

# Quivers and the Euclidean Group

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# Euclidean group

## Definition (Euclidean group)

Group of orientation-preserving isometries of  $n$ -dim Euclidean space:

$$E(n) = \mathbb{R}^n \rtimes SO(n)$$

Study (at least for  $n = 2, 3$ ) predates even concept of group.

We will focus on  $E(2)$  – much still unknown about rep theory.

# Representations of the Euclidean group

- $E(2)$  solvable  $\Rightarrow$  all finite-dim irreps are 1-dim
- finite-dim **unitary** reps (of interest in quantum mechanics) are completely reducible  $\Rightarrow$  isom to direct sum of one-dim reps
- infinite-dim unitary reps have received considerable attention
- $\exists$  finite-dim **nonunitary** indecomp reps (not irreducible)
  - much less known about these
  - play important role in math physics and rep theory of Poincaré group

# A little mathematical physics

## Poincaré Group

Group of isometries of Minkowski spacetime

$$\text{Poincaré group} = \{\text{translations}\} \rtimes \text{Lorentz group}$$

## The Little Group (Wigner 1939)

**Def:** maximal subgroup of Lorentz group leaving invariant the four-momentum of a particle

- governs internal space-time symmetries of particle
- **massive particles:** little group locally isom to  $O(3)$
- **massless particles:** little group locally isom to  $E(2)$

# A bit more mathematical physics

## Gravity

Consider

- Chern-Simons formulation of Einstein gravity
- $2 + 1$  dimensions
- space-time with Euclidean signature
- vanishing cosmological constant

Then phase space of gravity is moduli space of flat  $E(2)$ -connections

# The Euclidean algebra

Recall  $E(2) = \mathbb{R}^2 \rtimes SO(2)$

## The Euclidean Algebra

$\mathfrak{e}(2)$  = complexification of Lie alg of  $E(2)$

Has basis  $\{p_+, p_-, l\}$  and relations

$$[p_+, p_-] = 0, \quad [l, p_{\pm}] = \pm p_{\pm}$$

## Representation Theory

$SO(2)$  compact  $\Rightarrow$  finite-dim  $E(2)$ -modules equiv to finite-dim  $\mathfrak{e}(2)$ -modules where  $l$  acts semisimply with integer eigenvalues

We use term  **$\mathfrak{e}(2)$ -module** to mean such a module

# Weight decompositions

$V$  an  $\mathfrak{e}(2)$ -module

We have **weight decomposition** into  $l$ -eigenspaces

$$V = \bigoplus_{k \in \mathbb{Z}} V_k, \quad V_k = \{v \in V \mid l \cdot v = kv\}$$

and

$$p_+ V_k \subseteq V_{k+1}, \quad p_- V_k \subseteq V_{k-1}$$

We define

$$\mathbf{dim} V = (\dim V_k)_{k \in \mathbb{Z}} \in (\mathbb{Z}_{\geq 0})^{\mathbb{Z}}$$

# Modified enveloping algebra

$U$  = universal enveloping algebra of  $\mathfrak{e}(2)$

$U^+$ ,  $U^-$ ,  $U^0$  subalgebras generated by  $p_+$ ,  $p_-$ ,  $l$

Have triangular decomp  $U \cong U^+ \otimes U^0 \otimes U^-$

## Modified enveloping algebra

Following idea of Lusztig, define **modified enveloping algebra**

$$\tilde{U} = U^+ \otimes \left( \bigoplus_{k \in \mathbb{Z}} \mathbb{C} a_k \right) \otimes U^-$$

with multiplication

$$a_k a_l = \delta_{kl} a_k$$

$$p_+ a_k = a_{k+1} p_+, \quad p_- a_k = a_{k-1} p_-,$$

$$p_+ p_- a_k = p_- p_+ a_k$$

# Representation theory

$$\tilde{U} = U^+ \otimes \left( \bigoplus_{k \in \mathbb{Z}} \mathbb{C} a_k \right) \otimes U^-$$

$a_k \sim$  projection to  $k$ th weight space

## Definition

A  $\tilde{U}$ -module is **unital** if

- 1  $\forall v \in V, a_k v = 0$  for almost all  $k \in \mathbb{Z}$
- 2  $\forall v \in V, \sum_{k \in \mathbb{Z}} a_k v = v$

$\tilde{U}$ -module  $\sim$   $U$ -module with weight decomp

## Proposition

$$\mathbf{Mod} \tilde{U} \cong \mathbf{Mod} U \cong \mathbf{Mod} \epsilon(2)$$

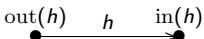
# Quivers

quiver = directed graph

$$Q = (I, H)$$

$I$  = vertex set

$H$  = (directed) edge set



## Representations of quivers

- $I$ -graded vector space  $V = (V_i)_{i \in I}$
- linear map  $x_h : V_{\text{out}(h)} \rightarrow V_{\text{in}(h)}$  for each  $h \in H$

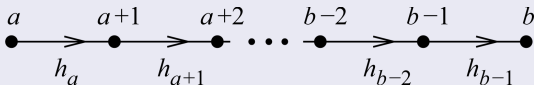
$$\text{rep}(Q, V) = \bigoplus_{h \in H} \text{Hom}_{\mathbb{C}}(V_{\text{out}(h)}, V_{\text{in}(h)})$$

# Quivers

## The quiver $Q_{a,b}$

$$I = \{k \in \mathbb{Z} \mid a \leq k \leq b\}$$

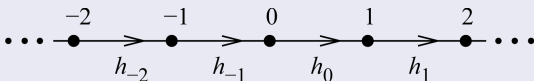
$$H = \{h_i \mid a \leq i \leq b-1\}, \quad \text{out}(h_i) = i, \quad \text{in}(h_i) = i+1$$



## The quiver $Q_\infty$

$$I = \mathbb{Z}$$

$$H = \{h_i \mid i \in \mathbb{Z}\}, \quad \text{out}(h_i) = i, \quad \text{in}(h_i) = i+1$$



# Path algebra and double quiver

## Path algebra

$\mathbb{C}Q$  = algebra spanned by paths with multiplication given by concatenation

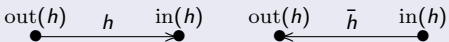
cat of reps of  $Q \cong \mathbf{Mod} \mathbb{C}Q$

## Double quiver

$Q^*$  = double quiver of  $Q$

$$I_{Q^*} = I_Q,$$

$$H_{Q^*} = H_Q \cup \bar{H}_Q, \quad \bar{H}_Q = \{\bar{h} \mid h \in H_Q\}$$



# Preprojective algebra

For  $i \in I$ , define

$$r_i = \sum_{h \in H, \text{out}(h)=i} \bar{h}h - \sum_{h \in H, \text{in}(h)=i} h\bar{h}$$

## Preprojective algebra

$$P(Q) = \mathbb{C}Q^*/J$$

$J =$  two-sided ideal generated by  $r_i, i \in I$

## Representations of the preprojective algebra

$\text{mod}(P(Q), V) = \{P(Q)\text{-modules with underlying v.s. } V\}$

Equivalent to set of elements of  $\text{rep}(Q^*, V)$  such that

$$\sum_{h \in H, \text{out}(h)=i} x_{\bar{h}}x_h - \sum_{h \in H, \text{in}(h)=i} x_hx_{\bar{h}} = 0 \quad \forall i \in I$$

# Representation theory of the preprojective algebra

## Proposition (Crawley-Boevey, Lusztig and others)

The following are equivalent for a quiver  $Q$ :

- 1  $P(Q)$  is finite-dimensional
- 2 All elements of  $\text{rep}(P(Q), V)$  are nilpotent
- 3  $Q$  is a Dynkin quiver (underlying graph of  $ADE$  type)

## Proposition

If  $Q$  is a finite quiver then

- 1  $P(Q)$  is of **finite rep type** iff  $Q$  is of Dynkin type  $A_n$ ,  $n \leq 4$
- 2  $P(Q)$  is of **tame rep type** iff  $Q$  is of Dynkin type  $A_5$  or  $D_4$
- 3  $P(Q)$  is of **wild rep type** for other types

# Representation theory of $P(Q_{a,b})$ and $P(Q_\infty)$

## Corollary

- $Q_{a,b}$  has finite rep type iff  $b - a \leq 3$ , and all reps are nilpotent
- $Q_\infty$  is of wild rep type and all reps are nilpotent

# Preprojective algebras and the Euclidean algebra

## Theorem

The map  $\psi : \mathbb{C}Q_\infty^* \rightarrow \tilde{U}$  given by ( $\epsilon_i =$  trivial path at  $i$ )

$$\psi(\epsilon_i) = a_i, \quad \psi(h_i) = p_+ a_i = a_{i+1} p_+, \quad \psi(\bar{h}_i) = a_i p_- = p_- a_{i+1}$$

extends to a surjective map of algebras with kernel  $J$ . Thus

$$P(Q_\infty) \cong \tilde{U}$$

## Corollary

$$\mathbf{Mod} \, \epsilon(2) \cong \mathbf{Mod} \, P(Q_\infty)$$

and

$$\mathbf{Mod}_{a,b} \, \epsilon(2) \cong \mathbf{Mod} \, P(Q_{a,b})$$

where  $\mathbf{Mod}_{a,b} \, \epsilon(2)$  is category of  $\epsilon(2)$ -modules with weights lying between  $a$  and  $b$

# Representation theory of the Euclidean algebra

## Theorem

- $\epsilon(2)$  (and hence  $E(2)$ ) has **wild representation type**
- for  $0 \leq b - a \leq 3$ ,  $\exists$  a finite number of isom classes of indecomposable  $\epsilon(2)$ -modules whose weights lie between  $a$  and  $b$

# Lusztig quiver variety

## Definition (Lusztig quiver variety)

$\Lambda_{V,Q}$  is set of all nilpotent  $(x_h) \in \text{mod}(P(Q), V)$

Recall, for  $Q$  of Dynkin type

$$\text{mod}(P(Q), V) = \Lambda_{V,Q}$$

## Relation to Kac-Moody algebras

Let  $\mathfrak{g}_Q =$  Kac-Moody algebra whose Dynkin graph is underlying graph of  $Q$ .

# irred comps of  $\Lambda_{V,Q} = \dim$  of  $(-\sum(\dim V_i)\alpha_i)$ -weight space of  $U(\mathfrak{g}_Q)^-$

# Nakajima quiver variety

Let  $Q$  be  $Q_\infty$  or  $Q_{a,b}$ .

For  $I$ -graded vec spaces  $V$  and  $W$ , define

$$L_Q(V, W) = \Lambda_{V,Q} \oplus \bigoplus_{i \in I} \text{Hom}_{\mathbb{C}}(W_i, V_i)$$

For  $(x, s) = ((x_h)_{h \in H}, (s_i)_{i \in I})$  we say

- $I$ -graded  $S \subseteq V$  is  **$x$ -invariant** if

$$x_h(S_{\text{out}(h)}) \subseteq S_{\text{in}(h)} \quad \forall h \in H$$

- $(x, s)$  is **stable** if  $\nexists$  proper  $x$ -invariant subspace of  $V$  containing  $\text{im } s$

Let  $L_Q(V, W)^{\text{st}} = \text{set of stable points}$

# Nakajima quiver variety

$G_V = \prod_{i \in I} GL(V_i)$  acts on  $L_Q(V, W)$  by

$$g \cdot (x, s) = ((g_{\text{in}(h)} x_h g_{\text{out}(h)}^{-1}), (g_i s_i))$$

Stabilizer in  $G_V$  of a stable point is trivial

Definition (Nakajima quiver variety)

$$\mathcal{L}_Q(V, W) = L_Q(V, W)^{\text{st}} / G_V$$

# Nakajima quiver varieties and Kac-Moody algebras

$$\bigoplus_V H_{\text{top}}(\mathcal{L}_Q(V, W)) \cong \text{irrep of } \mathfrak{g}_Q \text{ of hw } \sum_{i \in I} (\dim W_i) \omega_i$$

where  $\omega_i$  are fundamental weights

$$H_{\text{top}}(\mathcal{L}_Q(V, W)) = \sum_{i \in I} (\dim W_i) \omega_i - \sum_{i \in I} (\dim V_i) \alpha_i \text{ weight space}$$

# irred comps of  $\mathcal{L}_Q(V, W) = \dim$  of weight space

# Representation theory of $\epsilon(2)$

- recall  $\epsilon(2)$  has **wild** rep type
- restrict attention to subclasses of modules and attempt a classification
- impose restriction on number of generators of a module
- moduli spaces of such modules related to Nakajima quiver varieties

# Moduli spaces of representations of $\epsilon(2)$

Let  $V$  be a rep of  $\epsilon(2)$

We say  $\{u_1, \dots, u_n\} \subseteq V$  is a **set of generators** of  $V$  if

- ① each  $u_i$  is a weight vector
- ②  $\nexists$  proper submodule of  $V$  containing all  $u_i$

## Definition

For  $\mathbf{v}, \mathbf{w} \in (\mathbb{Z}_{\geq 0})^{\mathbb{Z}}$ , let  $E(\mathbf{v}, \mathbf{w})$  be set of all

$$(V, (u_k^j)_{k \in \mathbb{Z}, 1 \leq j \leq \mathbf{w}_k})$$

where

- $V$  is an  $\epsilon(2)$ -module with  $\dim V = \mathbf{v}$
- $(u_k^j)_{k \in \mathbb{Z}, 1 \leq j \leq \mathbf{w}_k}$  is a set of generators of  $V$  with  $\text{wt } u_k^j = k$

# Moduli spaces of representations of $\epsilon(2)$

## Definition

We say

$$(V, (u_k^j)) \sim (\tilde{V}, (\tilde{u}_k^j))$$

if  $\exists$   $\epsilon(2)$ -module isom

$$\phi : V \xrightarrow{\cong} \tilde{V}, \quad \phi(u_k^j) = \tilde{u}_k^j \quad \forall j, k$$

Let

$$\mathcal{E}(\mathbf{v}, \mathbf{w}) = E(\mathbf{v}, \mathbf{w}) / \sim$$

# Moduli spaces of representations of $\epsilon(2)$

## Theorem

There is a natural one-to-one correspondence

$$\mathcal{E}(\mathbf{v}, \mathbf{w}) \leftrightarrow \mathcal{L}_{Q_\infty}(V, W)$$

if  $\dim V = \mathbf{v}$ ,  $\dim W = \mathbf{w}$

## Idea of proof

Given  $(V, (u_k^j)) \in \mathcal{E}(\mathbf{v}, \mathbf{w})$ , define a point  $(x, s) \in \mathcal{L}_{Q_\infty}(V, W)$  by

$$\begin{aligned} x_{h_i} &= p_+ |_{V_i}, & x_{\bar{h}_i} &= p_- |_{V_{i+1}}, & k \in \mathbb{Z} \\ s(w_k^j) &= u_k^j, & k \in \mathbb{Z}, & 1 \leq j \leq \mathbf{w}_k \end{aligned}$$

where  $\{w_k^j\}_{1 \leq j \leq \mathbf{w}_k}$  is a basis of  $W_k$ . Then

generating set  $\leftrightarrow$  stability  $\sim \leftrightarrow G_V$  - orbits

# Remarks

- relationship between rep theory of Euclidean group and rep theory of  $\mathfrak{sl}_\infty$  (or groups  $SL(n)$ )
- moduli space of reps of Euclidean group along with a set of generators closely related to rep theory of  $\mathfrak{sl}_\infty$  and  $SL(n)$
- although Euclidean group has wild rep type, we have method of approaching classification:
  - fix cardinality and weights of a generating set
  - resulting moduli space enumerated by countable number of varieties – one variety for reps of each graded dimension

## Further directions

### Positive characteristic

- consider Euclidean group over field of characteristic  $p$
- weights now lie in  $\mathbb{Z}/p\mathbb{Z}$  instead of  $\mathbb{Z}$
- category of reps equivalent to category of reps of preprojective algebra of quiver of affine type  $\hat{A}_{p-1}$
- quiver varieties related to moduli spaces of solutions to anti-self-dual Yang-Mills equations and Hilbert schemes of points in  $\mathbb{C}^2$

# Further directions

## Crystals and Jordan-Hölder decompositions

- can define crystal structure on set of irred comps of Lusztig and Nakajima quiver varieties
- each irred comp can be identified with a sequence of crystal operators
- sequence corresponds to Jordan-Hölder decomposition of  $\mathfrak{e}(2)$ -modules