to be homogeneous if it belongs to $\mathfrak{f}'_a$ for some a; we then set $|x| = a$.

We denote by ${}_i\partial$ the linear map ${}_i\partial: \mathfrak{f}' \to \mathfrak{f}'$ such that

$${}_i\partial(1) = 0, \quad {}_i\partial\left(F_j\right) = \delta_{i,j} \qquad \text{for all } j$$

and

$${}_i\partial(xy) = {}_i\partial(x)y + q^{(|x|,\,\alpha_i)}x\,{}_i\partial(y)$$

for all homogeneous x, y. Similarly, we denote by ∂_i the linear map $\partial_i: \mathfrak{f}' \to \mathfrak{f}'$ such that

$$\partial_i(1) = 0, \quad \partial_i\left(F_j\right) = \delta_{i,j} \qquad \text{for all } j$$

and

$$\partial_i(xy) = q^{(|y|,\,\alpha_i)}\partial_i(x)y + x\partial_i(y)$$

for all homogeneous x, y. These maps are examples of the so-called Kashiwara operators [9]. If $x \in \mathfrak{f}'_\alpha$, then ${}_i\partial(x)$ and $\partial_i(x)$ are in $\mathfrak{f}'_{\alpha-\alpha_i}$ if $a_i \geq 1$ and ${}_i\partial(x) = \partial_i(x) = 0$ if $a_i = 0$.

In [9] it is shown that the maps ${}_i\partial$ and ∂_i stabilize the radical $\mathscr{I}$ of a certain bilinear inner product on $\mathfrak{f}'$; this radical turns out to contain (even to be generated by) the Serre relations. Therefore they are also defined on the quotient $\mathfrak{f}'/\mathscr{I}$. Here we will check directly that ${}_i\partial$ and ∂_i conserve the Serre relations, because the inner product will not be of any use in this section.

PROPOSITION 4.1. *For k, i, and j in $\{1, \dots, N\}$, $i \neq j$, we have*

$$ {}_k\partial\left(\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} F_i^{1-a_{ij}-s} F_j F_i^s \right) = 0, \tag{4.8} $$

$$ \partial_k\left(\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} F_i^{1-a_{ij}-s} F_j F_i^s \right) = 0. \tag{4.9} $$

Proof. A simple induction shows that

$$ {}_k\partial\left(F_j^n\right) = \partial_k\left(F_j^n\right) = \delta_{kj} q_j^{n-1} [n]_j F_j^{n-1} \qquad (n \in \mathbb{N}). $$

It follows that for $k \neq i$ and $k \neq j$, the proposition is clear. If $k = i$:

$$ \begin{aligned} &{}_i\partial\left(F_i^{1-a_{ij}-s} F_j F_i^s\right) \\ &\quad = q^{(|F_i^s|,\,\alpha_i)}\,{}_i\partial\left(F_i^{1-a_{ij}-s}\right) F_i^s + F_i^{1-a_{ij}-s} F_{j\,i}\partial\left(F_i^s\right) \\ &\quad = q^{(|F_i^s|,\,\alpha_i)+(|F_j|,\,\alpha_i)}\,{}_k\partial\left(F_i^{1-a_{ij}-s}\right) F_j F_i^s + q_i^{s-1}[s]_i F_i^{1-a_{ij}-s} F_j F_i^{s-1}. \end{aligned} $$

Since $|F_i^s| = s\alpha_i$, $(\alpha_i, \alpha_i) = 2d_i$, $|F_j| = \alpha_j$, and $(\alpha_j, \alpha_i) = d_i a_{ij}$, we have

$$ \begin{aligned} &{}_i\partial\left(F_i^{1-a_{ij}-s} F_j F_i^s\right) \\ &\qquad = q_i^s \left[1 - a_{ij} - s\right]_i F_i^{-a_{ij}-s} F_j F_i^s + q_i^{s-1}[s]_i F_i^{1-a_{ij}-s} F_j F_i^{s-1}. \end{aligned} $$

At this point we set $a = 1 - a_{ij}$, and we have to show that

$$ \sum_{s=0}^{a} (-1)^s \begin{bmatrix} a \\ s \end{bmatrix}_{d_i} \left(q_i^s [a-s]_i F_i^{a-s-1} F_j F_i^s + q_i^{s-1}[s]_i F_i^{a-s} F_j F_i^{s-1} \right) = 0. $$

The left-hand side is equal to

$$\begin{aligned}
&\sum_{s=0}^{a-1}(-1)^s\begin{bmatrix} a \\ s\end{bmatrix}_{d_i}[a-s]_i q_i^s F_i^{a-s-1}F_jF_i^s \\
&\quad+\sum_{s=1}^{a}(-1)^s\begin{bmatrix} a \\ s\end{bmatrix}_{d_i}[s]_i q_i^{s-1} F_i^{a-s}F_jF_i^{s-1} \\
&\quad=\sum_{s=0}^{a-1}(-1)^s\begin{bmatrix} a \\ s\end{bmatrix}_{d_i}[a-s]_i q_i^s F_i^{a-s-1}F_jF_i^s \\
&\qquad-\sum_{s=0}^{a-1}(-1)^s\begin{bmatrix} a \\ s+1\end{bmatrix}_{d_i}[s+1]_i q^s F_i^{a-s-1}F_jF_i^s
\end{aligned}$$

which is equal to 0 because $[{a \atop s}]_{d_i}[a-s]_i = [{a \atop s+1}]_{d_i}[s+1]_i$.

If $k = j$:

$$\begin{aligned}
{}_j\partial\left(F_i^{1-a_{ij}-s}F_jF_i^s\right) &= q^{(|F_i^s|,\,\alpha_j)}\partial_j\left(F_i^{1-a_{ij}-s}F_j\right)F_i^s \\
&= q_j^{sa_{ij}}F_i^{1-a_{ij}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&{}_j\partial\left(\sum_{s=0} 1-a_{ij}(-1)^s\begin{bmatrix} 1-a_{ij} \\ s\end{bmatrix}_{d_i} F_i^{1-a_{ij}-s}F_jF_i^s\right) \\
&\qquad=\left(\sum_{s=0}^{1-a_{ij}}(-1)^s\begin{bmatrix} 1-a_{ij} \\ s\end{bmatrix}_{d_i} q^{sa_{ij}}\right)F_i^{1-a_{ij}},
\end{aligned}$$

which is equal to 0 by (4.2). This proves (4.8). The relation (4.9) is obtained in the same way. ∎

COROLLARY 4.2. *The maps ∂_i and ${}_i\partial$ extend to well-defined linear maps on the algebra $\mathfrak{f}$ generated by F_i $(1 \le i \le N)$ satisfying the Serre relations.*

The following proposition will be useful in all that follows.

PROPOSITION 4.3. *For $x \in \mathfrak{f}$ homogeneous and $1 \le i \le N$, we have*

$$K_i x = q^{-(|x|,\,\alpha_i)}xK_i \tag{4.10}$$

$$E_i x = xE_i + \frac{K_i\,{}_i\partial(x) - \partial_i(x)K_i^{-1}}{q_i - q_i^{-1}} \tag{4.11}$$

Proof. The relation (4.10) is clear for $x = 1$, and for $x = F_j$ it follows from (4.4), since $(|x|, \alpha_i) = d_i a_{ij}$ and $q^{(|x|, \alpha_i)} = q_i^{a_{ij}}$. Assume that (4.10) is true for homogeneous x' and x'' in $\mathfrak{f}$; then

$$\begin{aligned} K_i(x'x'') &= q^{-(|x'|, \alpha_i)} x' K_i x'' \\ &= q^{-(|x'|+|x''|, \alpha_i)} x' x'' K_i \\ &= q^{-(|x'x''|, \alpha_i)} x' x'' K_i \qquad \text{since } |x'x''| = |x'| + |x''|. \end{aligned}$$

Therefore (4.10) is true for any homogeneous $x \in \mathfrak{f}$. The relation (4.11) is a version of Proposition 3.1.6 in [9]. We will prove it using (4.10). For $x = 1$ it is clear, and for $x = F_j$ it follows from (4.5), since $\partial_i(x) = {}_i\partial(x) = \delta_{ij}$ and

$$\frac{1}{q_i - q_i^{-1}} \left(K_{ii}\partial(x) - \partial_i(x) K_i^{-1} \right) = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}.$$

Assume that (4.11) is true for homogeneous x' and x'' in $\mathfrak{f}$; then

$$\begin{aligned} &E_i x' x'' - x' x'' E_i \\ &\quad = x' E_i x'' + \frac{1}{q_i - q_i^{-1}} \left(K_{ii}\partial(x') - \partial_i(x') K_i^{-1} \right) x'' - x' x'' E_i \\ &\quad = \frac{1}{q_i - q_i^{-1}} x' \left(K_{ii}\partial(x'') - \partial_i(x'') K_i^{-1} \right) \\ &\qquad + \frac{1}{q_i - q_i^{-1}} \left(K_{ii}\partial(x') - \partial_i(x') K_i^{-1} \right) x'' \\ &\quad = \frac{1}{q_i - q_i^{-1}} \Big(q^{(|x'|, \alpha_i)} K_i x'_{i}\partial(x'') - x' \partial_i(x'') K_i^{-1} \\ &\qquad\qquad + K_{ii}\partial(x') x'' - q^{(|x''|, \alpha_i)} \partial_i(x') x'' K_i^{-1} \Big) \\ &\quad = \frac{K_{ii}\partial(x'x'') - \partial_i(x'x'') K_i^{-1}}{q_i - q_i^{-1}}. \end{aligned}$$

Therefore (4.11) is true for $x'x''$. This completes the proof. ∎

Remark 4.1. Since ∂_i and ${}_i\partial$ are linear, the relation (4.11) is in fact valid for every $x \in \mathfrak{f}$.

The Operators $V_i(z)$ and $V_i(x, z)$

The notations used here are those of the previous subsection. Fix $\lambda \in \mathfrak{h}^*$ and let $\mathscr{M}(\lambda)$ denote the Verma module over $\mathscr{U}_q$ of highest weight $\lambda - \rho$ and highest weight vector v_λ satisfying, for $i = 1, \ldots, N$,

$$E_i v_\lambda = 0,$$

$$K_i v_\lambda = q^{(\lambda-\rho,\, \alpha_i)} v_\lambda = q_i^{\langle \lambda-\rho,\, \alpha_i^\vee \rangle} v_\lambda.$$

The fundamental reflexions r_i of $\mathfrak{h}^*$ were defined by $r_i\lambda = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$. We fix an index $i \in \{1, \ldots, N\}$ for the remainder of this section, and we consider $\lambda' = r_i\lambda$. For $j = 1, \ldots, N$, we introduce the following q-numbers associated with an integer n:

$$\mu_{ij}^{\pm}(n) = \frac{q_i^{\pm a_{ij}(\langle \lambda, \alpha_i^\vee \rangle - n)} - 1}{\left[\langle \lambda, \alpha_i^\vee \rangle - n\right]_i} \qquad \text{if } n \neq \langle \lambda, \alpha_i^\vee \rangle,$$

$$\mu_{ij}^{\pm}(n) = \left(q_i - q_i^{-1}\right) \frac{q_i^{\pm a_{ij}} - 1}{2} \qquad \text{if } n = \langle \lambda, \alpha_i^\vee \rangle,$$

and

$$\eta_{ij}(n) = \frac{\left[\left(-n + \langle \lambda, \alpha_i^\vee \rangle\right) a_{ji}\right]_j}{\left[-n + \langle \lambda, \alpha_i^\vee \rangle\right]_i} q_j^{1 - \langle \lambda, \alpha_j^\vee \rangle + \langle \lambda, \alpha_i^\vee \rangle a_{ji}}.$$

We have

$$\eta_{ij}(n) = q_j^{1 - \langle \lambda, \alpha_j^\vee \rangle + \langle \lambda, \alpha_i^\vee \rangle a_{ji}} \frac{\mu_{ij}^+ - \mu_{ij}^-}{q_j - q_j^{-1}}$$

and

$$\lim_{q \to 1} \eta_{ij}(n) = \lim_{q \to 1} \frac{\mu_{ij}^+ - \mu_{ij}^-}{q_j - q_j^{-1}} = a_{ji}.$$

We treat these expressions as deformations of the entries of the matrix $(a_{ij})_{i,j}$. For each $n \geq 0$, we define an operator $V_{i,n}\colon \mathscr{M}(\lambda') \to \mathscr{M}(\lambda)$ by

$$V_{i,n}(x v_{\lambda'}) = x F_i^n v_\lambda \qquad (x \in \mathfrak{f}).$$

And for each x, a generator of $\mathscr{U}_q$, we define the operator

$$V_{i,n}(x)\colon \mathscr{M}(\lambda') \to \mathscr{M}(\lambda),$$

given by

$$V_{i,n}(E_j)(xv_{\lambda'}) = \eta_{ij}(n)\,\partial_j(x)F_i^n v_\lambda + x\,{}_j\partial(F_i^n)v_\lambda,$$

$$= \eta_{ij}(n)\,\partial_j(x)F_i^n v_\lambda + \delta_{ij}[n]_j x F_i^{n-1}v_\lambda,$$

$$V_{i,n}(K_j^{\pm})(xv_{\lambda'}) = \mu_{ij}^{\pm}(n)\,xF_i^n v_\lambda,$$

$$V_{i,n}(F_j) = 0,$$

for $1 \le j \le N$.

PROPOSITION 4.4. *For* $x = E_j,\ F_j,\ K_j^{\pm}$ $(1 \le i \le N)$, *one has*

$$\langle x, V_{i,n}\rangle = \left[-n + \langle \lambda, \alpha_i^\vee\rangle\right]_i V_{i,n}(x). \tag{4.12}$$

Proof. If x is homogeneous in $\mathfrak{f}$, one has

$$\langle E_j, V_{i,n}\rangle(xv_{\lambda'}) = E_j V_{i,n}(xv_{\lambda'}) - K_j V_{i,n}\left(K_j^{-1}E_j x v_{\lambda'}\right) \tag{4.13}$$

and

$$E_j V_{i,n}(xv_{\lambda'}) = E_j x F_i^n v_\lambda = \frac{1}{q_j - q_j^{-1}}\left(K_j\,{}_j\partial(xF_i^n) - \partial_j(xF_i^n)K_j^{-1}\right)v_\lambda,$$

using (4.11).

By definition of ∂_j and ${}_j\partial$, this is equal to

$$\frac{1}{q_j - q_j^{-1}}\Big(K_j\,{}_j\partial(x)F_i^n + q^{(|x|,\alpha_j)}K_j x\,{}_j\partial(F_i^n)$$

$$-q^{(|F_i^n|,\alpha_j)}\partial_j(x)F_i^n K_j^{-1} - x\,\partial_j(F_i^n)K_j^{-1}\Big)v_\lambda.$$

The second and the fourth terms give

$$\frac{1}{q_j - q_j^{-1}}\left(\delta_{ij}[n]_j q_j^{n-1} x K_j F_i^{n-1}v_\lambda - q^{-\langle\lambda-\rho,\alpha_j^\vee\rangle}\delta_{ij}[n]_j q_j^{n-1}xF_j^{n-1}v_\lambda\right)$$

$$= \frac{1}{q_j - q_j^{-1}}\Big(q_j^{-2(n-1)+\langle\lambda-\rho,\alpha_j^\vee\rangle}\delta_{ij}[n]_j q_j^{n-1}xF_i^{n-1}v_\lambda$$

$$-q_j^{-\langle\lambda-\rho,\alpha_j^\vee\rangle}\delta_{ij}[n]_j q_j^{n-1}xF_j^{n-1}v_\lambda\Big)$$

$$= \frac{\delta_{ij}}{q_j - q_j^{-1}}[n]_j\left(q_j^{-(n-1)+\langle\lambda-\rho,\alpha_i^\vee\rangle} - q_j^{n-1-\langle\lambda-\rho,\alpha_i^\vee\rangle}\right)xF_i^{n-1}v_\lambda$$

$$= \left[-n + \langle\lambda,\alpha_i^\vee\rangle\right]_i\delta_{ij}[n]_j xF_i^{n-1}v_\lambda.$$

Using (4.10), the second term in (4.13) gives

$$
\begin{aligned}
&K_j V_{i,n}\left(K_j^{-1}E_j\right)xv_{\lambda'}\\
&\qquad = \frac{1}{q_j - q_j^{-1}} K_j V_{i,n}\left({}_j\partial(x) - K_j^{-1}\partial_j(x)K_j^{-1}\right)v_{\lambda'}\\
&\qquad = \frac{1}{q_j - q_j^{-1}}\left(K_{jj}\partial(x)F_i^n v_\lambda - q_j^{(|\partial_j(x)|,\,\alpha_j) - 2\langle \lambda' - \rho,\, \alpha_j^\vee\rangle} K_j V_{i,n}\partial_j(x)v_{\lambda'}\right)\\
&\qquad = \frac{1}{q_j - q_j^{-1}}\left(K_{jj}\partial(x)F_i^n v_\lambda - q_j^{-2\langle \lambda' - \rho,\, \alpha_j^\vee\rangle + \langle \lambda - \rho,\, \alpha_j^\vee\rangle}\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad \left.\cdot q^{-(|F_i^n|,\,\alpha_j)}\partial_j(x)F_i^n v_\lambda\right)\\
&\qquad = \frac{1}{q_j - q_{ij}^{-1}}\left(K_{jj}\partial(x)F_i^n v_\lambda - q_j^{1-\langle \lambda,\, \alpha_j^\vee\rangle + 2\langle \lambda,\, \alpha_i^\vee\rangle a_{ji} - n a_{ji}}\partial_j(x)F_i^n v_\lambda\right),
\end{aligned}
$$

where we have used the fact that if x is homogeneous, then ${}_j\partial(x)$ and $\partial_j(x)$ are also homogeneous; the fact that $(|F_i^n|, \alpha_j) = n(\alpha_i, \alpha_j) = nd_i a_{ij} = nd_j a_{ji}$; and that

$$
\langle \lambda', \alpha_j^\vee\rangle = \langle \lambda, \alpha_j^\vee\rangle - \langle \lambda, \alpha_i^\vee\rangle\langle \alpha_i, \alpha_j^\vee\rangle = \langle \lambda, \alpha_j^\vee\rangle - a_{ji}\langle \lambda, \alpha_i^\vee\rangle.
$$

Summing both terms of (4.13), we get

$$
\langle E_j, V_{i,n}\rangle(xv_{\lambda'}) = \left[-n + \langle \lambda, \alpha_i^\vee\rangle\right]_i\left(\delta_{ij}[n]_j x F_i^{n-1} + \eta_{ij}(n)\partial_j(x)F_i^n\right)v_\lambda.
$$

This proves the proposition for $x = E_j$. For $x = F_j$, the proposition is clear, since $\langle F_j, V_{i,n}\rangle = 0$. And for $x = K_j$ we have

$$
\begin{aligned}
\langle K_j, V_{i,n}\rangle(xv_{\lambda'}) &= K_j V_{i,n} K_j^{-1} x v_{\lambda'} - \varepsilon(K_j)V_{i,n}(xv_{\lambda'})\\
&= q^{(|x|,\,\alpha_j)} q_j^{-\langle \lambda' - \rho,\, \alpha_j^\vee\rangle} K_j x F_i^n v_\lambda - xF_i^n v_\lambda\\
&= \left(q_j^{a_{ji}(\langle \lambda,\, \alpha_i^\vee\rangle - n)} - 1\right) xF_i^n v_\lambda\\
&= \left[-n + \langle \lambda, \alpha_i^\vee\rangle\right]_i V_{i,n}\left(K_j^{\pm}\right)(xv_{\lambda'}).
\end{aligned}
$$

The case $x = K_j^{-1}$ is similar. ∎

A Cocycle Condition

Having defined the operators $V_{i,n}(\cdot)$ for the generators E_j, F_j, $K_j^{\pm}$, we can also define them for any element of the free associative algebra generated by E_j, F_j, and $K_j^{\pm}$ $(1 \leq i \leq N)$. Indeed, assume this is done for two elements x and y; then using (2.7), we have

$$\langle xy, V_{i,n}\rangle = x \cdot \langle y, V_{i,n}\rangle + \varepsilon(y)\langle x, V_{i,n}\rangle; \tag{4.14}$$

therefore, if we set

$$V_{i,n}(xy) = x \cdot V_{i,n}(y) + \varepsilon(y)V_{i,n}(x),$$

then

$$\langle xy, V_{i,n}\rangle = \left[-n + \langle \lambda, \alpha_i^{\vee}\rangle\right]_i V_{i,n}(xy).$$

It remains to extend this construction to the algebra $\mathscr{U}_q$.

PROPOSITION 4.5. *For any x in $\mathscr{U}_q$, there is an operator $V_{i,n}(x)$ from $\mathscr{M}(\lambda')$ to $\mathscr{M}(\lambda)$ for any nonnegative integer n satisfying*

$$\langle x, V_{i,n}\rangle = \left[-n + \langle \lambda, \alpha_i^{\vee}\rangle\right]_i V_{i,n}(x).$$

Moreover, for y in $\mathscr{U}_q$, we have

$$V_{i,n}(xy) = x \cdot V_{i,n}(y) + \varepsilon(y)V_{i,n}(x).$$

Proof. As in the previous section, we need to show that the cocycle condition (4.14) leaves the defining relations of $\mathscr{U}_q$ invariant.

We have

$$V_{i,n}(K_k K_l) = K_k V_{i,n}(K_l) K_k^{-1}.$$

Hence

$$\begin{aligned} V_{i,n}(K_k K_l)(xv_\lambda) &= \left(\mu_{il}^{+}(n) q_k^{-\langle \lambda', \alpha_k^{\vee}\rangle + \langle \lambda, \alpha_k^{\vee}\rangle} q^{-n(\alpha_i, \alpha_k)} + \mu_{ik}^{+}(n)\right) x F_i^n v_\lambda \\ &= \frac{q_i^{(-n+\langle \lambda, \alpha_i^{\vee}\rangle)(a_{ik}+a_{il})} - 1}{\left[-n + \langle \lambda, \alpha_i^{\vee}\rangle\right]_i} x F_i^n v_\lambda, \end{aligned}$$

which is symmetric in k and l. Therefore, $V_{i,n}(K_k K_l) = V_{i,n}(K_l K_k)$. For the relation $K_l K_l^{-1} = K_l^{-1} K_l$, it is straightforward (see Section 2). This shows the compatibility with (4.3). For the relation (4.6) we have

$$V_{i,n}(K_k F_l) = K_k \cdot V_{i,n}(F_l) + \varepsilon(F_l) V_{i,n}(K_k) = 0,$$

and for x homogeneous,

$$\begin{aligned} V_{i,n}(F_l K_k)(xv_{\lambda'}) &= (F_l \cdot V_{i,n}(K_k))(xv_{\lambda'}) \\ &= (F_l V_{i,n}(K_k)K_k - V_{i,n}(K_k)F_l K_k)(xv_{\lambda'}) \\ &= q^{-(|x|,\alpha_k)+(\alpha_k,\lambda'-\rho)}(\mu_{ik}^+(n)F_l x F_i^n v_\lambda - \mu_{ik}^+(n)F_l x F_i^n v_\lambda) \\ &= 0. \end{aligned}$$

Also for x homogeneous we have

$$\begin{aligned} & V_{i,n}(E_k F_l - F_l E_k)(xv_{\lambda'}) \\ &\quad = -(F_l \cdot V_{i,n}(E_k))(xv_{\lambda'}) \\ &\quad = -(F_l V_{i,n}(E_k)K_l - V_{i,n}(E_k)F_l K_l)xv'_\lambda \\ &\quad = -q^{-(|x|,\alpha_l)+(\lambda'-\rho,\alpha_l)}(F_l V_{i,n}(E_k)xv_{\lambda'} - V_{i,n}(E_k)F_l xv_{\lambda'}). \end{aligned}$$

The expression between parentheses is equal to

$$\begin{aligned} &\eta_{ik}(n)F_l\partial_k(x)F_i^n v_\lambda + \delta_{ik}[n]_k F_l x F_i^{n-1} v_\lambda \\ &\qquad - \eta_{ik}(n)\partial_k(F_l x)F_i^n v_\lambda - \delta_{ik}[n]_k F_l x F_i^{n-1} v_\lambda. \end{aligned}$$

Since $\partial_k(F_l x) = q^{(|x|,\alpha_k)}\delta_{kl}x + F_l\partial_k(x)$, we obtain

$$V_{i,n}(E_k F_l - F_l F_k)(xv_{\lambda'}) = \delta_{kl}q^{(\lambda'-\rho,\alpha_l)}\eta_{ik}(n)xF_i^n v_\lambda.$$

On the other hand,

$$\begin{aligned} \delta_{kl}V_{i,n}\left(\frac{K_k - K_k^{-1}}{q_k - q_k^{-1}}\right)(xv_{\lambda'}) &= \frac{\delta_{kl}}{q_k - q_k^{-1}}\left(V_{i,n}(K_k)xv_{\lambda'} - V_{i,n}\left(K_k^{-1}\right)xv_{\lambda'}\right) \\ &= \frac{\delta_{kl}}{q_k - q_k^{-1}}(\mu_{ik}^+(n) - \mu_{ik}^-(n))xF_i^n v_\lambda \\ &= \delta_{kl}q^{(\lambda'-\rho,\alpha_k)}\eta_{ik}(n)xF_i^n v_\lambda, \end{aligned}$$

which shows the compatibility with the relation (4.5). The verification for the remaining relations is done using the same calculations. ∎

For any $\mathscr{U}_q$-module $\mathscr{M}$, we define a shifted differential,

$$d_{\lambda,i}\colon \mathscr{M}[[z^{-1}]] \to \mathscr{M}[[z^{-1}]]\frac{dz}{z},$$

given linearly by

$$d_{\lambda,i}(z^{-n}) = [-n + \langle\lambda, \alpha_i^\vee\rangle]_i z^{-n}\frac{dz}{z}.$$

Let us consider the operators

$$V_i(z) = \sum_{n=0}^{\infty} V_{i,n} z^{-n-1} dz \in \operatorname{Hom}\big(\mathscr{M}(r_i\lambda), \mathscr{M}(\lambda)\big)\big[\big[z^{-1}\big]\big] \frac{dz}{z}$$

and

$$V_i(x,z) = \sum_{n=0}^{\infty} V_{i,n}(x) z^{-n} \in \operatorname{Hom}\big(\mathscr{M}(r_i\lambda), \mathscr{M}(\lambda)\big)\big[\big[z^{-1}\big]\big] \qquad \big(x \in \mathscr{U}_q\big).$$

The results of this section can be reformulated in the following:

THEOREM 4.6. *For any x, y in $\mathscr{U}_q$ and $i \in \{1, \dots, N\}$, we have*

$$\langle x, V_i(z)\rangle = d_{\lambda,i} V_i(x,z), \tag{4.15}$$

$$V_i(xy, z) = x \cdot V_i(y,z) + \varepsilon(y) V_i(x,z). \tag{4.16}$$

4.3. *q-de Rham Cocycles*

In this section we fix an element w of the Weyl group of the Lie algebra $\mathfrak{g}$. We assume that $w = r_{i_a} \cdots r_{i_1}$ is a reduced decomposition of w into fundamental reflexions. We also fix an element $\lambda \in \mathfrak{h}^*$, and we consider the following elements of $\mathfrak{h}^*$:

$$\lambda_p = r_{i_{p-1}} \cdots r_{i_1} \lambda \qquad (1 \le p \le a).$$

Let $\mathscr{A} = \mathbb{C}[[z_1^{-1}, \dots, z_a^{-1}]]$, and let Ω^p, $1 \le p \le a$, be the free $\mathscr{A}$-module generated by $\{dz_{j_1}/z_{j_1} \wedge \cdots \wedge dz_{j_p}/z_{j_p}, 1 \le j_1 < \cdots < j_p \le a\}$. We consider the sequence of $\mathscr{A}$-modules

$$\Omega^\bullet \colon 0 \to \Omega^0 \to \Omega^1 \to \cdots \to \Omega^a \to 0,$$

and we define a linear map $d''\colon \Omega^k \to \Omega^{k+1}$ as follows.

If

$$\eta = f\big(z_1^{-1}, \dots, z_a^{-1}\big) \frac{dz_{j_1}}{z_{j_1}} \wedge \cdots \wedge \frac{dz_{j_k}}{z_{j_k}}, \tag{4.17}$$

where f is a monomial in $\mathscr{A}$, and if n_p is the exponent of z_p^{-1} in f, then

$$d''\eta = \sum_{p=1}^{a} \big[-n_p + \langle \lambda_p, \alpha_{i_p}^\vee\rangle\big]_{i_p} f\big(z_1^{-1}, \dots, z_a^{-1}\big) \frac{dz_p}{z_p} \wedge \frac{dz_{j_1}}{z_{j_1}} \wedge \cdots \wedge \frac{dz_{j_k}}{z_{j_k}}.$$

Using the notations of the previous subsection, this can be expressed as

$$d''\eta = \sum_{p=1}^{a} d_{\lambda_p, i_p} f\left(z_1^{-1}, \dots, z_a^{-1}\right) \wedge \frac{dz_{j_1}}{z_{j_1}} \wedge \cdots \wedge \frac{dz_{j_k}}{z_{j_k}}.$$

We extend d'' by linearity to any η in Ω^k.

PROPOSITION 4.7. *We have $d''^2 = 0$, so $(\Omega^\bullet, d)$ is a complex.*

Proof. Let $f(z) = f(z_1^{-1}, z_2^{-1}, \dots, z_a^{-1})$ be a monomial, and let η be a form given by (4.17). Then

$$d''\eta = \sum_{p=1}^{a} \left[-n_p + \langle \lambda_p, \alpha_{i_p}^{\vee} \rangle\right]_{i_p} f(z) \frac{dz_p}{z_p} \wedge \frac{dz_{j_1}}{z_{j_1}} \wedge, \dots, \wedge \frac{z_{j_k}}{z_{j_k}}$$

and

$$\begin{aligned} d''^2 = {} & \sum_{p=1}^{a} \left[-n_p + \langle \lambda_p, \alpha_{i_p}^{\vee} \rangle\right]_{i_p} \\ & \cdot \left(\sum_{s=1}^{a} \left[-n_s + \langle \lambda_s, \alpha_{i_s}^{\vee} \rangle\right]_{i_s} f(z) \frac{dz_s}{z_s} \wedge \frac{dz_p}{z_p} \wedge \cdots \wedge \frac{dz_{j_k}}{z_{j_k}} \right) \\ = {} & \left(\sum_{p=1}^{a} \sum_{s=1}^{a} \left[-n_p + \langle \lambda_p, \alpha_{i_p}^{\vee} \rangle\right]_{i_p} \right. \\ & \left. \cdot \left[-n_s + \langle \lambda_s, \alpha_{i_s}^{\vee} \rangle\right]_{i_s} f(z) \frac{dz_s}{z_s} \wedge \frac{dz_p}{z_p} \right) \wedge \frac{dz_{j_1}}{z_{j_1}} \wedge \cdots \wedge \frac{dz_{j_k}}{z_{j_k}} \\ = {} & 0 \end{aligned}$$

∎

For each $p = 1, \dots, a$, we consider the operators defined previously:

$$V_p(z_p) \colon \mathscr{M}\left(\lambda_{p+1}\right) \to \mathscr{M}\left(\lambda_p\right) z_p^{-1}\left[\left[z_p^{-1}\right]\right]$$

and

$$V_p(x, z_p) \colon \mathscr{M}\left(\lambda_{p+1}\right) \to \mathscr{M}\left(\lambda_p\right)\left[\left[z_p^{-1}\right]\right] \qquad (x \in \mathscr{U}_q)$$

We recall from Section 1 that we have the following complex:

$$\operatorname{Hom}\left(\mathscr{U}_q^{\otimes \bullet}, \operatorname{Hom}(\mathscr{M}(w\lambda), \mathscr{M}(\lambda))\right),$$

and the differential d' of this complex is defined on the cochains as follows:

$$\begin{aligned} d'\phi(x_1, \ldots, x_n) &= x_1 \cdot \phi(x_2, \ldots, x_n) \\ &\quad + \sum_{i=1}^{n-1} (-1)^i \phi(x_1, \ldots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \ldots, x_n) \\ &\quad + (-1)^n \phi(x_1, \ldots, x_{n-1}) \varepsilon(x_n). \end{aligned}$$

We consider the double complex

$$\begin{aligned} C^{\bullet\bullet} &= \operatorname{Hom}\Big(\mathscr{U}_q^{\otimes\bullet}, \operatorname{Hom}(\mathscr{M}(w\lambda), \mathscr{M}(\lambda) \otimes \Omega^\bullet)\Big) \\ &\cong \operatorname{Hom}\Big(\mathscr{U}_q^{\otimes\bullet} \otimes \mathscr{M}(w\lambda), \mathscr{M}(\lambda) \otimes \Omega^\bullet\Big). \end{aligned}$$

The first differential for this double complex is d'; the second differential is the shifted de Rham differential d'' defined above. In the following, we will construct an a-cocycle in the simple complex associated with $C^{\bullet\bullet}$.

For $x_1, x_2, \ldots, x_m$ in $\mathscr{U}_q$ and $p_1, p_2, \ldots, p_m$ being an increasing sequence in $\{1, \ldots, a\}$, we define $\mathscr{G}(x_1, \ldots, x_m;\ p_1, \ldots, p_m)$ as equal to the following expression:

$$\begin{aligned} &\sum_{(x)} V_1(z_1) \cdots V_{p_1-1}(z_{p_1-1}) V_{p_1}(x_1', z_{p_1}) x_1'' \\ &\qquad \cdot \Big(V_{p_1+1}(z_{p_1+1}) \cdots V_{p_2-1}(z_{p_2-1}) V_{p_2}(x_2', z_{p_2}) x_2'' \\ &\qquad\quad \cdot \Big(\cdots V_{p_m}(x_m', z_{p_m}) x_m'' \Big(V_{p_m+1}(z_{p_m+1}) \cdots V_{i_a}(z_a)\Big) \cdots \Big)\Big) dz_1 \\ &\qquad \wedge \widehat{dz}_{p_1} \wedge \cdots \wedge \widehat{dz}_{p_m} \wedge \cdots \wedge dz_a, \end{aligned}$$

where ^ means omission, and the summation is over all of the terms involved in the comultiplication of $x_1, \ldots, x_m$ in the Sweedler notation. In each summand, we consider initially the composition $V_1(z_1)V_2(z_2) \cdots V_a(z_a)$, and for each p_k we substitute $V_{p_k}(z_{p_k})$ by $V_{p_k}(x_k', z_{p_k})x_k'' \cdot (\ldots,$ where x_k'' acts on all of the remaining factors to the right, if there are any.

For each m, $0 \le m \le a$, we define the operators

$$V^{m, a-m} \in \operatorname{Hom}\Big(\mathscr{U}_q^{\otimes m}, \operatorname{Hom}(\mathscr{M}(w\lambda), \mathscr{M}(\lambda) \otimes \Omega^{a-m})\Big) \qquad (a)$$

as follows:

$$\begin{aligned} &V^{m, a-m}(x_1, \ldots, x_m) \\ &\quad = (-1)^{m(m+1)/2} \\ &\qquad \cdot \sum_{1 \le p_1 < \cdots < p_m \le a} (-1)^{p_1 + \cdots + p_m} \mathscr{G}(x_1, \ldots, x_m;\ p_1, \ldots, p_m). \end{aligned}$$

To clarify these rather complicated expressions, we will give some examples:

$$V^{0,a} = V_1(z_1)\cdots V_a(z_a)dz_1 \wedge \cdots dz_a.$$

$$\begin{aligned} &V^{a,0}(x_1,\dots,x_n)\\ &\quad = \sum_{(x_1),\dots,(x_{a-1})} V_1(x_1',z_1)x_1''\\ &\qquad \cdot\big(V_2(x_2',z_2)x_2''(\cdots(V_{a-1}(x_{a-1}',z_{a-1})x_{a-1}''\cdot V_a(x_a,z_a))\cdots)\big). \end{aligned}$$

For $a=2$:

$$V^{0,2} = V_1(z_1)V_2(z_2)dz_1\wedge dz_2$$

$$V^{1,1}(x) = \sum_{(x)} \big(V_1(x',z_1)x''\cdot V_2(z_2)dz_2 - V_1(z_1)V_2(x,z_2)dz_1\big)$$

$$V^{2,0}(x_1,x_2) = \sum_{(x_1)} V_1(x_1',z_1)x_1''\cdot V_2(x_2,z_2).$$

For $a=3$:

$$V^{0,3} = V_1(z_1)V_2(z_2)V_3(z_3)dz_1\wedge dz_2\wedge dz_3.$$

$$\begin{aligned} V^{1,2}(x) &= \sum_{(x)} V_1(x',z_1)x''\cdot\big(V_2(z_2)V_3(z_3)\big)dz_2\wedge dz_3\\ &\quad -\sum_{(x)} V_1(z_1)V_2(x',z_2)x''\cdot V_3(z_3)dz_1\wedge dz_3\\ &\quad + V_1(z_1)V_2(z_2)V_3(x,z_3)dz_1\wedge dz_2. \end{aligned}$$

$$\begin{aligned} V^{2,1}(x_1,x_2) &= \sum_{(x_1),(x_2)} V_1(x_1',z_1)x_1''\cdot\big(V_2(x_2',z_2)x_2''\cdot V_3(z_3)\big)dz_3\\ &\quad -\sum_{(x_1)} V_1(x_1',z_1)x_1''\cdot\big(V_2(z_2)V_3(x_2,z_3)\big)dz_2\\ &\quad +\sum_{(x_1)} V_1(z_1)V_2(x_1',z_2)x_1''\cdot V_2(x_2,z_3)dz_1. \end{aligned}$$

$$V^{3,0}(x_1,x_2,x_3) = \sum_{(x_1),(x_2)} V_1(x_1',z_1)x_1''\cdot\big(V_2(x_2',z_2)\big)x_2''\cdot V_3(x_3,z_3).$$

PROPOSITION 4.8. *We have*

(1) $d''V^{0,a} = d'V^{a,0} = 0$.

(2) $d'V^{k,a-k} = (-1)^k d''V^{k+1,a-k-1}$ *for* $k=0,\dots,a-1$.

Proof. We will prove (1); (2) is proved in the same way after changing the right side using the relation (4.15) of Theorem 4.6. The fact that $d''V^{0,a} = 0$ is clear, since $\Omega^\bullet$ is of length a.

By definition of d' we have

$$\begin{aligned} d'V^{a,0}&(x_1, \dots, x_a, x_{a+1}) \\ &= x_1 \cdot V^{a,0}(x_2, \dots, x_{a+1}) && (a) \\ &\quad + \sum_{k=1}^{a} (-1)^k V^{a,0}(x_1, \dots, x_k x_{k+1}, \dots, x_{a+1}) && (b) \\ &\quad + (-1)^{a+1} \varepsilon(x_{a+1}) V^{a,0}(x_1, \dots, x_a). && (c) \end{aligned}$$

Using the composition lemma, we have

$$(a) = \sum_{(x_1), \dots, (x_a)} x_1' \cdot V_1(x_2', z_1) x_1'' \cdot (x_2'' \cdot [\cdots]),$$

where the terms in $[\cdots]$ are the same as in $V^{a,0}(x_2, \dots, x_{a+1})$, except for the first factor. Hence

$$(a) = \sum_{(x_1), \dots, (x_a)} (x_1' \cdot V_1(x_2', z_1))((x_1'' x_2'') \cdot [\cdots]).$$

Using Theorem 4.6, the summand corresponding to k in (b), for $k < a$, is equal to

$$\begin{aligned} &(-1)^k V_1(x_1', z_1) x_1'' \\ &\quad \cdot \big(V_2(x_2', z_2) x_2'' \cdot \cdots x_{k-1}'' \\ &\quad\quad \cdot \big((x_k' \cdot V_k(x_{k+1}', z_k) + \varepsilon(x_{k+1}') V_k(x_k', z_k)) x_k'' x_{k+1}'' \cdot \cdots\big)\big) \end{aligned}$$

Using the counit axiom, this is equal to

$$\begin{aligned} &(-1)^k V_1(x_1', z_1) x_1'' \\ &\quad \cdot \big(V_2(x_2', z_2) x_2'' \cdot \cdots x_{k-1}'' \cdot \big(x_k' \cdot V_k(x_{k+1}', z_k)(x_k'' x_{k+1}'') \cdot (\cdots)\big)\big) \\ &\quad + (-1)^k V_k(x_1', z_1) x_1'' (\cdots x_{k-1} \cdot (V_k(x_k', z_k)(x_k'' x_{k+1}) \cdots)). \end{aligned}$$

Using the composition lemma one more time in the second term, we obtain

$$\begin{aligned} &(-1)^k V_1(x_1', z_1) x_1'' \\ &\quad \cdot \big(V_2(x_2', z_2) x_2'' \cdot \cdots x_{k-1}'' \cdot \big(x_k' \cdot V_k(x_{k+1}', z_k)(x_k'' x_{k+1}'') \cdot (\cdots)\big)\big) \\ &\quad + (-1)^k V_k(x_1', z_1) x_1'' \\ &\quad \cdot (\cdots x_{k-1} \cdot (V_k(x_k', z_k) x_k'' \cdot (x_{k+1}' \cdot (\cdots) x_{k+1}'' \cdot) \cdots)). \end{aligned}$$

The term corresponding to $k = a$ in (b) is

$$(-1)^a V^{a,0}(x_1, \dots, x_a x_{a+1}),$$

which is equal to

$$(-1)^a \sum V_1(x_1', z_1) x_1'' \cdot (\cdots (x_{a-1}'' x_a) \cdot V_{a+1}(x_{a+1}, z_a) \cdots)$$
$$+ (-1)^a \varepsilon(x_{a+1}) V^{a,0}(x_1, \dots, x_a).$$

It follows that all the terms in (b) cancel, except the first and the last, and it is clear that the first term cancels with (a) and the last one cancels with (c). ∎

As a consequence of the preceding results, we have the following:

THEOREM 4.9. *The element* $\mathscr{V} = (V^{0,1}, \dots, V^{a,0})$ *is an a-cocycle in the simple complex associated with the double complex* $C^\bullet(\mathscr{U}_q, \mathrm{Hom}(\mathscr{M}(w\lambda), \mathscr{M}(\lambda) \cap \Omega^\bullet))$.

And we have

THEOREM 4.10. *The cocycle* $\mathscr{V}$ *induces linear maps,*

$$f_m \colon \mathscr{H}^m(\Omega^\bullet)^* \to \mathrm{Ext}_{\mathscr{U}_q}^{a-m}(\mathscr{M}(w\lambda), \mathscr{M}(\lambda)) \qquad 0 \le m \le a.$$

Proof. Let $\phi_m \colon \mathscr{H}^m(\Omega^\bullet) \to \mathbb{C}$ be a linear form. We extend ϕ to the space $\mathscr{Z}^m(\Omega^\bullet)$ of cocycles up to homotopy. We obtain a linear form $\phi_m \colon \mathscr{Z}^m(\Omega^\bullet) \to \mathbb{C}$. Then we extend ϕ_m to Ω^m by taking it to be zero on a complement of $\mathscr{Z}^m$ in Ω^m. Let $x_1, \dots, x_{a-m}$ be elements of $\mathscr{U}_q$, and set

$$\Phi_m = V^{a-m,m}(x_1 \otimes \cdots \otimes x_{a-m}) \in \mathrm{Hom}(\mathscr{M}(w\lambda), \mathscr{M}(\lambda) \otimes \Omega^m).$$

We consider the composition

$$\begin{array}{ccc} \mathscr{M}(w\lambda) & \xrightarrow{\Phi_m} & \mathscr{M}(\lambda) \otimes \Omega^m \\ & & \Big\downarrow {\scriptstyle Id \otimes \phi_m} \\ & & \mathscr{M}(\lambda) \otimes \mathbb{C} \cong \mathscr{M}(\lambda). \end{array}$$

The map obtained is then an element of $\mathrm{Hom}(\mathscr{U}_q^{\otimes a-m}, \mathrm{Hom}(\mathscr{M}(w\lambda), \mathscr{M}(\lambda)))$. Since V is an a-cocycle by the preceding theorem, we have $d' V^{a-m,m} = (-1)^{a-m} d'' V^{a-m+1,m-1}$; therefore $d'((Id \otimes \phi_m) \circ \Phi_m) = 0$. Meanwhile, ϕ_m was chosen up to homotopy, satisfying $d'' \circ \phi_m = 0$; it follows that the resulting map is a cochain in the complex $C^\bullet(\mathscr{U}_q, \mathrm{Hom}(\mathscr{M}(w\lambda), \mathscr{M}(\lambda)))$. And since $d'' V^{a-m,m} = (-1)^{a-m-1} d' V^{a-m-1,m+1}$ (by Proposition 4.8), the class of this cochain modulo coboundaries does not depend on the choice of ϕ_m up to homotopy, because the above relation implies that the image of $\phi_m \circ d''$ is $d'((Id \otimes (\phi_m \circ d'')) \circ \Phi_m)$, and hence is a coboundary. Since the cohomology spaces of the cochain complex $C^\bullet(\mathscr{U}_q, \mathrm{Hom}(\mathscr{M}(w\lambda), \mathscr{M}(\lambda)))$ are the Ext-spaces $\mathrm{Ext}^\bullet_{\mathscr{U}_q}(\mathscr{M}(w\lambda), \mathscr{M}(\lambda))$, the theorem is proved. ∎

To illustrate this theorem, we assume that the numbers $\langle \lambda_p, \alpha_{i_p}^\vee \rangle$, $p = 1, \dots, a$, are nonnegative integers. The homology space $\mathscr{H}^a(\Omega^\bullet)$ is one-dimensional (q is not a root of unity), generated by (the image of) the linear form $r \in \Omega^{a*}$ defined by

$$r(\eta) = \operatorname{Res}_{z_a=0} \cdots \operatorname{Res}_{z_1=0}\Big(z_1^{\langle \lambda_1, \alpha_{i_1}^\vee \rangle} \cdots z_a^{\langle \lambda_a, \alpha_{i_a}^\vee \rangle} \eta \Big).$$

The element $f_a(r)$, in $\operatorname{Hom}_{\mathscr{U}_q}(\mathscr{M}(w\lambda), \mathscr{M}(\lambda))$, is the unique intertwiner between $\mathscr{M}(w\lambda)$ and $\mathscr{M}(\lambda)$, sending $v_{w\lambda}$ to $F_{i_a}^{\langle \lambda_a, \alpha_{i_a}^\vee \rangle + 1} \cdots F_{i_1}^{\langle \lambda_1, \alpha_{i_1}^\vee \rangle + 1} v_\lambda$.

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