Classification of irreducible quasifinite modules over map Virasoro algebras

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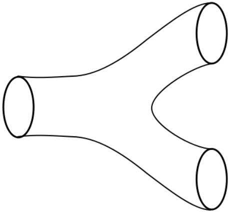

Slides: www.mathstat.uottawa.ca/∼asavag2

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Outline

Summary: We classify the irreducible quasifinite (i.e. possessing finite-dimensional weight spaces) modules of the map Virasoro algebras.

Overview:

1. The Virasoro algebra
2. Virasoro map algebras
3. Important classes of modules for the Virasoro algebra
4. Generalized evaluation modules
5. Classification Theorem
6. Reducibility of Verma modules
7. Further directions

The Virasoro algebra

Definition (Witt algebra $\operatorname{Der}\mathbb{C}[t, t^{-1}]$)

The Lie algebra of polynomial vector fields on the circle.

Basis: $d_n := t^{n+1}\frac{d}{dt}$, $n \in \mathbb{Z}$

Lie bracket: $[d_m, d_n] = (n-m)d_{n+m}$

Definition (Virasoro algebra)

Universal central extension of the Witt algebra

$$\mathrm{Vir} := \operatorname{Der}\mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

$$[d_n, c] = 0, \quad [d_m, d_n] = (n-m)d_{m+n} + \delta_{m,-n}\frac{m^3-m}{12}c, \quad m, n \in \mathbb{Z}.$$

Plays a fundamental role in VOAs, conformal field theory, string theory, rep theory of affine Lie algebras.

Map algebras

Notation

X - scheme (or algebraic variety) over $\mathbb{C}$

$A = A_X = \mathcal{O}_X(X)$ - coordinate ring of X

We assume A is Noetherian.

Definition (Virasoro map algebra)

The Virasoro map algebra is $\mathrm{Vir} \otimes A$, with Lie bracket

$$[u \otimes f, v \otimes g] = [u, v] \otimes fg, \quad u, v \in \mathrm{Vir},\ f, g \in A$$

(extended by linearity).

$\mathrm{Vir} \otimes A$ is just the Lie algebra of regular maps from X to Vir with pointwise multiplication.

Note: If $A = \mathbb{C}$, then $\mathrm{Vir} \otimes A = \mathrm{Vir}$.

Previous results (equivariant map algebras)

If a finite group Γ acts on X and a Lie algebra $\mathfrak{g}$, we can consider the Lie algebra of equivariant maps $X \to \mathfrak{g}$. This is called an equivariant map algebra (EMA).

Examples

Twisted loop algebras: X a 1-d torus, $\mathfrak{g}$ simple.

Twisted multiloop algebras: X a higher dim tours, $\mathfrak{g}$ simple.

Current algebras: $X = \mathbb{C}$, $\mathfrak{g}$ simple.

Generalized Onsager algebra

Representation theory

- If $\mathfrak{g}$ is finite-dimensional, the irreducible finite-dimensional reps of EMAs have been classified (Neher-S.-Senesi 2009)
- If, in addition, A is finitely generated, extensions and block decompositions have been computed (Neher-S. 2011)

Weight decompositions

Set $\mathcal{V} = \mathrm{Vir} \otimes A$. We have a decomposition

$$\mathcal{V} = \bigoplus_{i \in \mathbb{Z}} \mathcal{V}_i, \quad \mathcal{V}_0 = (d_0 \otimes A) \oplus (c \otimes A), \quad \mathcal{V}_i = d_i \otimes A,\ i \neq 0.$$

(This is just weight decomposition w.r.t. action of d_0.)

We set

$$\mathcal{V}_- = \bigoplus_{i<0} \mathcal{V}_i, \quad \mathcal{V}_+ = \bigoplus_{i>0} \mathcal{V}_i.$$

Definition (Weight module)

V is a weight module if

$$V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda, \quad V_\lambda = \{v \in V \mid d_0 v = \lambda v\}.$$

Some important types of modules

Definition (Quasifinite module)

A $\mathcal{V}$-module is quasifinite (or Harish-Chandra, or admissible) if

- it is a wieght module, and
- all weight spaces are finite-dimensional.

Definition (Highest and lowest weight module)

V is highest-weight (resp. lowest weight) if there exists $v \in V$ such that

- $\mathcal{V}_{>0} v = 0$ (resp. $\mathcal{V}_{<0} v = 0$), and
- $U(\mathcal{V})v = V$.

Such a v is called a highest weight vector (resp. lowest weight vector).

Definition (Uniformly bounded module)

A weight $\mathcal{V}$-module V is uniformly bounded if there exists $N \in \mathbb{N}$ such that

$$\dim V_\lambda < N \quad \forall\, \lambda \in \mathbb{C}.$$

Modules of the intermediate series

Fix $a, b \in \mathbb{C}$.

Define $V(a,b)$ to be the Vir-module with underlying vector space $\mathbb{C}[t, t^{-1}]$, with c acting by zero, and

$$u \cdot v = (u + a\,\mathrm{div}(u) + bt^{-1}ut)v, \quad u \in \mathrm{Der}\,\mathbb{C}[t, t^{-1}],\ v \in V(a,b).$$

We have:

- if $b \notin \mathbb{Z}$ or $a \neq 0, 1$, then $V(a,b)$ is irreducible,
- otherwise $V(a,b)$ has two irreducible subquotients: the trivial submodule $\mathbb{C}$ and $V(a,b)/\mathbb{C}$.

The nontrivial subquotients of the $V(a,b)$ are called modules of the intermediate series (or tensor density modules).

Remark: The modules of the intermediate series are weight modules with $\dim V_\lambda = 1$ for all nonzero weight spaces.

Quasifinite Vir-modules

Theorem (Mathieu 1992)

Any irreducible quasifinite module over Vir is

- a highest weight module,
- a lowest weight module, or
- a module of the intermediate series.

Remarks

1. The unitary irreducible quasifinite Vir-modules were classified earlier by Chari and Pressley (1988).
2. It is known that the nontrivial highest and lowest weight modules are not uniformly bounded.

Generalized evaluation modules

Definition (Generalized evaluation module)

Suppose we have

- maximal ideal $\mathfrak{m} \trianglelefteq A$,
- $n \in \mathbb{Z}_{>0}$,
- V a $(\mathrm{Vir} \otimes (A/\mathfrak{m}^n))$-module with rep $\rho : \mathrm{Vir} \otimes (A/\mathfrak{m}^n) \to \mathrm{End}\, V$.

Then the composition

$$\mathrm{Vir} \otimes A \twoheadrightarrow (\mathrm{Vir} \otimes A)/(\mathrm{Vir} \otimes \mathfrak{m}^n) \cong \mathrm{Vir} \otimes (A/\mathfrak{m}^n) \xrightarrow{\rho} \mathrm{End}\, V$$

is a (single point) generalized evaluation representation/module of $\mathrm{Vir} \otimes A$.

If $n = 1$, it is a (single point) evaluation representation/module.

Classification of quasifinite modules for the map Virasoro algebra

Classification Theorem (S. 2011)

Any irreducible quasifinite $(\mathrm{Vir} \otimes A)$-module is one of the folowing:

1. a single point evaluation module corresponding to a Vir-module of the intermediate series (or tensor density module),
2. a finite tensor product of single point generalized evaluation modules corresponding to irreducible highest weight Vir-modules, or
3. a finite tensor product of single point generalized evaluation modules corresponding to irreducible lowest weight Vir-modules.

In particular, they are all tensor products of generalized evaluation modules.

Remark

If $A = \mathbb{C}[t, t^{-1}]$, then $\mathcal{V} = \mathrm{Vir} \otimes \mathbb{C}[t, t^{-1}]$ is the loop-Virasoro algebra. In this case, we recover results of Guo-Lu-Zhao.

Comparison with equivariant map algebras (EMAs)

Important differences between the classification for quasifinite $\mathcal{V}$-modules and f.d. modules for EMAs.

For EMAs, all irreducible f.d. modules are tensor products of evaluation modules and one-dimensional modules (Neher-S.-Senesi 2009).

Differences:

1. For the intermediate series, one can only have single point evaluation modules. This is because a tensor product of intermediate series modules is no longer quasifinite.
2. For the highest/lowest weight modules, we need to allow generalized evaluation modules. Evaluation modules are not enough.

Reducibility of Verma modules

Definition (Verma module)

For $\varphi \in \mathcal{V}_0^*$, we define a one-dimensional $(\mathcal{V}_0 \oplus \mathcal{V}_+)$-module $\mathbb{C}_\varphi$ by declaring $\mathcal{V}_+$ to act by zero.

Then the Verma module corresponding to φ is

$$M(\varphi) := U(\mathcal{V}) \otimes_{U(\mathcal{V}_0 \oplus \mathcal{V}_+)} \mathbb{C}_\varphi.$$

Theorem (S. 2011)

$M(\varphi)$ is reducible if there exists a nontrivial ideal $J \trianglelefteq A$ such that $\varphi(d_0 \otimes J) = 0$.

If A is an infinite-dimensional integral domain, the reverse implication also holds.

Remark

Again, for the loop-Virasoro algebra, we recover results of Guo-Lu-Zhao.

Irreducible highest-weight modules

It follows from the Classification Theorem that the irreducible highest weight module

$$V(\varphi) = M(\varphi)/\text{largest proper submodule}$$

is quasifinite if and only if there exist an ideal $J \trianglelefteq A$ of finite-codimension (as a v.s.) such that

$$\varphi(\mathrm{Vir}_0 \otimes J) = 0$$

and, in this case,

$$\varphi(\mathrm{Vir} \otimes J) V(\varphi) = 0.$$

So we get a complete characterization of the highest weight modules that are quasifinite (this problem is trivial for the usual Virasoro algebra).

Further directions

Extensions between irreducible quasifinite modules.

Classification of irreducible quasifinite modules for twisted (or equivariant) versions of map Virasoro algebras.

Classification of (appropriate class of) modules when one replaces Vir by other important infinite-dimensional Lie algebras (Heisenberg algebra, Lie algebra of all differential operators on the circle, etc.)