Twisted $L$-Functions and Complex Multiplication

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1. INTRODUCTION

The study of algebraicity of special values of the $L$-function of a grossencharacter of an imaginary quadratic field $K$ was first initiated by Eisenstein. His work appeared much later in a different formulation in the work of Birch and Swinnerton-Dyer [1] and Damerell [2]. Shimura generalized this aspect to more general CM fields [7]. Using the same language, Goldstein and Schappacher [4] related the work of Eisenstein and Damerell to the conjecture of Birch and Swinnerton-Dyer on elliptic curves and to the Deligne conjecture.

The purpose of this article is to study, with the same methods, the special values of certain twisted $L$-functions as follows:

Let $E$ be an elliptic curve defined over $\mathbb{Q}$, of conductor $N$, and let $L(E, s)$ be its usual $L$-function. Let us write $L(E, s) = \sum_{n \geq 1} a_n n^{-s}$ (for $\Re(s) > 3/2$).

For every $b \in \mathbb{Z}/N\mathbb{Z}$, we set

$$L(E, b, s) = \sum_{n \geq 1} e^{2\pi i b n^2/N} a_n n^{-s} \quad (\Re(x) > 3/2). \quad (1.1)$$

It seems reasonable to conjecture that the series $L(E, b, s)$ admit an analytic continuation to every $s \in \mathbb{C}$, and we would like to study the values at $s = 1$ of these Dirichlet series. For $b = 0$, this is essentially one aspect of the Birch and Swinnerton-Dyer conjecture. To be more precise, we need a conjecture of the form

$$L(E, b, 1) = \zeta_E \omega^+ + \beta_b \omega^-, \quad (1.2)$$

with $\zeta_E$, $\omega^+$, $\omega^-$ expressed in terms of the arithmetic of $E$, and $b$. The periods $\omega^+$, $\omega^-$ are defined in a modular language. For example, they are the so-called $u^+$ in the article of Shimura [8].
We study the algebraic properties of $L(E, b, 1)$ in the case when $E$ has complex multiplications by $O_K$. There exists a finite number (say $r$) of torsion points $x_i$ of $C/L$ (for a lattice $L$ corresponding to $E$ over $C$) and a simple explicit function $C(E, s)$ (depending on the choice of $x_i$) such that

$$C(E, s) L_p(\chi, s) = \sum_{i=1}^{r} K_1(x_i, 0, s, L),$$

where $\chi$ is the grossencharacter associated to $E$, $L_p$ is a certain partial Hecke $L$-function depending on an integral ideal of $O_K$ relatively prime to $N O_K$, and $K_1$ denotes the Kronecker series. The Hecke $L$-function is recovered by taking the sum of the partial $L$-series corresponding to integral ideals for which the Artin symbols describe the Galois group of the ray class field modulo $N$. Because of the complex multiplication, the ray class field is equal to the extension of $K$ generated by the $N$-torsion points of $E/K$. Our results concerning $L(E, b, 1)$ follow then from the properties of the Hasse–Weil and Hecke $L$-functions and the relationship between the Kronecker and the Eisenstein series.

The next two sections deal with the necessary background concerning Hecke characters, $L$-functions, complex multiplication, and torsion points. The last section presents our study of the values at 1 of the twisted $L$-function.

2. HECKE CHARACTERS AND L-FUNCTIONS

In this section we recall some important properties which will be used in subsequent sections.

Let $F$ be a number field with $O_F$ its ring of integers, and let $\mathfrak{m}$ be an integral ideal of $K$ and $I_{\mathfrak{m}}$ be the group of fractional ideals relatively prime to $\mathfrak{m}$. Let $K$ be another number field in which $F$ can be embedded. If $G$ is the set of embeddings of $F$ in $K$, we denote by $\mathbb{Z}[G]$ the free abelian group over $G$. A Hecke character $\chi$ of the field $F$ with values in $K^*$ having conductor $\mathfrak{m}$ and of type at infinity $(n_\infty)_\chi$ element of $\mathbb{Z}[G]$ is:

1. A homomorphism from the group of fractional ideals of $F$ relatively prime to $\mathfrak{m}$ into $K^*$.

2. For every $\sigma \in F^*$ such that the ideal $\sigma O_F$ is relatively prime to $\mathfrak{m}$ and $\sigma = \beta/\gamma$, with $\beta, \gamma \in O_K$ relatively prime to $\mathfrak{m}$ and $\beta \equiv \gamma \mod \mathfrak{m}$ (we say $\sigma \equiv 1 \mod \mathfrak{m}$), we have

$$\chi(\sigma O_F) = \prod \sigma(x_i)^{n_i} \in K^*.$$
If $\chi_1$ is defined over $I_m$ and $\chi_2$ is defined over $I_n$ and if the two give the same map from $I_m I_n$ to $K^*$, we identify them to one character $\chi$. The conductor of $\chi$ is the smallest ideal $m$ of $O_F$ such that $\chi$ is identified with a Hecke character of $I_m$.

Let $E$ be an elliptic curve defined over a field $F$ with complex multiplications by $O_K$ where $K$ is an imaginary quadratic field. There is an embedding of $K$ into $F$ which we fix, and we look at $F$ as embedded into $\bar{K}$. We define a character attached to these data as a map

$$\chi : I_I \to K^*,$$

where $I$ is an integral ideal which is divisible by all prime ideals at which $E$ has bad reduction, as follows:

If $p$ is a prime ideal in $I$, we consider the embedding

$$\mathbb{Q} \otimes \mathbb{Z} \operatorname{End}(E) \hookrightarrow \mathbb{Q} \otimes \mathbb{Z} \operatorname{End}_\kappa(E_p),$$

where $\kappa_p$ is the residue field of $K$ at $p$. Let

$$\mathcal{F} : E_p \to E_p$$

$$(x, y) \mapsto (x^{1/p}, y^{1/p})$$

be the Frobenius map at $p$. Here $\mathbb{N} = \langle (O_K/pO_K) \rangle$. Then $\mathcal{F}_p$ is in the center of $\operatorname{End}_\kappa(E)$ and hence is in the image of the above embedding. Therefore, there exists $\gamma(p)$ in $\mathbb{Q} \otimes \operatorname{End}_\kappa(E)$ such that the image of $\gamma(p)$ is $\mathcal{F}_p$. Since $E$ has complex multiplications, one can show that the map $\chi : I_I \to K^*$ is a Hecke character (the grossencharacter of $E$). We talk about $N_{FK}$ as the type at infinity of $\chi$ since it acts on a principal fractional ideal $(x)$ of $F^*$ relatively prime to $I$ by

$$\chi(x) = \iota(x) N_{FK}(x),$$

where $\iota$ is a homomorphism of $(O_K/I)^*$ into the groups of units of $O_K$ and $N_{FK}$ is the norm of the extension of $F/K$. In particular, if $x \equiv 1 \mod I$ then $\chi(x) = N_{FK}(x)$. Moreover, $\chi$ is unramified outside the places of bad reduction of $E$.

Let $L(\chi, s)$ denote the Hecke $L$-function attached to $\chi$ and $L(E/F, s)$ denote the Hasse–Weil $L$-function of the elliptic curve $E/F$. From the properties of $\chi$ one can show that

$$L(E/F, s) = L(\chi, s) L(\bar{\gamma}, s),$$

where $\bar{\gamma}$ is the complex conjugate of the character $\chi$. 

From now on we assume that $E$ is defined over $\mathbb{Q}$ and has complex multiplications by $O_K$. Using the fact that $\tilde{\varphi}(p) = \tilde{\varphi}(p)$ for $p$ prime in $O_K$ and comparing the local factors of the $L$-functions according to whether $p$ is ramified, inert, or splits in $O_K$, the relation (2.2) yields

$$L(E/\mathbb{Q}, s) = L(\chi, s).$$

We finish this section by an important relation which will be useful later on. Let $N_E$ be the conductor of $E$, $f$ the conductor of the grossencharacter $\chi$ and $d_K$ the discriminant of the field $K$. Then

$$|d_K|, N = N_E.$$  

This follows from the functional equations satisfied by the $L$-functions and from (2.3).

3. TORSION POINTS OF AN ELLIPTIC CURVE

Let $E/\mathbb{Q}$ be an elliptic curve with complex multiplications by the ring of integers $O_K$ of a quadratic imaginary $K$. We fix an embedding $K \hookrightarrow \mathbb{C}$. There is a lattice $A$ of $\mathbb{C}$ such that we have the Weierstrass isomorphism

$$\tilde{\varphi}: \mathbb{C} \to E(\mathbb{C})$$

$$(z, A) \mapsto (\varphi(z, A), \varphi'(z, A)),$$

where $\varphi$ is the Weierstrass elliptic function. Since $E/\mathbb{Q}$ has complex multiplications, then $A = \Omega \cdot O_K$ for $\Omega \in \mathbb{C}^*$. In fact, we can choose $\Omega$ to be a nonzero real number.

Let $N$ be a nonnegative integer; the $N$-torsion group of $E$ over $\mathbb{C}$ is the subgroup of $E$ given by

$$E_N = \{ \tilde{\varphi}(z, A), z \in N^{-1}A \} \subseteq \mathbb{P}_2(\mathbb{C}).$$

If we add the coordinates of all the points in $E_N$ to $K$, we obtain a finite extension $K(E_N)$ of $K$. If $S$ denotes the set of places of $K$ at which $E$ has bad reduction, then the extension $K(E_N)/K$ is abelian, nonramified outside the places in $K$ dividing $N, O_K$ and the places in $S$. Moreover, if $b$ is an integral ideal of $K$ relatively prime to $N$ and to $S$, and if $\rho \in N^{-1}A/A$, then the action of the grossencharacter $\chi$ on torsion points is given by

$$\tilde{\varphi}(\rho, A) \cdot K(E_N/K) = \tilde{\varphi}(\chi(b) \rho, A),$$

(3.1)
where \((b, K(E_\chi))\) is the Artin symbol of \(b\) and \(\chi\) is the Hecke character attached to \(E\).

Since \(E\) is defined over \(\mathbb{Q}\) with complex multiplications by \(O_K\), the class number is \(h_K = 1\), so that all fractional ideals are principal. Let \(N\) be the conductor of \(E/\mathbb{Q}\) and \(\mathfrak{f}\) the conductor of the grossencharacter of \(E\). A representative \(a\mathcal{O}_K\) of the principal ray modulo \(N\) satisfies \(a \equiv 1 \mod N\), and from (2.4) we deduce that \(N, O_K = [d_K]/\mathfrak{f} \leq \mathfrak{f}\); therefore \(a \equiv 1 \mod \mathfrak{f}\). Hence, any character of \((O_K/\mathfrak{f})^*\) acts trivially on an representative of the principal ray mod \(N\). Using (2.1) and (3.1), we deduce that we have an abelian extension \(K(E_\chi)/K\) nonramified outside \(N\) and such that the Artin symbols of ideals in the ray class group modulo \(N\) act trivially on \(K(E_\chi)\).

From class field theory we have \(K(E_\chi) \subseteq K_N\), where \(K_N\) is the ray class field modulo \(N\). On the other hand, using [6, 5.10, p. 124], we have \(K_N = K(\Phi_K(P), P \in E_N)\) where \(\Phi\) is the Weber function \(\Phi_K(P) = x(p)^i\), with \(x\) being the \(x\)-coordinate and \(i = \frac{1}{2}[O_K^\times]\). Hence \(K_N \subseteq K(E_\chi)\). Therefore, the field \(K(E_\chi)\) is the ray class field mod \(N\).

Let us denote by \(\sigma_a\) the Artin symbol \((a, K(E_\chi)/K)\). Since \(\chi\) is unramified outside \(N\), then, for \(a = (\chi)\) and \(b = (\beta)\), \(\sigma_a = \sigma_b\) if and only if for some unit \(\epsilon\) in \(O_K^\times\) we have \(\epsilon a / b \equiv 1 \mod N\). It follows that

\[
\text{if } \sigma_a = \sigma_b \quad \text{then } \|\rho a \equiv \rho b \mod N. \quad (3.2)
\]

Let \(\mu_N\) be the group of the \(N\)th root of unity. Consider the Weil pairing \(\epsilon_N : E_N \times E_N \to \mu_N\) (see [9]). The set \(\{\epsilon_N(S, T), S, T \in E_N\}\) is a subgroup \(\mu_d\) of \(\mu_N\). It follows that for every \(S\) and \(T\), \(1 = \epsilon_N(S, T)^d = \epsilon_N([d] S, T)\). By the nondegeneracy of the Weil pairing, we must have \([d] S = 0\); i.e., \(S\) is a \(d\)-torsion point. Since \(S\) was arbitrary, it follows that \(d = N\). Moreover, the pairing \(\epsilon_N\) is equivariant under the Galois action. Then for every \(\sigma \in \text{Gal}(K(E_\chi)/K(E_\chi))\), we have \(\epsilon_N(S, T)^\sigma = \epsilon_N(S^\sigma, T^\sigma) = \epsilon_N(S, T)\) since \(S\) and \(T\) are in \(K(E_\chi)\). It follows that \(\epsilon_N(S, T) \in K(E_\chi)\) and therefore \(\mu_N \subseteq K(E_\chi)^*\). Hence, we have the following inclusions

\[
\mathbb{Q} \xrightarrow{2} K \to K(E_\chi)\]
\[
\mathbb{Q} \xrightarrow{\kappa(N)} \mathbb{Q}(\mu_N) \to K(E_\chi).
\]

Since the cyclotomic field \(\mathbb{Q}(\mu_N)\) is the ray class field modulo \(N\) of \(\mathbb{Q}\), then if \(\zeta_N\) is a primitive \(N\)th root of unity, we have

\[
\zeta_{N}^{(a, K(E_\chi)/K)} = \zeta_{N}^{(\sigma_a, \mathbb{Q}(\mu_N)/\mathbb{Q}} = \zeta_{N}^{\rho a}.
\quad (3.3)
\]
4. TWISTED L-FUNCTIONS

Let \( A \) be a lattice in \( \mathbb{C} \). If \((u, v)\) is a \( \mathbb{Z} \)-basis of \( A \) such that \( \Im(u/v) > 0 \) then the real number \((iv - w)/2\pi i\) is positive and independent of the basis; we denote it by \( A(A) \). Let us consider the following homomorphisms of the additive group \( \mathbb{C} \) into the unit circle parametrized by the variable \( z_0 \):

\[
\psi(z, z_0, A) = \exp \left( \frac{z_0 z - z_0\xi}{A(A)} \right).
\]

For \( k \geq 0 \), we define the holomorphic functions on the domain \( \Re(s) > 1 + k/2 \) by:

\[
\mathcal{X}_k(z, z_0, s, A) = \sum_{\omega} \psi(\omega, z_0, A) \frac{(\bar{z} + \bar{\omega})^k}{|\bar{z} + \omega|^{2s}}.
\]

The sum is extended to every \( \omega \) in \( A \) except \( -z \) if \( z \in A \). For \( i \) and \( j \) integers satisfying \( j > i > 0 \) and \( z \in \mathbb{C}/A \) we set

\[
E_{i,j}^s(z, A) = \mathcal{X}_{i+j}(z, 0, j, A)
\]

and

\[
E_{k}^s(z, A) = E_{0,k}^s(z, A).
\]

The \( \mathcal{X}_k \) are the Kronecker double series and the \( E_{ij}^s \) are the Eisenstein series; see [11].

We consider now an elliptic curve \( E/\mathbb{Q} \) with complex multiplications by \( \mathcal{O}_K \), \( K \) being an imaginary quadratic field. Recall that

\[
L(E/\mathbb{Q}, s) = L(\chi, s) = \sum_{a=1}^{\mathcal{A}} \frac{\chi(a)}{\mathfrak{n}_a^s},
\]

where \( \mathfrak{f} \) is the conductor of the Hecke character \( \chi \). We are interested in nonprimitive \( L \)-functions in which the sums in the \( L \)-functions are extended over prime ideals which are relatively prime to \( NO_K \), \( N \) being the conductor of \( E/\mathbb{Q} \). Recall also that \( N \in \mathfrak{f} \). For \( \sigma \in \text{Gal}(K(E_K)/K) \), we define the partial \( L \)-series associated to \( \chi \) and relative to \( \sigma \) by

\[
L(\chi, \sigma, s) = \sum_{a=1}^{\mathcal{A}} \frac{\chi(a)}{\mathfrak{n}_a^s},
\]

where the sum ranges over integral ideals \( a \subseteq \mathcal{O}_K \) with \( \sigma_a = \sigma \). Each \( \sigma \in \text{Gal}(K(E_K)/K) \) corresponds to a \( \sigma_a \) for some integral ideal \( a \subseteq \mathcal{O}_K \); let \( \mathcal{A} \) be a complete set of integral ideals in \( \mathcal{O}_K \) representatives for all elements.
in \( \text{Gal}(K(E_K)/K) \). If we denote the series \( L(\chi, \sigma_a, s) \) by \( L_a(\chi, s) \), it is clear that
\[
L(\chi, s) = \sum_{a \in \mathfrak{a}} L_a(\chi, s).
\]

Let \( \Omega \) be a fixed nonzero real number such that \( E(\mathbb{C}) \cong \mathbb{C}/A \) with \( A = \Omega O_K \).

**Proposition 4.1.** Let \( a = p_0 O_K \) be an ideal relatively prime to \( \mathfrak{a} \). Let \( \rho \in \Omega K^* \subset \mathbb{C}^* \) such that \( \rho \Omega^{-1} O_K = \mathfrak{a}^{-1} \). Then for \( \text{Re}(s) > 3/2 \), we have:
\[
\frac{\mathcal{X}(p_0 O_K)}{\mathcal{N}(p_0 O_K)} \cdot \frac{\overline{\rho} \rho}{|\rho|^2} \cdot L_{\mathfrak{a}}(\mathfrak{a}, s) = \sum_{p O_K \mathfrak{a}} \mathcal{X}(\mathfrak{a}) \frac{\mathcal{X}(\rho_0 \rho, 0, s, A)}{|\mathcal{X}(\rho_0 \rho, 0, s, A)|^2}.
\]

**Proof.** We note first that \( p_0 \rho \) is an \( \mathfrak{a} \)-torsion point. We have \( \mathcal{X}(\rho_0 \rho) = \mathcal{X}(\mathfrak{a}) \) if \( b \) is relatively prime to \( \mathfrak{a} \), then \( \mathcal{X}(\mathfrak{b}) O_K = \mathfrak{b} \). Thus
\[
\mathcal{X}(\mathfrak{a}) \frac{\mathcal{X}(\rho_0 \rho, 0, s, A)}{|\mathcal{X}(\rho_0 \rho, 0, s, A)|^2} = \mathcal{X}(\rho_0 \rho) \sum_{p O_K \mathfrak{a}} \sum_{\mathfrak{a} \in \mathfrak{P}_K} \frac{\mathcal{X}(\rho_0 \rho, 0, s, A)}{|\mathcal{X}(\rho_0 \rho, 0, s, A)|^2}.
\]

It remains to show that
\[
L_{\mathfrak{a}}(\mathfrak{a}, s) = \sum_{p O_K \mathfrak{a}} \sum_{\mathfrak{a} \in \mathfrak{P}_K} \frac{\mathcal{X}(\mathfrak{a})}{\mathcal{N}(\mathfrak{a})} \frac{\mathcal{X}(\rho_0 \rho, 0, s, A)}{|\mathcal{X}(\rho_0 \rho, 0, s, A)|^2}.
\]

For this, it is enough to check that for \( p O_K \mathfrak{a} \) and \( \rho \mathfrak{a} \in \mathfrak{a}^{-1} \), we have \( \sigma_\mathfrak{a} = \sigma_\mathfrak{a}(\rho_0 \rho, \mathfrak{a}) x (x \in \mathfrak{a}^{-1} \mathfrak{a}) \). This follows from class field theory.

**Remark 4.1.** There is a similar decomposition in [5]. See also [4]. The series \( L_a \) has an analytic continuation to the whole complex plane in the same way as the Hecke \( L \)-series. From the above proposition and the definition of \( E_1^\mathfrak{a} \) we have

**Corollary 4.2.** With the same notations, we have:
\[
\frac{\mathcal{X}(\mathfrak{a})}{\mathcal{N}(\mathfrak{a})} L_{\mathfrak{a}}(\mathfrak{a}, 1) = \sum_{p O_K \mathfrak{a}} E_1^\mathfrak{a}(\mathfrak{a}) E_1^\mathfrak{a}(p O_K, \rho_0 \rho, 0, A).
\]

**Proposition 4.3.** We have
1. \( E_1^\mathfrak{a}(\mathfrak{a}) E_1^\mathfrak{a}(p O_K, \rho_0 \rho, 0, A) \) \( \in K(E_K) \).
2. \( E_1^\mathfrak{a}(\mathfrak{a}) E_1^\mathfrak{a}(p O_K, \rho_0 \rho, 0, A) = E_1^\mathfrak{a}(\mathfrak{a}) E_1^\mathfrak{a}(p O_K, 0) \).

This a particular case of [4, Theorem 6.2].

Since $\rho \in \Omega K^*$, the corollary implies that $\Omega^{-1}L_{\alpha}(\bar{\gamma}, 1)$ and $\sum K^*_\alpha(\gamma(p), \rho, A)$ are $K$-proportional. And using the above proposition, we obtain:

**Proposition 4.4.** For every $a$, we have

1. $L_{\alpha}(\bar{\gamma}, 1)/\Omega \in K(E_N)$.

2. If $a$ is relatively prime to $N$, then
   $$\frac{L_{\alpha}(\bar{\gamma}, 1)}{\Omega} = \left( \frac{L_{\alpha}(\bar{\gamma}, 1)}{\Omega} \right)^{\sigma_a}.$$ 

**Proposition 4.5.** We have

$$\frac{L(E, 1)}{\Omega} = \text{Trace}_{K(E_N)/K} \left( \frac{L_{\alpha}(\bar{\gamma}, 1)}{\Omega} \right).$$

**Proof.** Since $\Omega^{-1}L_{\alpha}(\bar{\gamma}, 1) \in K(E_N)$, the expression makes sense. Since $E$ is defined over $\mathbb{Q}$, for a real number $s$ we have

$$L_{\alpha}(\bar{\gamma}, s) = L_{\alpha}(\bar{\gamma}, s).$$

Since $L(E, s) = \sum L_{\alpha}(\bar{\gamma}, s)$, it follows that

$$\frac{L(E, 1)}{\Omega} = \sum a \frac{L_{\alpha}(\bar{\gamma}, 1)}{\Omega} = \sum \left( \frac{L_{\alpha}(\bar{\gamma}, 1)}{\Omega} \right)^{\sigma_a},$$

where $\sigma_a$ describes $\text{Gal}(K(E_N)/K)$. This proves the proposition. 

The quantity $L_{\alpha}(\bar{\gamma}, 1)/\Omega$ is real; indeed, $\bar{\Omega} = \Omega$ and $L_{\alpha}(E, s) = \sum \chi(a)/n^a$, where the sum is over all $a = (\gamma)$ such that $\gamma \equiv 1(\mod N)$ and $\sigma_a = 1$. Hence

$$L_{\alpha}(E, s) = \sum_{a = (\gamma)} \frac{\gamma(a)}{n^a} = L_{\alpha}(E, s).$$ (4.1)

We now introduce the twisted $L$-function associated with $E/\mathbb{Q}$ which still has complex multiplications by $O_K$ and $N$ is its conductor.

If the Hasse–Weil $L$-function of $E$ is given by

$$L(E, s) = \sum_{a \geq 1} a_n n^{-s} \quad \text{for} \quad \Re(s) > 3/2,$$
then, for every \( b \in \mathbb{Z}/\mathbb{Z} \), we set

\[
L(E, b, s) = \sum_{\sigma \in \mathcal{A}} e^{2\pi i b n / N} \sigma_n^{-s}.
\]

**Proposition 4.6.** For \( \Re(s) > 3/2 \), we have

\[
L(E, b, s) = \sum_{a \in \mathcal{A}} e^{2\pi i a N / N} L_{\mathcal{A}}(\bar{z}, s).
\]

**Proof.** This follows from the definition and from (3.2) which says that if \( \sigma_a = \sigma_b \), then \( \% a \equiv \% b \mod N \).

Let \( \zeta_N = e^{2\pi i / N} \) be a primitive \( N \)th root of unity. From (4.1) we have

\[
\zeta_{\sigma_a} = \zeta_{(Na, \mathbb{Q}(\sigma_a)/\mathbb{Q})} = \zeta_{Na}.
\]

Therefore, using (4.4), we have

\[
\frac{L(E, b, 1)}{\Omega} = \sum_{a} \left( \zeta_{Na} \right) L_{\mathcal{A}}(\bar{z}, 1)
\]

\[
= \sum_{a} \left( \zeta_{Na} \right) \left( \frac{L_{\mathcal{A}}(\bar{z}, 1)}{\Omega} \right)^{\sigma_a}
\]

\[
= \sum_{a} \left( \zeta_{Na} \right) \left( \frac{L_{\mathcal{A}}(\bar{z}, 1)}{\Omega} \right)^{\sigma_a}.
\]

The above sums are over ideals \( a \) such that \( \sigma_a \) runs over the Galois group \( \text{Gal}(K(E_n)/K) \). Therefore, we have

**Theorem 4.7.** The value \( L(E, b, 1) \) is a \( K \)-rational multiple of the period \( \Omega \); more precisely:

\[
\frac{L(E, b, 1)}{\Omega} = \text{Trace}_{K(E_n)/K} \left( \zeta_{Na} \frac{L_{\mathcal{A}}(\bar{z}, 1)}{\Omega} \right).
\]

**Remark 4.2.** We should note that \( \Omega \) is well determined up to a scalar in \( K^* \). It also has a homological interpretation; see [4]. The formula in the theorem gives the expression (1.2). It remains to express these quantities in terms of the arithmetic of \( E \).
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REFERENCES