

Irreducible finite-dimensional representations of equivariant map algebras

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Outline

Goal: Classify the irreducible finite-dimensional representations of a certain class of Lie algebras.

Overview:

- 1 Equivariant map algebras
- 2 Examples
- 3 Evaluation representations
- 4 Classification theorem
- 5 Applications
 - ▶ recover some known classifications (often in a simplified manner)
 - ▶ produce some new classifications

Terminology:

small = irreducible finite-dimensional

(Untwisted) Map algebras

Notation

k - algebraically closed field of characteristic zero

X - scheme (or algebraic variety) over k

$A = A_X = \mathcal{O}_X(X)$ - coordinate ring of X

\mathfrak{g} - finite-dimensional Lie algebra over k

Definition (Untwisted map algebra)

$M(X, \mathfrak{g}) =$ Lie algebra of regular maps from X to \mathfrak{g}

Pointwise multiplication:

$$[\alpha, \beta]_{M(X, \mathfrak{g})}(x) = [\alpha(x), \beta(x)]_{\mathfrak{g}} \text{ for } \alpha, \beta \in M(X, \mathfrak{g})$$

Note: $M(X, \mathfrak{g}) \cong \mathfrak{g} \otimes A_X$

Examples

Discrete spaces

If X is a discrete variety, then

$$M(X, \mathfrak{g}) \cong \prod_{x \in X} \mathfrak{g}, \quad \alpha \mapsto (\alpha(x))_{x \in X}, \quad \alpha \in M(X, \mathfrak{g}).$$

In particular, if $X = \{x\}$ is a point, then

$$M(X, \mathfrak{g}) \cong \mathfrak{g}, \quad \alpha \mapsto (\alpha(x)), \quad \alpha \in M(X, \mathfrak{g}).$$

The isomorphisms are given by **evaluation**.

Current algebras

$$X = k^n \implies A_X = k[t_1, \dots, t_n]$$

Thus, $M(X, \mathfrak{g}) \cong \mathfrak{g} \otimes k[t_1, \dots, t_n]$ is a **current algebra**.

Untwisted multiloop algebras

$$X = (k^\times)^n \implies A_X = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$$

Thus,

$$M(X, \mathfrak{g}) \cong \mathfrak{g} \otimes k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$$

is the **untwisted multiloop algebra**.

If $n = 1$, this is called the **untwisted loop algebra** and plays an important role in the theory of (untwisted) affine Lie algebras.

Examples

Three point algebras

$$\begin{aligned} X &= k \setminus \{0, 1\} = \mathbb{P}^1 \setminus \{0, 1, \infty\} \\ \implies A_X &\cong k[t, t^{-1}, (t-1)^{-1}] \end{aligned}$$

Thus,

$$M(X, \mathfrak{sl}_2) \cong \mathfrak{sl}_2 \otimes k[t, t^{-1}, (t-1)^{-1}]$$

is the **three point \mathfrak{sl}_2 loop algebra**.

Remarks

- Removing any 2 points from k results in an isomorphic map algebra.
- $M(X, \mathfrak{sl}_2)$ is isomorphic to the **tetrahedron Lie algebra** and to a direct sum of 3 copies of the **Onsager algebra** (Hartwig-Terwilliger 2007).

Equivariant map algebras

- Γ - finite group
- Suppose Γ acts on X and \mathfrak{g} by automorphisms

Definition (equivariant map algebra)

The **equivariant map algebra** is the Lie algebra of Γ -equivariant maps from X to \mathfrak{g} :

$$M(X, \mathfrak{g})^\Gamma = \{\alpha \in M(X, \mathfrak{g}) : \alpha(g \cdot x) = g \cdot \alpha(x) \forall x \in X, g \in \Gamma\}$$

Note: If X is any scheme, then $M(X, \mathfrak{g})^\Gamma \cong M(X_{\text{aff}}, \mathfrak{g})^\Gamma$ where $X_{\text{aff}} = \text{Spec } A_X$ is the affine scheme with the same coordinate ring as X . So we often assume X is affine.

Equivariant map algebras – algebraic description

- Induced action on A_X given by

$$(g \cdot f)(x) = f(g^{-1} \cdot x), \quad f \in A_X, \quad x \in X, \quad g \in \Gamma$$

- Γ acts diagonally on $\mathfrak{g} \otimes A_X$:

$$g \cdot (u \otimes f) = (g \cdot u) \otimes (g \cdot f)$$

- Then

$$M(X, \mathfrak{g})^\Gamma \cong (\mathfrak{g} \otimes A_X)^\Gamma$$

Example: Trivial Γ -action on \mathfrak{g}

If Γ acts trivially on \mathfrak{g} , then

$$M(X, \mathfrak{g})^\Gamma \cong M(X//\Gamma, \mathfrak{g}) \cong \mathfrak{g} \otimes A_X^\Gamma$$

where $X//\Gamma = \text{Spec } A_X^\Gamma$ is the quotient of X by Γ .

Thus $M(X, \mathfrak{g})^\Gamma$ is isomorphic to an untwisted map algebra.

Example: multiloop algebras

$$\Gamma = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}, \quad X = (k^\times)^n$$

- For $i = 1, \dots, n$, let ξ_i be a primitive m_i -th root of unity.
- Define action of Γ on X by

$$(a_1, \dots, a_n) \cdot (z_1, \dots, z_n) = (\xi_1^{a_1} z_1, \dots, \xi_n^{a_n} z_n)$$

- Define action of Γ on \mathfrak{g} by specifying commuting automorphisms σ_i , $i = 1, \dots, n$, such that $\sigma_i^{m_i} = 1$.

Then $M(X, \mathfrak{g})^\Gamma$ is the **(twisted) multiloop algebra**.

If $n = 1$, this is the **(twisted) loop algebra**.

Affine Lie algebras

The affine Lie algebras can be constructed as central extensions of loop algebras plus a differential:

$$\widehat{\mathfrak{g}} = M(X, \mathfrak{g})^\Gamma \oplus kc \oplus kd \quad (n = 1)$$

Example: generalized Onsager algebra

$$\Gamma = \mathbb{Z}_2 = \{1, \sigma\}, \quad X = k^\times, \quad \mathfrak{g} = \text{simple Lie algebra}$$

- Γ acts on X by $\sigma \cdot x = x^{-1}$
- Γ acts on \mathfrak{g} by any involution

When Γ acts on \mathfrak{g} by the Chevalley involution, we write

$$\mathcal{O}(\mathfrak{g}) = M(X, \mathfrak{g})^\Gamma$$

Remarks

- If $k = \mathbb{C}$, $\mathcal{O}(\mathfrak{sl}_2)$ is isomorphic to the **Onsager algebra** (Roan 1991)
 - ▶ Key ingredient in Onsager's original solution of the 2D Ising model
- For $k = \mathbb{C}$, $\mathcal{O}(\mathfrak{sl}_n)$ was studied by Uglov and Ivanov (1996)

Evaluation

If $\mathbf{x} = \{x_1, \dots, x_n\} \subseteq X$, we have the **evaluation map**

$$\text{ev}_{\mathbf{x}} : M(X, \mathfrak{g})^{\Gamma} \rightarrow \mathfrak{g}^{\oplus n}, \quad \alpha \mapsto (\alpha(x_i))_i$$

Important: This map is not surjective in general!

For $x \in X$, define

$$\begin{aligned}\Gamma_x &= \{g \in \Gamma : g \cdot x = x\} \\ \mathfrak{g}^x &= \{u \in \mathfrak{g} : \Gamma_x \cdot u = u\}\end{aligned}$$

Lemma

For X affine, $\mathbf{x} = \{x_1, \dots, x_n\} \subseteq X$, $x_i \notin \Gamma \cdot x_j$ for $i \neq j$,

$$\text{im ev}_{\mathbf{x}} = \bigoplus_i \mathfrak{g}^{x_i}.$$

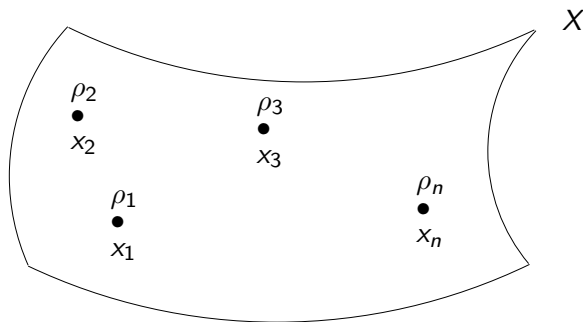
Evaluation representations

Given

- $\mathbf{x} = \{x_1, \dots, x_n\} \subseteq X$, and
- representations $\rho_i : \mathfrak{g}^{x_i} \rightarrow \text{End}_k V_i$, $i = 1, \dots, n$,

we define the **(twisted) evaluation representation** as the composition

$$M(X, \mathfrak{g})^\Gamma \xrightarrow{\text{ev}_{\mathbf{x}}} \bigoplus_i \mathfrak{g}^{x_i} \xrightarrow{\bigotimes_i \rho_i} \text{End}_k(\bigotimes_i V_i).$$



Important remarks

This notion of evaluation representation differs from the classical definition.

- Some authors use the term **evaluation representation** only for the case when evaluation is at a single point and call the general case a tensor product of evaluation representations.
- To a point $x \in X$, we associate a representation of \mathfrak{g}^x instead of \mathfrak{g} . If Γ acts freely, this coincides with the usual definition.
- Recall that (when $\mathfrak{g}^x \subsetneq \mathfrak{g}$) not all reps of \mathfrak{g}^x extend to reps of \mathfrak{g} – so the new definition is more general.
- We do not require the representations ρ_i to be faithful.

We will see that the more general definition allows for a more uniform classification of representations.

Evaluation representations

$\mathcal{R}_x = \{\text{isomorphism classes of small reps of } \mathfrak{g}^x\}$

$$\mathcal{R}_X = \bigsqcup_{x \in X} \mathcal{R}_x$$

Since $\Gamma_{g \cdot x} = g\Gamma_x g^{-1}$, we have

$$g \cdot \mathfrak{g}^x = \mathfrak{g}^{g \cdot x}.$$

We have an action of Γ on \mathcal{R}_X : if $[\rho] \in \mathcal{R}_x$, then

$$g \cdot [\rho] = [\rho \circ g^{-1}] \in \mathcal{R}_{g \cdot x}.$$

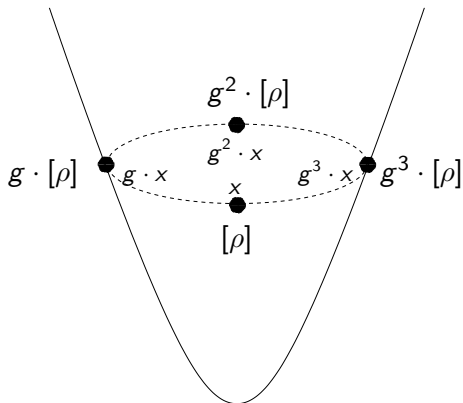
Definition (\mathcal{F})

\mathcal{F} is set of all $\Psi : X \rightarrow \mathcal{R}_X$ such that

- 1 Ψ is Γ -equivariant,
- 2 $\Psi(x) \in \mathcal{R}_x$ for all $x \in X$, and
- 3 $\text{supp } \Psi = \{x \in X : \Psi(x) \neq 0\}$ is finite.

Evaluation representations

We think of $\Psi \in \mathcal{F}$ as assigning a finite number of (isom classes of) reps of \mathfrak{g}^x to points $x \in X$ in a Γ -equivariant way.



Evaluation representations

For each $\Psi \in \mathcal{F}$, define

$$\text{ev}_\Psi = \text{ev}_x(\Psi(x_i))_{i=1}^n = \text{ev}_{x_1} \Psi(x_1) \otimes \cdots \otimes \text{ev}_{x_n} \Psi(x_n)$$

where $\mathbf{x} = (x_1, \dots, x_n)$ is an n -tuple of points of X containing one point from each Γ -orbit in $\text{supp } \Psi$ (the isom class is independent of this choice).

For $\Psi \in \mathcal{F}$, ev_Ψ is the isomorphism class of a small representation of $M(X, \mathfrak{g})^\Gamma$.

Proposition

The map

$$\mathcal{F} \longrightarrow \{\text{isom classes of small reps of } M(X, \mathfrak{g})^\Gamma\}, \quad \Psi \mapsto \text{ev}_\Psi$$

is injective.

One-dimensional representations

Recall: Any 1-dimensional rep of a Lie algebra L corresponds to a linear map $\lambda : L \rightarrow k$ such that $\lambda([L, L]) = 0$.

We identify such 1-dimensional reps with elements

$$\lambda \in (L/[L, L])^*.$$

Two 1-dimensional reps are isomorphic if and only if they are equal as elements of $(L/[L, L])^*$.

Classification Theorem

Theorem (Neher-S.-Senesi 2009)

Suppose Γ is a finite group acting on an affine scheme (or variety) X and a finite-dimensional Lie algebra \mathfrak{g} . Let $\mathfrak{M} = M(X, \mathfrak{g})^\Gamma$.

Then the map

$$(\lambda, \Psi) \mapsto \lambda \otimes \text{ev}_\Psi, \quad \lambda \in (\mathfrak{M}/[\mathfrak{M}, \mathfrak{M}])^*, \quad \Psi \in \mathcal{F}$$

gives a surjection

$$(\mathfrak{M}/[\mathfrak{M}, \mathfrak{M}])^* \times \mathcal{F} \rightarrow \{\text{isom classes of small representations of } \mathfrak{M}\}.$$

In particular, all small representations are of the form

$$(1\text{-dim rep}) \otimes (\text{evaluation rep}).$$

Classification – Remarks

$$(\lambda, \Psi) \mapsto \lambda \otimes \text{ev}_\Psi, \quad \lambda \in (\mathfrak{M}/[\mathfrak{M}, \mathfrak{M}])^*, \quad \Psi \in \mathcal{F}$$

- 1 This map is not injective in general since we can have nontrivial evaluation reps which are 1-dimensional. This happens when \mathfrak{g}^X is not perfect (e.g. reductive but not semisimple).

Example: $\mathfrak{g} = \mathfrak{sl}_2$, $\Gamma = \mathbb{Z}_2$, $X = k = \mathbb{C}$

- ▶ Γ acts on \mathfrak{g} by the Chevalley involution.
 - ▶ Γ acts on X by multiplication by -1 .
 - ▶ Then $\mathfrak{g}^0 = \mathfrak{g}^\Gamma$ is one-dimensional and so has nontrivial 1-dim reps.
- 2 However, we can specify precisely when $\lambda \otimes \text{ev}_\Psi \cong \lambda' \otimes \text{ev}_{\Psi'}$.
 - 3 The restriction of the map to either factor is injective.

Classification

$$(\lambda, \Psi) \mapsto \lambda \otimes \text{ev}_\Psi, \quad \lambda \in (\mathfrak{M}/[\mathfrak{M}, \mathfrak{M}])^*, \quad \Psi \in \mathcal{F}$$

Corollary

- ① *If \mathfrak{M} is perfect (i.e. $\mathfrak{M} = [\mathfrak{M}, \mathfrak{M}]$), then we have a bijection*

$$\mathcal{F} \leftrightarrow \{\text{isom classes of small reps}\}, \quad \Psi \mapsto \text{ev}_\Psi.$$

In particular, all small reps are evaluation reps.

- ② *If $[\mathfrak{g}^\Gamma, \mathfrak{g}] = \mathfrak{g}$, then \mathfrak{M} is perfect and the above bijection holds.*
- ③ *If Γ acts on \mathfrak{g} by diagram automorphisms, then $[\mathfrak{g}^\Gamma, \mathfrak{g}] = \mathfrak{g}$ and the above bijection holds.*

Note: Being perfect is not a necessary condition for the all small reps to be evaluation reps (as we will see).

Application: untwisted map algebras

If Γ is trivial, then

$$M(X, \mathfrak{g})^\Gamma = M(X, \mathfrak{g}), \quad \mathfrak{g}^\Gamma = \mathfrak{g}.$$

Thus, if \mathfrak{g} is perfect,

$$[\mathfrak{g}^\Gamma, \mathfrak{g}] = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$$

and so all small reps are evaluation reps.

Application: multiloop algebras

Corollary

If \mathfrak{M} is a (twisted) multiloop algebra, then \mathfrak{M} is perfect and so we have a bijection

$$\mathcal{F} \leftrightarrow \{\text{isom classes of small reps}\}, \quad \Psi \mapsto \text{ev}_\Psi.$$

In particular, all small reps are evaluation reps.

Remarks

- 1 This recovers results of Chari-Pressley (for loop algebras) and Batra, Lau (multiloop algebras), but with a different description.
- 2 The description given above (in terms of \mathcal{F}) gives a simple and uniform description of the somewhat technical conditions appearing in previous classifications.
- 3 Action of Γ on X is free and so $\mathfrak{g}^x = \mathfrak{g}$ for all $x \in X$. So the more general notion of evaluation rep does not play a role.

Application: generalized Onsager algebra

$$\Gamma = \mathbb{Z}_2 = \{1, \sigma\}, \quad X = k^\times, \quad \mathfrak{g} = \text{simple Lie algebra}$$

- Γ acts on X by $\sigma \cdot x = x^{-1}$
- Γ acts on \mathfrak{g} by any involution

Corollary

With Γ , X , \mathfrak{g} as above, we have a bijection

$$\mathcal{F} \leftrightarrow \{\text{isom classes of small reps}\}, \quad \Psi \mapsto \text{ev}_\Psi.$$

In particular, all small reps are evaluation reps.

Remarks – generalized Onsager algebra

- There are two types of points of X :
 - ▶ $x \in \{\pm 1\} \implies \Gamma_x = \Gamma = \mathbb{Z}_2, \mathfrak{g}^x = \mathfrak{g}^\Gamma$
 - ▶ $x \notin \{\pm 1\} \implies \Gamma_x = \{1\}, \mathfrak{g}^x = \mathfrak{g}$
- \mathfrak{g}^Γ can be semisimple or have a one-dimensional center

When \mathfrak{g}^Γ has a one-dimensional center:

- the generalized Onsager algebra is not perfect
- we can place (nontrivial) one-dim reps of \mathfrak{g}^Γ at the points ± 1
- under our more general definition of evaluation rep, all small reps are evaluation reps
- under classical notion of evaluation rep, there are small reps which are **not** evaluation reps

Moral: The more general definition of evaluation rep allows for a more uniform classification.

Special case: Onsager algebra

- When $k = \mathbb{C}$ and Γ acts on $\mathfrak{g} = \mathfrak{sl}_2$ by the Chevalley involution, then

$$\mathcal{O}(\mathfrak{sl}_2) \stackrel{\text{def}}{=} M(X, \mathfrak{sl}_2)^\Gamma$$

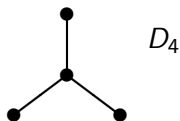
is the **Onsager algebra**.

- $\mathfrak{g}^{\{\pm 1\}}$ is one-dimensional abelian and $\mathcal{O}(\mathfrak{sl}_2)$ is not perfect.
- Small reps of $\mathcal{O}(\mathfrak{sl}_2)$ were classified previously (Date-Roan 2000)
 - ▶ classical definition of evaluation rep was used
 - ▶ not all small reps were evaluation reps
 - ▶ this necessitated the introduction of the **type** of a representation

Note: For the other cases, the classification seems to be new.

Application: a nonabelian example

$$\Gamma = S_3, \quad X = \mathbb{P}^1 \setminus \{0, 1, \infty\}, \quad \mathfrak{g} = \mathfrak{so}_8 \quad (\text{type } D_4)$$



- symmetry group of Dynkin diagram of \mathfrak{g} is S_3
- so Γ acts on \mathfrak{g} by diagram automorphisms
- for any permutation of the set $\{0, 1, \infty\}$, $\exists!$ Möbius transformation of \mathbb{P}^1 inducing that permutation
- so Γ acts naturally on X

Thus we can form the equivariant map algebra $M(X, \mathfrak{g})^\Gamma$ and show that it is perfect.

Our classification tells us all small reps are eval reps and gives a bijection between these and the set \mathcal{F} .

Application: a nonabelian example

It is straightforward to find the points with nontrivial stabilizer:

x	Γ_x	Type of \mathfrak{g}^x
-1	$\{\text{Id}, (0 \infty)\} \cong \mathbb{Z}_2$	B_3
2	$\{\text{Id}, (1 \infty)\} \cong \mathbb{Z}_2$	B_3
$\frac{1}{2}$	$\{\text{Id}, (0 1)\} \cong \mathbb{Z}_2$	B_3
$e^{\pm\pi i/3}$	$\{\text{Id}, (0 1 \infty), (0 \infty 1)\} \cong \mathbb{Z}_3$	G_2

The sets

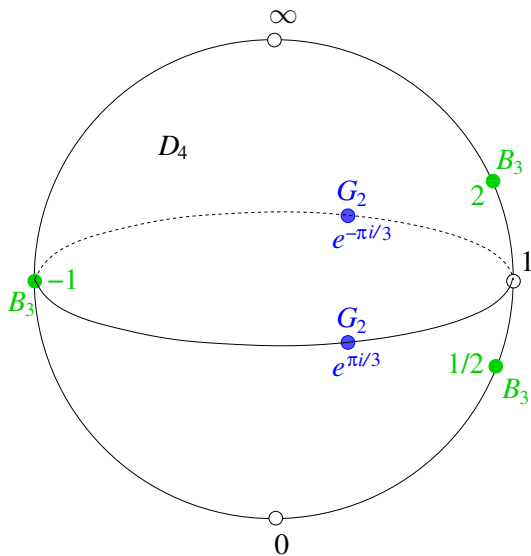
$$\left\{-1, 2, \frac{1}{2}\right\} \quad \text{and} \quad \left\{e^{\pi i/3}, e^{-\pi i/3}\right\}$$

are Γ -orbits.

So elements of \mathcal{F} can assign

- irreps of type B_3 to the 3-element orbit,
- irreps of type G_2 to the 2-element orbit,
- irreps of type D_4 to the other points (6-element orbits).

Application: a nonabelian example



Question

Question

When are all small representations evaluation representations?

We have seen

- perfect (i.e. $\mathfrak{M} = [\mathfrak{M}, \mathfrak{M}]$) \implies all small reps are eval reps
- the converse is not true (e.g. the Onsager algebra)

Reduction

Since all reps are of the form

$$(1\text{-dim rep}) \otimes (\text{eval rep})$$

it suffices to know when there are **one-dimensional** reps that are not evaluation reps.

When are all small reps eval reps?

Definition

$$\tilde{X} = \{x \in X \mid \mathfrak{g}^x \neq [\mathfrak{g}^x, \mathfrak{g}^x]\}$$

Recall

$\mathfrak{g}^x = [\mathfrak{g}^x, \mathfrak{g}^x] \iff$ all one-dimensional reps of \mathfrak{g}^x are trivial

Thus, \tilde{X} is precisely the set of points where we can place nontrivial one-dimensional evaluation representations.

Proposition (Neher-S-Senesi 2009)

If X is a Noetherian scheme (i.e. A is finitely generated) and $|\tilde{X}| = \infty$, then $M(X, \mathfrak{g})^\Gamma$ has one-dimensional representations that are not evaluation representations.

When are all small reps eval reps?

Example

$$\Gamma = \mathbb{Z}_2 = \{1, \sigma\}, \quad X = k^2, \quad \mathfrak{g} = \mathfrak{sl}_2(k)$$

- $\sigma \cdot (x_1, x_2) = (x_1, -x_2), (x_1, x_2) \in k^2$
- σ acts as Chevalley involution on \mathfrak{g}

Then

- $x = (x_1, x_2) \in k^2, x_2 \neq 0 \implies \Gamma_x = \{1\} \implies \mathfrak{g}^x = \mathfrak{g}$
- $x = (x_1, 0) \in k^2 \implies \Gamma_x = \Gamma \implies \mathfrak{g}^x = \mathbb{C}$ (abelian)

Thus

$$\tilde{X} = \{(x_1, 0) \mid x_1 \in k\} \quad \text{and so} \quad |\tilde{X}| = \infty.$$

Therefore $M(X, \mathfrak{g})^\Gamma$ has one-dimensional reps that are not eval reps.

When are all small reps eval reps?

Question

$|\tilde{X}| < \infty$ is a necessary condition for all small reps to be eval reps.

Is it sufficient?

Answer: **NO**

When are all small reps eval reps?

Let \mathfrak{x} be a finite subset of X and consider the commutative diagram:

$$\begin{array}{ccccc} \mathfrak{M} & \xrightarrow{\text{ev}_{\mathfrak{x}}} & \bigoplus_{x \in \mathfrak{x}} \mathfrak{g}^x & \longrightarrow & \bigoplus_{x \in \mathfrak{x}} \mathfrak{g}^x / [\mathfrak{g}^x, \mathfrak{g}^x] \\ & \searrow & & \nearrow \gamma & \\ & & \mathfrak{M} / [\mathfrak{M}, \mathfrak{M}] & & \end{array}$$

Theorem

If $|\tilde{X}| < \infty$, then

$$(\lambda, \Psi) \mapsto \lambda \otimes \text{ev}_{\Psi}, \quad \lambda \in (\ker \gamma)^*, \quad \Psi \in \mathcal{F},$$

is a bijection

$$(\ker \gamma)^* \times \mathcal{F} \longleftrightarrow \{\text{isom classes of small reps}\}$$

Corollary

If $|\tilde{X}| < \infty$, all small reps are eval reps if and only if $\ker \gamma = 0$. This is true if and only if

$$[\mathfrak{M}, \mathfrak{M}] = \mathfrak{M}^d := \{\alpha \in \mathfrak{M} \mid \alpha(x) = [\mathfrak{g}^x, \mathfrak{g}^x] \forall x \in X\}.$$

Note: $[\mathfrak{M}, \mathfrak{M}] \subseteq \mathfrak{M}^d$ is always true.

Example: $|\tilde{X}| < \infty$ with small reps that are not eval reps

- $\mathfrak{g} = \mathfrak{sl}_2(k)$
- $X = Z(y^2 - x^3) = \{(x, y) \mid y^2 = x^3\} \subseteq k^2$
- So $A = k[y, x]/(y^2 - x^3)$
- $\Gamma = \mathbb{Z}_2 = \{1, \sigma\}$
- $\sigma \cdot (y, x) = (-y, x)$
- This action fixes $y^2 - x^3$ and so induces an action of Γ on X .
- Only fixed point is the origin.
- Thus $\tilde{X} = \{0\}$ and so $|\tilde{X}| < \infty$

Then one can easily show that

$$\mathfrak{m}^d / [\mathfrak{m}, \mathfrak{m}] \cong xk[x]/(x^3) \neq 0$$

Thus $[\mathfrak{m}, \mathfrak{m}] \subsetneq \mathfrak{m}^d$ and so \mathfrak{m} has small reps that are not eval reps.

Note: X has a singularity at 0.

Further directions (work in progress)

- The category of finite-dimensional representations of an equivariant map algebra is not semisimple in general.
- Can one describe the finite-dimensional representations (not necessarily irreducible)?
 - ▶ (twisted) Weyl modules (untwisted case considered by Chari-Fourier-Khandai 2009)
 - ▶ block decompositions
 - ★ untwisted loop (Chari-Moura)
 - ★ twisted loop (Senesi)
 - ★ untwisted map algebras (Kodera)
- **Current work (with E. Neher):** Describe extensions between small reps (and then block decompositions). Untwisted case done by Kodera.