Moduli spaces of sheaves and the boson-fermion correspondence

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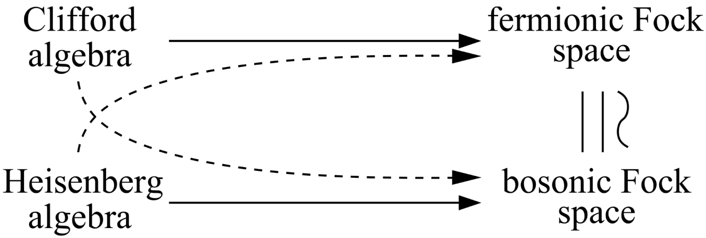
Algebraic boson-fermion correspondence



Geometric representation theory

Basic idea

vector space $\rightsquigarrow$ "(co)homology" of some space(s)

algebra action $\rightsquigarrow$ geometrically defined operators (e.g. via correspondences)

Examples

Space	**Algebraic object**
Lusztig quiver varieties	$U_q(\mathfrak{g})^-$
Nakajima quiver varieties	Reps of $\mathfrak{g}$
$\mathrm{Hilb}_n(\mathbb{C}^2)$	Bosonic Fock space
Affine Grassmannian of G	Rep theory of G^L

Geometric representation theory

Benefits and uses

- use rep theory to study geometry of spaces involved
- use geometry to study rep theory
- geometric realizations often produce nice bases with integrality and positivity properties
- geometrization $\longrightarrow$ categorification

Hilbert schemes and Heisenberg algebras (Nakajima, Grojnowksi)

$\mathrm{Hilb}_n(\mathbb{C}^2)$ = Hilbert scheme of n points in $\mathbb{C}^2$

Consider $\bigoplus_n H^*(\mathrm{Hilb}_n(\mathbb{C}^2))$

Correspondences in $\mathrm{Hilb}_n(\mathbb{C}^2) \times \mathrm{Hilb}_{n+k}(\mathbb{C}^2)$ yield action of Heisenberg algebra

$\bigoplus_n H^*(\mathrm{Hilb}_n(\mathbb{C}^2)) \cong$ bosonic Fock space (1 color)

Important idea: Consider all the Hilbert schemes together

Geometric operators via correspondences

Spaces X, Y

Correspondence $Z \subset X \times Y$ and natural projections

$$X \xleftarrow{p_X} X \times Y \xrightarrow{p_Y} Y$$

Define an operator $H^*(X) \to H^*(Y)$ by

$$H^*(X) \ni \alpha \mapsto (p_Y)_!(p_X^*(\alpha) \cup [Z]) \in H^*(Y)$$

Overview

Our goal

Find space(s) such that "(co)homology" has natural actions of Heisenberg algebra and Clifford algebra

Operators

Carlsson-Okounkov:

$$\text{correspondences} \rightsquigarrow \text{(virtual) vector bundles}$$

Oscillator/Heisenberg algebra

r-colored oscillator algebra

- $$\mathfrak{s}^r = \bigoplus_{m\in\mathbb{Z},\, l\in\{1,\dots,r\}} \mathbb{C}p^l(m) \oplus \mathbb{C}K$$
- commutation relations
$$[\mathfrak{s}^r, K] = 0, \quad [p^k(m), p^l(n)] = \frac{1}{m}\delta_{m,-n}\delta_{k,l}K.$$

$$\mathrm{Span}_{\mathbb{C}}\{p^l(m), K \mid l \in \{1,\dots,r\},\ m \in \mathbb{Z}\setminus\{0\}\}$$

is an r-colored infinite-dimensional Heisenberg algebra

Bosonic Fock space

r-colored bosonic Fock space

$$\mathbf{B} = B^{\otimes r}, \ \ B = \mathbb{C}[p_1, p_2, \ldots; q, q^{-1}] \cong \Lambda \otimes \mathbb{C}[q, q^{-1}]$$

$$\Lambda = \text{ring of symmetric functions}, \ \ p_n = \sum_i x_i^n$$

Charge-energy decomposition:

$$\mathbf{B} = \bigoplus_{\mathbf{c} \in \mathbb{Z}^r, j \in \mathbb{Z}_{\geq 0}} \mathbf{B}_j^{\mathbf{c}}$$

$$\mathbf{B}_j^{\mathbf{c}} = \{(q^{c^1} f_1, \ldots, q^{c^r} f_r) \mid f_i \in \Lambda, \sum \deg f_i = j\}$$

Bosonic Fock space

Representation of $\mathfrak{s}^r$ on **B**

$$p^l(m) \mapsto \mathrm{id}^{\otimes(l-1)} \otimes \frac{\partial}{\partial p_m} \otimes \mathrm{id}^{\otimes(r-l)}, \;\; m > 0,$$

$$p^l(-m) \mapsto \mathrm{id}^{\otimes(l-1)} \otimes \frac{1}{m} p_m \otimes \mathrm{id}^{\otimes(r-l)}, m > 0,$$

$$p^l(0) \mapsto \mathrm{id}^{\otimes(l-1)} \otimes q\frac{\partial}{\partial q} \otimes \mathrm{id}^{\otimes(r-l)},$$

$$K \mapsto \mathrm{id}.$$

Physical interpretation

- $p^l(-m)$ creates a particle of "color" l in state m
- $p^l(m)$ annihilates a particle of "color" l in state m

Clifford algebra

r-colored Clifford algebra Cl^r

- generators

$$\psi^l(j),\ \psi^l(j)^*, \quad j \in \mathbb{Z},\ 1 \le l \le r$$

- relations

$$\psi^l(i)\psi^l(j)^* + \psi^l(j)^*\psi^l(i) = \delta_{ij},$$
$$\psi^l(i)\psi^l(j) + \psi^l(j)\psi^l(i) = 0 = \psi^l(i)^*\psi^l(j)^* + \psi^l(j)^*\psi^l(i)^*,$$
$$[\psi^k(i), \psi^l(j)] = [\psi^k(i), \psi^l(j)^*] = [\psi^k(i)^*, \psi^l(j)^*] = 0$$

Fermionic Fock space

Semi-infinite monomials

$$I = i_1 \wedge i_2 \wedge \dots, \qquad i_j \in \mathbb{Z}$$

such that

- $i_1 > i_2 > i_3 > \dots$
- $i_k = i_{k-1} - 1$ for $k \gg 0$

Fermionic Fock space

$$\mathbf{F} = F^{\otimes r}$$

$$F = \mathrm{Span}_{\mathbb{C}}\{\text{semi-infinite monomials}\}$$

r-colored fermionic Fock space

Charge

The charge of I is the integer $c(I)$ such that

$$i_k = c(I) + 1 - k \quad \text{for } k \gg 0$$

For $\mathbf{I} = (I^1, \ldots, I^r)$,

$$\mathbf{c}(\mathbf{I}) = (c(I^1), \ldots, c(I^r)) \in \mathbb{Z}^r$$

Partitions and semi-infinite monomials

For a partition I of charge c, define a partition $\lambda(I)$ by

$$I = i_1 \wedge i_2 \wedge \ldots, \quad i_k = (c(I) + 1 - k) + \lambda(I)_k$$

For $\mathbf{I} = (I^1, \ldots, I^r)$,

$$\boldsymbol{\lambda}(\mathbf{I}) = (\lambda(I^1), \ldots, \lambda(I^r))$$

Bijection

$$\{\text{semi-infinite monomials}\} \leftrightarrow \{\text{partitions}\} \times \mathbb{Z}$$

$$I \mapsto (\lambda(I), c(I))$$

Energy

The energy of $\mathbf{I}$ is

$$|\mathbf{I}| = \sum_{k=1}^{r} |\lambda(I^k)| \in \mathbb{Z}_{\geq 0}$$

Charge-energy decomposition

$$\mathbf{F} = \bigoplus_{\mathbf{c} \in \mathbb{Z}^r, j \in \mathbb{Z}_{\geq 0}} \mathbf{F}_j^{\mathbf{c}}$$

$$\mathbf{F}_j^{\mathbf{c}} = \mathrm{Span}_{\mathbb{C}}\{\mathbf{I} \mid \mathbf{c}(I) = \mathbf{c},\ |\mathbf{I}| = j\}$$

Fermionic Fock space

Wedging and contracting operators

$$\psi(j)(i_1 \wedge i_2 \wedge \dots) = \begin{cases} 0 & \text{if } j = i_s \text{ for some } s, \\ (-1)^s i_1 \wedge \dots \wedge i_s \wedge j \wedge i_{s+1} \wedge \dots & \text{if } i_s > j > i_{s+1}. \end{cases}$$

$$\psi(j)^*(i_1 \wedge i_2 \wedge \dots) = \begin{cases} 0 & \text{if } j \neq i_s \text{ for all } s, \\ (-1)^{s-1} i_1 \wedge i_2 \wedge \dots \wedge i_{s-1} \wedge i_{s+1} \wedge \dots & \text{if } j = i_s. \end{cases}$$

Fermionic Fock space

Representation of Cl^r on $\mathbf{F}$

$$\psi^l(j) \mapsto \mathrm{id}^{\otimes l-1} \otimes \psi(j) \otimes \mathrm{id}^{\otimes r-l}$$

$$\psi^l(j)^* \mapsto \mathrm{id}^{\otimes l-1} \otimes \psi(j)^* \otimes \mathrm{id}^{\otimes r-l}$$

We have

$$\psi^l(j)(F^{\mathbf{c}}) \subseteq F^{\mathbf{c}+1_l}$$

$$\psi^l(j)^*(F^{\mathbf{c}}) \subseteq F^{\mathbf{c}-1_l}$$

Physical interpretation

- $\psi^l(j)$ creates a particle of “color” l in state j
- $\psi^l(j)^*$ annihilates a particle of “color” l in state j

Boson-fermion correspondence

Bosonization

Define an $\mathfrak{s}^r$-structure on $\mathbf{F}$ by

$$p^l(n) \mapsto \frac{1}{n}\sum_{j\in\mathbb{Z}} \psi^l(j)\psi^l(j+n)^*, \quad n \in \mathbb{Z}\setminus\{0\},$$

$$p^l(0) \mapsto \sum_{j>0} \psi(j)\psi(j)^* - \sum_{j\leq 0} \psi(j)^*\psi(j)$$

We have isomorphism $\mathbf{F} \cong_{\mathfrak{s}^r-\mathrm{mod}} \mathbf{B}$, $\mathbf{F}_j^{\mathbf{c}} \leftrightarrow \mathbf{B}_j^{\mathbf{c}}$

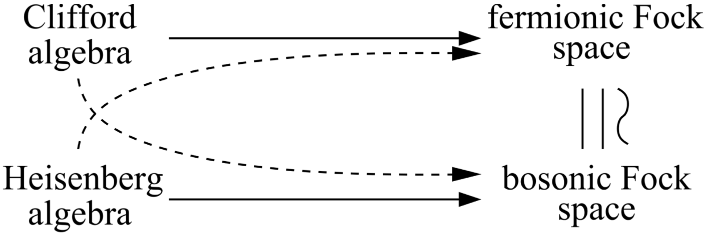
Fermionization

Can define Cl^r-structure on $\mathbf{B}$ (vertex operators)

Have isomorphism

$$\mathbf{F} \cong_{\mathrm{Cl}^r-\mathrm{mod}} \mathbf{B}$$

Algebraic boson-fermion correspondence

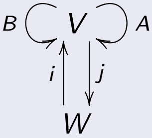


The moduli space $\mathcal{M}(r, n)$

Definition

$\mathcal{M}(r, n)$ = moduli space of framed rank r torsion-free sheaves on $\mathbb{CP}^2$ with $c_2 = n$

Alternative description



$W = \mathbb{C}^r, \ \ V = \mathbb{C}^n$

$\mathcal{M}(r, n) \cong \{(A, B, i, j) \mid [A, B]+ij = 0, \ (A, B, i, j) \text{ is stable}\}/GL(V)$

(A, B, i, j) is stable if $\nexists$ proper A, B-invariant subspace of V containing im i

$r = 1$: $\mathcal{M}(1, n) \cong$ Hilbert scheme of n points in $\mathbb{C}^2$

Previous work

Consider $r = 1$ case (Hilbert schemes)

Nakajima/Grojnowski defined action of Heisenberg algebra on cohomology via correspondences

Natural torus action – fixed points are Nakajima A_∞ quiver varieties (Nakajima defines action of gl_∞ via correspondences)

Localization Theorem: Equivariant cohomology of a space isomorphic to equivariant cohomology of fixed point set

Yields geometric boson-fermion correspondence ([S.])

Licata defined Clifford algebra action on the cohomology of the fixed point set

Drawback: Heisenberg operators defined globally but Clifford operators defined only on fixed point sets

Torus actions

Torus

Fix torus $T = (\mathbb{C}^*)^r \times \mathbb{C}^*$.

T-action on $\mathcal{M}(r, n)$

For $\mathbf{c} \in \mathbb{Z}^r$, let $\mathcal{M}_{\mathbf{c}}(r, n)$ be the moduli space with T-action

$$(e, t) \star_{\mathbf{c}} (A, B, i, j) = (tA, t^{-1}B, ie^{-1}t^{-\mathbf{c}}, et^{\mathbf{c}}j)$$

where

$$t^{\mathbf{c}} = (t^{c^1}, t^{c^2}, \dots, t^{c^r}) \in (\mathbb{C}^*)^r$$

and $(\mathbb{C}^*)^r$ acts diagonally on W (we've fixed a basis for W).

Equivariant cohomology

$H^*_T(\mathcal{M}_{\mathbf{c}}(r,n)) =$ T-equivariant cohomology of $\mathcal{M}_{\mathbf{c}}(r,n)$

$H^*_T(\mathcal{M}_{\mathbf{c}}(r,n))$ is a module over

$$H^*_T(\mathrm{pt}) = \mathbb{C}[\epsilon, b_1, \ldots, b_r]$$

where $\epsilon = c_2(t)$, $b_i = c_2(e_i)$ have degree 2.

Localized equivariant cohomology

We let

$$\mathcal{H}^*_T(\mathcal{M}_{\mathbf{c}}(r,n)) = H^*_T(\mathcal{M}_{\mathbf{c}}(r,n)) \otimes_{\mathbb{C}[b_1,\ldots,b_r,\epsilon]} \mathbb{C}(b_1,\ldots,b_r,\epsilon)$$

denoted the localized equivariant cohomology of $\mathcal{M}_{\mathbf{c}}(r,n)$.

Fixed points and tangent spaces

T-fixed points of $\mathcal{M}_{\mathbf{c}}(r,n)$

$$\mathcal{M}_{\mathbf{c}}(r,n)^T \leftrightarrow \{r\text{-colored semi-infinite monomials } \mathbf{l} \text{ of charge } \mathbf{c}\}$$

Tangent spaces

- $\mathcal{T}_{\mathbf{l}}$ = tangent space at $\mathbf{l}$
- T acts on $\mathcal{T}_{\mathbf{l}}$
- fix a splitting

$$\mathcal{T}_{\mathbf{l}} = \mathcal{T}_{\mathbf{l}}^{-} \oplus \mathcal{T}_{\mathbf{l}}^{+}$$

Inner product

Inclusion and projection maps

- inclusion

$$i : \mathcal{M}_{\mathbf{c}}(r,n)^T \hookrightarrow \mathcal{M}_{\mathbf{c}}(r,n)$$

- projection

$$p : \mathcal{M}_{\mathbf{c}}(r,n)^T \to \{\mathrm{pt}\}$$

Inner product

Define inner product on $\mathcal{H}_T^{2rn}(\mathcal{M}_{\mathbf{c}}(r,n),\mathbb{C})$ by

$$\langle a,b\rangle_{n,\mathbf{c}} = (-1)^{rn} p_*(i_*)^{-1}(a \cup b)$$

and extend to an inner product on $\bigoplus_{n,\mathbf{c}} \mathcal{H}_T^{2rn}(\mathcal{M}_{\mathbf{c}}(r,n),\mathbb{C})$

$$\langle\cdot,\cdot\rangle = \bigoplus_{n,\mathbf{c}} \langle\cdot,\cdot\rangle_{n,\mathbf{c}}$$

Our geometric vector space

Orthonormal classes

Define

$$[\mathbf{I}] = \frac{i_*(1_{\mathbf{I}})}{c^T_{\mathrm{top}}(\mathcal{T}^-_{\mathbf{I}})} \in \mathcal{H}^{2rn}_T(\mathcal{M}_{\mathbf{c}}(r,n),\mathbb{C})$$

The classes $\{[\mathbf{I}]\}$ are orthonormal

Definition

$$A_{\mathbf{c}}(r,n) = \mathrm{Span}_{\mathbb{C}}\{[\mathbf{I}]\} \subset \mathcal{H}^{2rn}_T(\mathcal{M}_{\mathbf{c}}(r,n),\mathbb{C})$$

$$\mathbf{A} = \bigoplus_{n,\mathbf{c}} A_{\mathbf{c}}(r,n)$$

Restriction of $\langle\cdot,\cdot\rangle$ to $\mathbf{A}$ is non-degenerate and $\mathbb{C}$-valued

Note: $\mathbf{A}$ is a $\mathbb{C}$-lattice of localized equivariant cohomology

Operators on equivariant cohomology

Inner product

Define inner product on $\mathcal{H}_T^{2r(n_1+n_2)}(\mathcal{M}_{\mathbf{c}}(r,n_1)\times\mathcal{M}_{\mathbf{d}}(r,n_2))$ analogously

$$\langle a,b\rangle_{\mathbf{c},\mathbf{d}}=(-1)^{rn_2}p_*((i_1\times i_2)_*)^{-1}(a\cup b)$$

Operators (Carlsson-Okounkov)

A class $\beta\in\mathcal{H}_T^{2r(n_1+n_2)}(\mathcal{M}_{\mathbf{c}}(r,n_1)\times\mathcal{M}_{\mathbf{d}}(r,n_2))$ gives a linear operator

$$\beta: A_{\mathbf{c}}(r,n_1)\to A_{\mathbf{d}}(r,n_2)$$

by

$$\langle\beta a,b\rangle_{\mathbf{d}}\stackrel{\text{def}}{=}\langle a\otimes b,\beta\rangle_{\mathbf{c},\mathbf{d}}$$

Tautological bundles

$$V \times_{GL(V)} \{(A,B,i,j) \mid [A,B]+ij=0,\ (A,B,i,j) \text{ stable}\}$$
$$\downarrow$$
$$\mathcal{M}_c(r,n)$$

is a T-equivariant vector bundle – denote it V

T-equivariant vector bundle $W \times \mathcal{M}_{\mathbf{c}}(r,n) \longrightarrow \mathcal{M}_{\mathbf{c}}(r,n)$ – denote it W

Have T-equivariant vector bundles

- $\mathrm{Hom}(V,V)$
- $\mathrm{Hom}(V,W)$
- $\mathrm{Hom}(W,V)$

T-equivariant complex

T-equivariant complex of vector bundles on $\mathcal{M}_{\mathbf{c}}(r, n_1) \times \mathcal{M}_{\mathbf{d}}(r, n_2)$

$$\mathrm{Hom}(V_1, V_2) \xrightarrow{\sigma} \begin{matrix} t\,\mathrm{Hom}(V_1, V_2) \oplus t^{-1}\,\mathrm{Hom}(V_1, V_2) \\ \oplus \\ \mathrm{Hom}(W_1, V_2) \oplus \mathrm{Hom}(V_1, W_2) \end{matrix} \xrightarrow{\tau} \mathrm{Hom}(V_1, V_2).$$

where

$$\sigma(\xi) = \begin{pmatrix} \xi A_1 - A_2\xi \\ \xi B_1 - B_2\xi \\ \xi i_1 \\ -j_2\xi \end{pmatrix}, \quad \tau \begin{pmatrix} C \\ D \\ I \\ J \end{pmatrix} = \left([A, D] + [C, B] + i_2 J + I j_1\right).$$

Notes:

1. When $\mathbf{c} = \mathbf{d}$, $n_1 = n_2$, cohomology of this complex is the tangent bundle.
2. Zero set of section of a complex similar to above yields Nakajima's correspondences for Heisenberg action on Hilbert schemes (rank 1 case).

Geometric Heisenberg operators

Definition

$\mathcal{K}_{\mathbf{c},\mathbf{d}}(n_1, n_2)$ = cohomology of the above complex (vector bundle)

Define operators

$$P^l(n) : \mathbf{A} \to \mathbf{A}$$

by

$$P^l(n)|_{A_{\mathbf{c}}(r,k)} = \pm \gamma^l c^T_{\mathrm{top}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(k, k-n)), \ \ n \neq 0$$

$$P^l(0)|_{A_{\mathbf{c}}(r,k)} = c^l \mathrm{id}$$

Notes:

- γ^l are orthogonal equivariant cohomology classes
- c^T_{top} denotes top non-vanishing Chern class

Geometric bosonic Fock space

Theorem [Licata-S.]

1. maps

$$p^l(n) \mapsto P^l(n), \quad n \in \mathbb{Z}, \quad l \in \{1, \ldots, r\}, \quad K \mapsto \mathrm{id}$$

   define a rep of $\mathfrak{s}^r$ on **A**

2. linear map

$$\mathbf{A} \xrightarrow{\cong} \mathbf{B}, \quad [\mathbf{I}] \mapsto (q^{c(I^1)} s_{\lambda(I^1)}, \ldots, q^{c(I^r)} s_{\lambda(I^r)})$$

   is an isometric isomorphism of $\mathfrak{s}^r$-modules

3. under this isomorphism

$$\mathcal{H}_T^{2rn}(\mathcal{M}_{\mathbf{c}}(r,n)) \supset A_{\mathbf{c}}(r,n) \longleftrightarrow \mathbf{B}_n^{\mathbf{c}}$$

Geometric Clifford operators

Definition

Define operators

$$\Psi^l(n) : \mathbf{A} \to \mathbf{A}$$

by

$$\Psi^l(n)|_{A_{\mathbf{c}}(r,k)} = c^T_{\mathrm{top}}(\mathcal{K}_{\mathbf{c},\mathbf{c}+1_l}(k, k+n-c^l-1))$$

Define $\Psi^l(n)^*$ to be adjoint to $\Psi^l(n)$

Geometric fermionic Fock space

Theorem [Licata-S.]

1. maps

$$\psi^l(n) \mapsto \Psi^l(n), \quad \psi^l(n)^* \mapsto \Psi^l(n)^*, \quad n \in \mathbb{Z}, \quad l \in \{1, \dots, r\},$$

   define a rep of Cl^r on $\mathbf{A}$

2. linear map

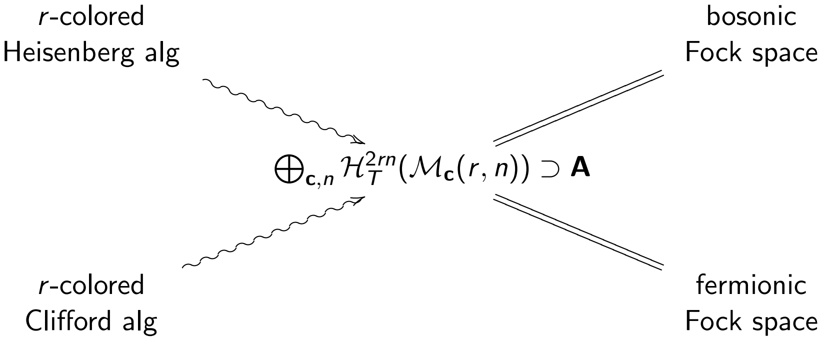
$$\mathbf{A} \xrightarrow{\cong} \mathbf{F}, \quad [\mathbf{l}] \mapsto \mathbf{l}$$

   is an isometric isomorphism of Cl^r-modules

3. under this isomorphism

$$\mathcal{H}_T^{2rn}(\mathcal{M}_{\mathbf{c}}(r,n)) \supset A_{\mathbf{c}}(r,n) \longleftrightarrow \mathbf{F}_n^{\mathbf{c}}$$

Summary



Important idea: consider different T-actions together

Further directions

- homogeneous realization of basic rep of $\widehat{\mathfrak{gl}_r}$
  - affine Lie algebra $\widehat{\mathfrak{gl}_r}$ embeds into Cl^r
  - slight modification of complexes yields action of $\widehat{\mathfrak{gl}_r}$ on $\mathbf{A}$
  - $\bigoplus_{n,\mathbf{c}\,:\,\sum c^\alpha=0} A_{\mathbf{c}}(r,n) \cong$ basic representation
- principal realization of basic rep?
- relation between these and other geometric constructions of basic rep?
  - explicit algebraic descriptions of nice geometric bases
  - level-rank duality
- other vertex operator constructions?
- categorification?