Quiver grassmannians, quiver varieties and the preprojective algebra

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Outline

Goal: Present a natural grassmannian type description of various quiver varieties that play an important role in geometric representation theory.

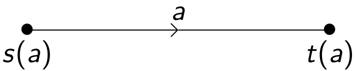
Overview:

1. Quivers, path algebras, and preprojective algebras
2. Modules of the path and preprojective algebras
3. Quiver varieties
4. Quiver grassmannians
5. Realization of quiver varieties as quiver grassmannians

Quivers

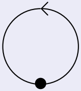
quiver = directed graph $Q = (Q_0, Q_1, s, t)$ (or just (Q_0, Q_1))

- Q_0 = set of vertices
- Q_1 = set of arrows (directed edges)
- $s, t : Q_1 \to Q_0$



Assumptions

- all quivers are finite: $|Q_0|, |Q_1| < \infty$
- no loops: no $a \in Q_1$ with $s(a) = t(a)$



The path algebra

A path in Q is a sequence $\beta = a_l a_{l-1} \dots a_1$ such that $t(a_i) = s(a_{i+1})$ for $1 \le i \le l-1$.

For each vertex i we have the trivial path e_i with $s(e_i) = t(e_i)$.

Definition (Path algebra)

The path algebra $\mathbb{C}Q$ is the $\mathbb{C}$-algebra with:

- basis of underlying vector space = set of paths
- product of paths given by concatenation:

$$(a_l \dots a_1)(b_m \dots b_1) = \begin{cases} a_l \dots a_1 b_m \dots b_1 & \text{if } t(b_m) = s(a_1), \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mathbb{C}Q$ is naturally graded by path lengths.

Preprojective algebra

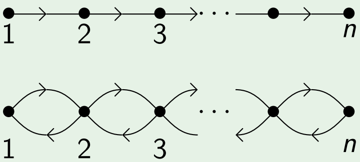
For $Q = (Q_0, Q_1)$ a quiver, we define the double quiver $\tilde{Q} = (Q_0, \tilde{Q}_1)$ where

$$\tilde{Q}_1 = \bigcup_{a \in Q_1} \{a, \bar{a}\}, \quad \text{where} \quad s(\bar{a}) = t(a),\ t(\bar{a}) = s(a).$$

Example

Q

$\tilde{Q}$



Definition (Preprojective algebra)

$$\mathcal{P} = \mathcal{P}(Q) = \mathbb{C}\tilde{Q} / \sum_{a \in Q_1} (a\bar{a} - \bar{a}a)$$

Grading $\mathcal{P} = \bigoplus_{n \geq 0} \mathcal{P}_n$ inherited from grading on $\mathbb{C}\tilde{Q}$.

Categories of modules

A an associative algebra:

- A-Mod $=$ category of A-modules
- A-mod $=$ category of f.d. A-modules

$\mathcal{P}_0$-modules as Q_0-graded vector spaces

$$\mathcal{P}_0 = \bigoplus_{i \in Q_0} \mathbb{C} e_i, \quad e_i e_j = \delta_{ij} e_i$$

Thus we have an equivalence of categories

$$\mathcal{P}_0\text{-mod} \simeq \text{f.d. } Q_0\text{-graded vector spaces}$$

and we identify the two categories.

Any $\mathcal{P}$-module is a $\mathcal{P}_0$-module (by restriction) and so we will also view $\mathcal{P}$-modules as Q_0-graded vector spaces.

$\mathcal{P}_0$-modules

Up to isom, the objects of $\mathcal{P}_0$-mod are classified by their graded dimension:

$$\dim_{Q_0} V = \sum_i (\dim V_i) i \in \mathbb{N}Q_0$$

For $V, W \in \mathcal{P}_0$-mod,

$$\begin{aligned} \operatorname{Hom}_{\mathcal{P}_0}(V, W) &= \bigoplus_{i \in Q_0} \operatorname{Hom}_{\mathbb{C}}(V_i, W_i), \\ \operatorname{End}_{\mathcal{P}_0} V &= \operatorname{Hom}_{\mathcal{P}_0}(V, V), \\ \operatorname{Aut}_{\mathcal{P}_0} V &= \prod_{i \in Q_0} GL(V_i). \end{aligned}$$

We write $V \subseteq W$ when V is a $\mathcal{P}_0$-submodule of W.

Nilpotency

$A = \bigoplus_{n \geq 0} A_n$ graded associative algebra.

V an A-module.

Definition (Nilpotent)

V is nilpotent if

$$A_n \cdot V = 0 \qquad \text{for } n \gg 0.$$

Definition (Locally nilpotent)

V is locally nilpotent if for all $v \in V$

$$A_n \cdot v = 0 \qquad \text{for } n \gg 0.$$

A-lnMod = category of locally nilpotent A-modules.

$\mathcal{P}$-mod and finite type

The representation theory of $\mathcal{P}$ is closely related to the representation theory of Lie algebras.

Proposition

For a quiver Q, the following are equivalent:

1. $\mathcal{P}(Q)$ *is finite-dimensional,*
2. *all finite-dimensional* $\mathcal{P}(Q)$*-modules are nilpotent,*
3. *all finite-dimensional* $\mathcal{P}(Q)$*-modules are locally nilpotent,*
4. Q *is of finite type (i.e. underlying graph is a Dynkin diagram of finite ADE type).*

Lusztig quiver varieties

$\mathfrak{g}$ a symmetric Kac-Moody algebra

Q a quiver obtained by orienting the Dynkin diagram of $\mathfrak{g}$

Definition (Lusztig quiver variety)

Suppose $V \in \mathcal{P}_0$-mod.

$\Lambda(V)$ is the variety of nilpotent representations $\mathcal{P} \to \mathrm{End}_{\mathbb{C}} V$ (or $\mathcal{P}$-module structures) compatible with the $\mathcal{P}_0$-module structure on V (i.e. such that $e_i V = V_i$).

$\Lambda(V)$ is called a Lusztig quiver variety.

Nakajima quiver varieties

For $V, W \in \mathcal{P}_0$-mod, let

$$\Lambda(V, W) = \Lambda(V) \times \mathrm{Hom}_{\mathcal{P}_0}(V, W).$$

Stable points:

$$\Lambda(V, W)^{\mathrm{st}} = \{(x, t) \in \Lambda(V, W) \mid x(U) \subseteq U \subseteq \ker t \implies U = 0\}.$$

There is a natural action of $\mathrm{Aut}_{\mathcal{P}_0} V$ on $\Lambda(V, W)$ and restriction to $\Lambda(V, W)^{\mathrm{st}}$ is free.

Definition (Lagrangian Nakajima quiver variety)

For $V, W \in \mathcal{P}_0$-mod, the lagrangian Nakajima quiver variety is

$$\mathfrak{L}(V, W) := \Lambda(V, W)^{\mathrm{st}} / \mathrm{Aut}_{\mathcal{P}_0} V.$$

Up to isom, $\mathfrak{L}(V, W)$ depends only on $v = \dim_{Q_0} V$ and $w = \dim_{Q_0} W$ and so we sometimes denote it by $\mathfrak{L}(v, w)$.

Quiver varieties and representation theory

We associate to $v, w \in \mathbb{N}Q_0$ certain weights of $\mathfrak{g}$:

$$\lambda_w = \sum_i w_i \omega_i \quad (\omega_i \text{ - fundamental weights})$$

$$\alpha_v = \sum_i v_i \alpha_i \quad (\alpha_i \text{ - simple roots})$$

Theorem (Nakajima)

$$\bigoplus_v H_{\mathrm{top}}(\mathfrak{L}(v,w)) \cong L(\lambda_w) = \textit{irred. h.w. rep of } \mathfrak{g} \textit{ of h.w. } \lambda_w$$

$$H_{\mathrm{top}}(\mathfrak{L}(v,w)) \cong L(\lambda_w)_{\lambda_w - \alpha_v} = (\lambda_w - \alpha_v) - \textit{weight space}$$

Action of $\mathfrak{g}$ defined geometrically (via correspondences).

Quiver varieties and crystals

Theorem (Kashiwara-Saito)

The set

$$\bigsqcup_{V} \{\textit{irreducible components of } \Lambda_V\}$$

can be given (geometrically) the structure of the crystal $B(\infty)$ *of* $U_q(\mathfrak{g})^-$.

Theorem (Saito)

The set

$$\bigsqcup_{v} \{\textit{irreducible components of } \mathfrak{L}(v, w)\}$$

can be given (geometrically) the structure of the crystal $B(\lambda_w)$ *of* $L(\lambda_w)$.

Demazure quiver varieties

Recall:

- Weyl group $\mathcal{W}$ of $\mathfrak{g}$ acts on weight lattice.
- For $\sigma \in \mathcal{W}$ and $\lambda \in P^+$,

$$\dim L(\lambda)_{\sigma \cdot \lambda} = 1.$$

We say the weight $\sigma \cdot \lambda$ is extremal.

Proposition (S. '06)

$\mathfrak{L}(v, w) = \{\mathrm{Aut}_{\mathcal{P}_0} \cdot (x^{w,\sigma}, t^{w,\sigma})\}$ *is a point if and only if*

$$\lambda_w - \alpha_v = \sigma \cdot \lambda_w \quad \textit{for some } \sigma \in \mathcal{W} \textit{ (i.e. is extremal).}$$

Definition (Demazure quiver variety)

For $\sigma \in \mathcal{W}$ and $v, w \in \mathbb{N}Q_0$, define the Demazure quiver variety

$$\mathfrak{L}_\sigma(v, w) = \{\mathrm{Aut}_{\mathcal{P}_0} V \cdot (x, t) \mid (x, t) \text{ is a subrep of } (x^{w,\sigma}, t^{w,\sigma})\}.$$

Demazure quiver varieties

Theorem (S. '06)

$$\bigoplus_{v} H_{\mathrm{top}}(\mathfrak{L}_\sigma(v, w)) \cong L_\sigma(\lambda_w)$$

where

$$L_\sigma(\lambda_w) = U(\mathfrak{g})^+ \cdot L(\lambda_w)_{\sigma \cdot \lambda_w}$$

is the Demazure module.

Remarks

Remarks

1. Term "lagrangian Nakajima quiver variety" comes from fact that these varieties are lagrangian subvarieties of (smooth) Nakajima quiver varieties.
2. Smooth Nakajima quiver varieties can also be described as hyper-Kähler quotients.
3. Sometimes moduli space description of quiver variety can hard to work with.

Goal

Give another description of these (and related) varieties.

Quiver grassmannians

Definition (Quiver grassmannian)

For V a $\mathbb{C}Q$-module, let

$$\mathrm{Gr}_Q(V) = \text{ variety of all } \mathbb{C}Q\text{-submodules of } V,$$
$$\mathrm{Gr}_Q(u, V) = \{U \in \mathrm{Gr}_Q(V) \mid \dim U = u\}, \quad u \in \mathbb{N}Q_0.$$

We call $\mathrm{Gr}_Q(u, V)$ a quiver grassmannian.

Example

If Q has a single vertex and no arrows, $\mathrm{Gr}_Q(u, V)$ is the usual grassmannian of u-dimensional subspaces of V.

Applications

- Morphisms of $\mathbb{C}Q$-modules (Crawley-Boevey, Schofield).
- Cluster algebras (Caldero-Chapoton, Caldero-Keller, Derksen-Weyman-Zelevinsky).

Quiver grassmannians

Remarks

- $\mathrm{Gr}_Q(u, V)$ is a closed subset of a product of grassmannians and thus is a projective variety.
- If V is a $\mathcal{P}$-module, then $\mathcal{P}$-submodules of V are the same as $\mathbb{C}\tilde{Q}$-submodules and so we sometimes write $\mathrm{Gr}_{\mathcal{P}}(u, V)$ instead of $\mathrm{Gr}_{\tilde{Q}}(u, V)$.

Goal

Realize various quiver varieties as quiver grassmannians.

Our description will involve injective hulls of semisimple modules.

Simple objects

Definition

For $i \in Q_0$, define s^i by

$$s^i_j = \begin{cases} \mathbb{C} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Then s^i is a simple $\mathbb{C}\tilde{Q}$-module and a simple $\mathcal{P}(Q)$-module.

Lemma

$\{s^i\}_{i \in Q_0}$ is a set of representatives of the isomorphism classes of simple objects of $\mathbb{C}\tilde{Q}\text{-lnMod}$ and $\mathcal{P}(Q)\text{-lnMod}$.

If Q is of finite type, $\{s^i\}_{i \in Q_0}$ is a set of representatives of the isomorphism classes of simple objects of $\mathbb{C}\tilde{Q}\text{-Mod}$ and $\mathcal{P}\text{-Mod}$.

Injective hulls

Definition

A an associative algebra, V an A-module

An injective hull of V is an injective A-module E that is an essential extension of V. That is,

- E is injective,
- V is a submodule of E, and
- any nonzero submodule of E intersects V nontrivially.

Eckmann-Schöpf Theorem

The category $\mathcal{P}$-Mod has enough injectives (for arbitrary Q).

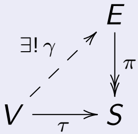
In particular, the simple modules s^i have injective hulls.

Key proposition

Suppose

- $A = \bigoplus_{n \geq 0} A_n$ *is a graded algebra*
- V *is a locally nilpotent* A*-module*
- S *is a semisimple* A*-module with injective hull* E

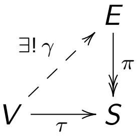
1. *If* $\pi : E \twoheadrightarrow S$, $\tau : V \to S$ *are* A_0*-module homomorphisms*



   γ *an* A*-module homomorphism. Furthermore,* γ *is injective iff* $\tau|_{\mathrm{socle}\, V}$ *is injective.*

2. *For two projections* $\pi_1, \pi_2 : E \twoheadrightarrow S$ *of* A_0*-modules,* $\exists!\, \gamma \in \mathrm{Aut}_{\mathcal{P}}\, E$ *such that* $\pi_2 = \pi_1 \gamma$.

Key proposition



Remark

The map $\pi : E \twoheadrightarrow S$ is equivalent to choosing an A_0-module decomposition $E = S \oplus T$.

Proposition states that any two such decompositions are related by a unique A-module automorphism of E fixing S.

Moral

Once we fix a projection $E \twoheadrightarrow S$, the data of an A-module V and an A_0-module homomorphism $V \to S$ is equivalent to the data of a map $V \to E$ whose image is a submodule of E.

Quiver varieties as quiver grassmannians

Definition

For $i \in Q_0$, let q^i be the injective hull of the simple module s^i.

Then for $w \in \mathbb{N}Q_0$, the injective hull of the semisimple module

$$s^w = \bigoplus_{i \in Q_0} \left(s^i\right)^{\oplus w_i}$$

is

$$q^w = \bigoplus_{i \in Q_0} \left(q^i\right)^{\oplus w_i}.$$

We can view $W \in \mathcal{P}_0$-mod as a semisimple module s^w, $w = \dim_{Q_0} W$ (we extend it trivially to a $\mathcal{P}$-module).

Quiver varieties as quiver grassmannians

Applying our key proposition

$$\begin{array}{ccc} & & q^w \\ & \overset{\exists!\,\gamma}{\nearrow} & \big\downarrow \pi \\ V & \xrightarrow[t]{} & W = s^w \end{array}$$

data of $(x, t) \in \Lambda(V) \times \mathrm{Hom}_{\mathcal{P}_0}(V, W)$ is equivalent to data of $\mathcal{P}_0$-module map $\gamma : V \to q^w$ whose image is a submodule of q^w.

Stability condition equivalent to injectivity of γ.

Quotient by $\mathrm{Aut}_{\mathcal{P}_0} V$ yields quiver grassmannian.

Theorem (S.-Tingley 2009)

There is a bijective algebraic map from $\mathrm{Gr}_{\mathcal{P}}(v, q^w)$ *to* $\mathfrak{L}(v, w)$*.*

In particular, $\mathrm{Gr}_{\mathcal{P}}(v, q^w)$ *is homeomorphic to* $\mathfrak{L}(v, w)$*.*

Defining the $\mathfrak{g}$-action

Goal

Define a natural $\mathfrak{g}$-action on homology of quiver grassmannians so that

$$\bigoplus_{v} H_{\mathrm{top}}(\mathrm{Gr}_{\mathcal{P}}(v, q^w)) \cong L(\lambda_w)$$

with

$$H_{\mathrm{top}}(\mathrm{Gr}_{\mathcal{P}}(v, q^w)) \cong L(\lambda_w)_{\lambda_w - \alpha_v}.$$

Method

Use the known action on the homology of quiver varieties and translate to the language of quiver grassmannians via our theorem.

Our homology theory will be in terms of constructible functions (also developed for quiver varieties).

Defining the $\mathfrak{g}$-action

To define a $\mathfrak{g}$-action it suffices to define action of the Chevalley generators $\{e_i, f_i\}_{i \in Q_0}$.

Let

$$\mathrm{Gr}_{\mathcal{P}}(u, u+i, V) = \{(U, U') \in \mathrm{Gr}_{\mathcal{P}}(u, V) \times \mathrm{Gr}_{\mathcal{P}}(u+i, V) \mid U \subseteq U'\}$$

and consider the natural projections

$$\mathrm{Gr}_{\mathcal{P}}(u, V) \xleftarrow{\pi_1} \mathrm{Gr}_{\mathcal{P}}(u, u+i, V) \xrightarrow{\pi_2} \mathrm{Gr}_{\mathcal{P}}(u+i, V).$$

Then we define

$$e_i : H_{\mathrm{top}}(\mathrm{Gr}_{\mathcal{P}}(u+i, V)) \to H_{\mathrm{top}}(\mathrm{Gr}_{\mathcal{P}}(u, V)), \quad e_i(\alpha) = (\pi_1)_!(\pi_2^* \alpha),$$
$$f_i : H_{\mathrm{top}}(\mathrm{Gr}_{\mathcal{P}}(u, V)) \to H_{\mathrm{top}}(\mathrm{Gr}_{\mathcal{P}}(u+i, V)), \quad f_i(\alpha) = (\pi_2)_1(\pi_1^* \alpha).$$

where $H_{\mathrm{top}}(X)$ is a certain space of constructible functions on X.

Defining the $\mathfrak{g}$-action

Theorem (S.-Tingley 2009)

The actions of e_i and f_i described above define a $\mathfrak{g}$-action on

$$\bigoplus_v H_{\mathrm{top}}(\mathrm{Gr}_{\mathcal{P}}(v, q^w))$$

and

$$\bigoplus_v H_{\mathrm{top}}(\mathrm{Gr}_{\mathcal{P}}(v, q^w)) \cong L(\lambda_w)$$

with

$$H_{\mathrm{top}}(\mathrm{Gr}_{\mathcal{P}}(v, q^w)) \cong L(\lambda_w)_{\lambda_w - \alpha_v}.$$

Demazure quiver grassmannians

Proposition (S. '06)

For $v, w \in \mathbb{N}Q_0$, the following are equivalent:

1. $\lambda_w - \alpha_v = \sigma \cdot \lambda_w$, $\sigma \in \mathcal{W}$, *is an extremal weight,*
2. $\mathrm{Gr}_{\mathcal{P}}(v, q^w)$ *is a single point,*
3. $\exists!$ *submodule* $q^{w,\sigma}$ *of* q^w *of graded dimension* v*.*

Definition (Demazure quiver grassmannian)

Let $v, w \in \mathbb{N}Q_0$ and $\sigma \in \mathcal{W}$. Then

$$\mathrm{Gr}_{\mathcal{P}}(v, q^{w,\sigma})$$

is a Demazure quiver grassmannian.

Proposition (S.-Tingley '09)

$\mathrm{Gr}_{\mathcal{P}}(v, q^{w,\sigma})$ *is homeomorphic to the Demazure quiver variety* $\mathfrak{L}_\sigma(v, w)$*.*

Nested quiver grassmannians

Suppose $V_1 \subseteq V_2$ are $\mathcal{P}$-modules.

Then we have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Gr}_{\mathcal{P}}(v, V_1) & \xleftarrow{\pi_1^1} & \mathrm{Gr}_{\mathcal{P}}(v, v+i, V_1) & \xrightarrow{\pi_2^1} & \mathrm{Gr}_{\mathcal{P}}(v+i, V_1) \\
\Big\downarrow \iota_v & & \Big\downarrow \iota_{v,v+i} & & \Big\downarrow \iota_{v+i} \\
\mathrm{Gr}_{\mathcal{P}}(v, V_2) & \xleftarrow{\pi_1^2} & \mathrm{Gr}_{\mathcal{P}}(v, v+i, V_2) & \xrightarrow{\pi_2^2} & \mathrm{Gr}_{\mathcal{P}}(v+i, V_2)
\end{array}$$

and one can show that

1. $e_i^1 = \iota_v^* \circ e_i^2 \circ (\iota_{v+i})_!$, and
2. $f_i^1 = \iota_{v+i}^* \circ f_i^2 \circ (\iota_v)_!$

where e_i^j is the e_i operator applied to quiver grassmannians in V_j.

Nested quiver grassmannians

In particular, for $v, w \in \mathbb{N}Q_0$ and $\sigma \in \mathcal{W}$, we have

$$\begin{array}{ccccc} \mathrm{Gr}_{\mathcal{P}}(v, q^{w,\sigma}) & \xleftarrow{\pi_1^1} & \mathrm{Gr}_{\mathcal{P}}(v, v+i, q^{w,\sigma}) & \xrightarrow{\pi_2^1} & \mathrm{Gr}_{\mathcal{P}}(v+i, q^{w,\sigma}) \\ \downarrow \iota_v & & \downarrow \iota_{v,v+i} & & \downarrow \iota_{v+i} \\ \mathrm{Gr}_{\mathcal{P}}(v, q^{w}) & \xleftarrow{\pi_1^2} & \mathrm{Gr}_{\mathcal{P}}(v, v+i, q^{w}) & \xrightarrow{\pi_2^2} & \mathrm{Gr}_{\mathcal{P}}(v+i, q^{w}) \end{array} \qquad (1)$$

Lemma

For $v, w \in \mathbb{N}Q_0$, there exists $\sigma \in \mathcal{W}$ such that $\mathrm{Gr}_{\mathcal{P}}(v, q^{w,\sigma'})$ is homeomorphic to $\mathfrak{L}(v, w)$ for all $\sigma' \succeq \sigma$.

Then the inclusions in (1) are isomorphisms.

Note: If $\mathfrak{g}$ is of finite type, we can take σ to be the longest element in the Weyl group.

Compatibility with nested quiver grassmannians

By the above lemma, with σ large enough, the inclusions in

$$\begin{array}{ccccc}
\mathrm{Gr}_{\mathcal{P}}(u, q^{w,\sigma}) & \xleftarrow{\pi_1^1} & \mathrm{Gr}_{\mathcal{P}}(u, u+i, q^{w,\sigma}) & \xrightarrow{\pi_2^1} & \mathrm{Gr}_{\mathcal{P}}(u+i, q^{w,\sigma}) \\
\big\downarrow \iota_u & & \big\downarrow \iota_{u,u+i} & & \big\downarrow \iota_{u+i} \\
\mathrm{Gr}_{\mathcal{P}}(u, q^{w}) & \xleftarrow{\pi_1^2} & \mathrm{Gr}_{\mathcal{P}}(u, u+i, q^{w}) & \xrightarrow{\pi_2^2} & \mathrm{Gr}_{\mathcal{P}}(u+i, q^{w})
\end{array}$$

are isomorphisms.

Even though the q^w may be infinite-dimensional, we always have

$$\dim q^{w,\sigma} < \infty.$$

Moral

We can always work with quiver grassmannians inside finite-dimensional modules if we wish.

Locally nilpotent modules

Recall the Bruhat order $\prec$ on $\mathcal{W}$.

One can show that

$$\sigma_1 \prec \sigma_2 \implies q^{w,\sigma_1} \subseteq q^{w,\sigma_2}.$$

So the $(q^{w,\sigma})_{\sigma \in \mathcal{W}}$ form a directed system.

Definition

Let $\tilde{q}^w$ be the direct limit of the directed system $(q^{w,\sigma})_{\sigma \in \mathcal{W}}$.

Theorem

$\tilde{q}^w$ is the injective hull of s^w in the category $\mathcal{P}$-lnMod.

Corollary

$\tilde{q}^w \cong q^w$ if and only if Q is of finite or affine (tame) type.

Group actions

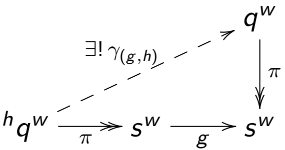
For $w \in \mathbb{N}Q_0$, define

$$G_w = \mathrm{Aut}_{\mathcal{P}_0}\, s^w$$
$$G_{\mathcal{P}} = \text{group of algebra automs of } \mathcal{P} \text{ that fix } \mathcal{P}_0 \text{ pointwise}$$

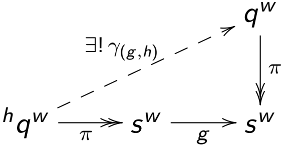
For $V \in \mathcal{P}$-Mod and $h \in G_{\mathcal{P}}$, define ${}^hV \in \mathcal{P}$-Mod by the action

$$(a, v) \mapsto h^{-1}(a) \cdot v, \quad a \in \mathcal{P}.$$

Fix $(g, h) \in G_w \times G_{\mathcal{P}}$. By our "key proposition"



Group actions



This defines a $G_w \times G_{\mathcal{P}}$-action on q^w and hence on $\mathrm{Gr}_{\mathcal{P}}(v, q^w)$ for all v.

One can define a $G_w \times G_{\mathcal{P}}$-action on lagrangian Nakajima quiver varieties.

Theorem (S.-Tingley '09)

The homeomorphism $\mathrm{Gr}_{\mathcal{P}}(v, q^w) \to \mathfrak{L}(v, w)$ *is* $G_w \times G_{\mathcal{P}}$*-equivariant.*

Graded quiver varieties and quiver grassmannians

Definition (Nakajima)

Graded quiver varieties are the fixed point sets

$$\mathfrak{L}(v, w)^A$$

under the action of certain subgroups $A \subseteq G_w \times G_{\mathcal{P}}$

Note: Another direct description in terms of graded vector spaces can be given.

Graded quiver varieties used by Nakajima to define t-analogs of q-characters of quantum affine algebras.

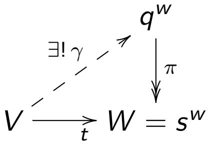
By the above, we can realize the graded quiver varieties as graded quiver grassmannians (fixed point sets of quiver grassmannians)

Recap

Recall that lagrangian Nakajima quiver varieties are quotients of stable points of

$$\Lambda(V, W) = \Lambda(V) \times \mathrm{Hom}_{\mathcal{P}_0}(V, W).$$

We view W as a semisimple module s^w, $w = \dim_{Q_0} W$, and our key proposition:



tells us that this data is equivalent to a linear map $V \to q^w$ whose image is a submodule of q^w.

Stability condition is equivalent to the map $V \to q^w$ being injective.

Quotient by $\mathrm{Aut}_{\mathcal{P}_0} V$ then yields the quiver grassmannian (space of injective maps modulo base change in the domain).

Remarks

1. In finite and affine type A and D, one can give explicit descriptions of the injective hulls q^w in terms of Young tableaux and Young walls/pyramids (Frenkel-S. '03, S. '06)

2. In the general case, one can give a direct (inductive) description of the q^w.

3. Lusztig has presented a canonical bijection between the points of the lagrangian Nakajima quiver variety and the points of a type of quiver grassmannian.
   - Lusztig construction used projective (instead of injective) objects.
   - Nilpotency condition needed to be added explicitly (unlike in injective construction).
   - In finite type, the projective objects are also injective – the two constructions are related by the Chevalley involution.