Geometric Methods in Representation Theory

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April 12, 2007

Overview

Representation Theory (Algebra)

- Important area of mathematics and physics
- Lots known but still a very active area

↕ Geometric Representation Theory

Geometry

- Ubiquitous in mathematics and physics
- Ancient field
- Active area of research

Groups

Group

Set with an operation that has an identity and inverses.

Examples

Set	Operation	Identity	Inverse of a
Integers	Addition	0	$-a$
$\mathbb{R} - \{0\}$	Multiplication	1	$1/a$
GL_n	Matrix mult	I_n	a^{-1}

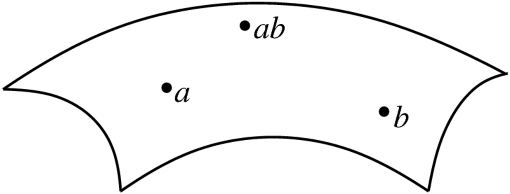
Note: GL_n - $n \times n$ invertible matrices

Lie Groups

Sophus Lie
(1842-1899)

Lie Group

- "Continuous" group
- Group which is also a smooth manifold (curve, surface, etc.)
- Can be considered a geometric object



Examples

Lie Groups in Geometry

- Euclidean space $\mathbb{R}^n$ with vector addition
- GL_n: $n \times n$ invertible matrices with matrix multiplication
- SO_n: rotations of n-dimensional space

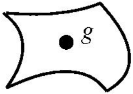
Lie Groups in Physics

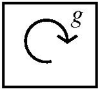
- Lorentz group & Poincaré group: symmetries of spacetime used in special relativity
- Heisenberg group: used in quantum mechanics
- Gauge group of the standard model in particle physics

Representations of Lie Groups

Representation of a Lie group

Vector space on which a Lie group acts naturally.



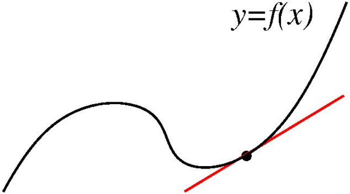


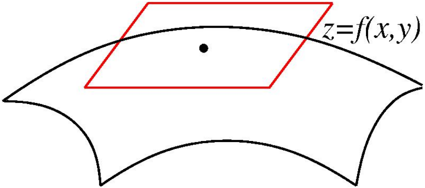
Examples

- SO_n rotates n-dimensional space
- Lorentz group and Poincaré group both act on *Minkowski spacetime*
- Heisenberg group acts on state space of one-dimensional quantum mechanical systems
- Gauge group acts on spaces involved in the description of fundamental particles

Tangent Spaces

Recall tangent lines and spaces from calculus:

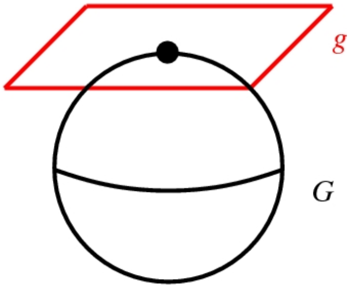




Allows us to consider linear spaces

Lie Algebras

Tangent space to the identity element of a Lie group G

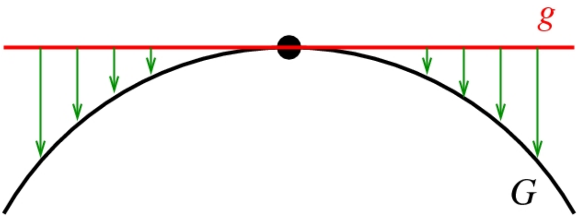


This is called the Lie algebra (denoted g) of G

Exponential Map

Exponential Map

Identifies the Lie algebra g (tangent space) with the Lie group G



Lie algebra encodes information about the group operation

Exponential Map: Example

Example

Group: GL_n - the $n \times n$ invertible matrices

Lie algebra: gl_n - *all* $n \times n$ matrices

Exponential map: If M is an $n \times n$ matrix,

$$e^M = 1 + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \dots$$

is an $n \times n$ invertible matrix.

Exponential Map: Dictionary

Dictionary

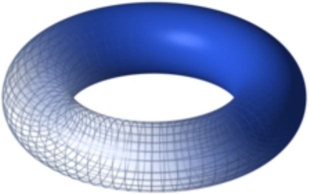
Exponential map allows us to translate:

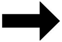
Representations of Lie groups	$\longleftrightarrow$	Representations of Lie algebras

Homology

Homology

Procedure for assigning algebraic objects (e.g. vector spaces) to geometric objects (e.g. curves, surfaces)





{line, plane, line}

Homology

Uses

Yields a way of distinguishing spaces:

Different algebraic objects (e.g. vector spaces) $\Longrightarrow$ Different geometric objects

Poincaré
(1854-1912)

Poincaré Conjecture

- Basic idea: Special case (3D sphere) of

  Same algebraic objects $\overset{?}{\Longrightarrow}$ Same geometric objects

- Proved by Grigori Perelmen in 2002-2003
- Awarded the Fields medal (declined)

Geometric Representation Theory

Representations (Algebra)

Lie group or Lie algebra acting on a vector space

Homology (Geometry)

Method for assigning vector spaces to geometric objects

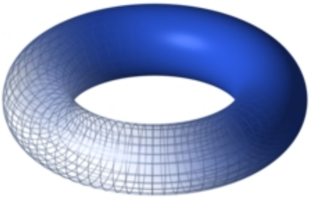
Geometric Representation Theory

Combine these two ideas!

Geometric Representation Theory

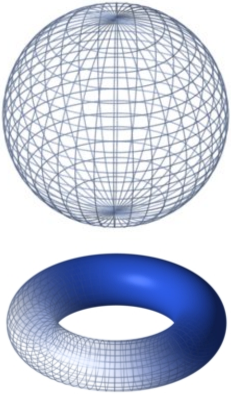
Geometric Representation Theory

Construct representations by natural Lie algebra actions on homology of geometric objects



$\Rightarrow$ Vector spaces $\circlearrowleft$ Lie algebra

Geometric Objects Involved



Mathematics

- Grassmannians
- Flag varieties
- Quiver varieties
- Hyper-Kähler manifolds

Physics

- Spaces appearing in gauge theory
- Moduli spaces of Yang-Mills instantons on gravitational instantons

Conclusion: Interplay Has Its Advantages

Geometry $\longrightarrow$ Representation Theory

- Prove conjectures (e.g. Kazhdan-Lusztig conjecture)
- Simplify existing proofs (new insights)

Representation Theory $\longrightarrow$ Geometry

- Representation theory *organizes* homology
- Identify new structure

Algebra $\Longleftrightarrow$ Geometry