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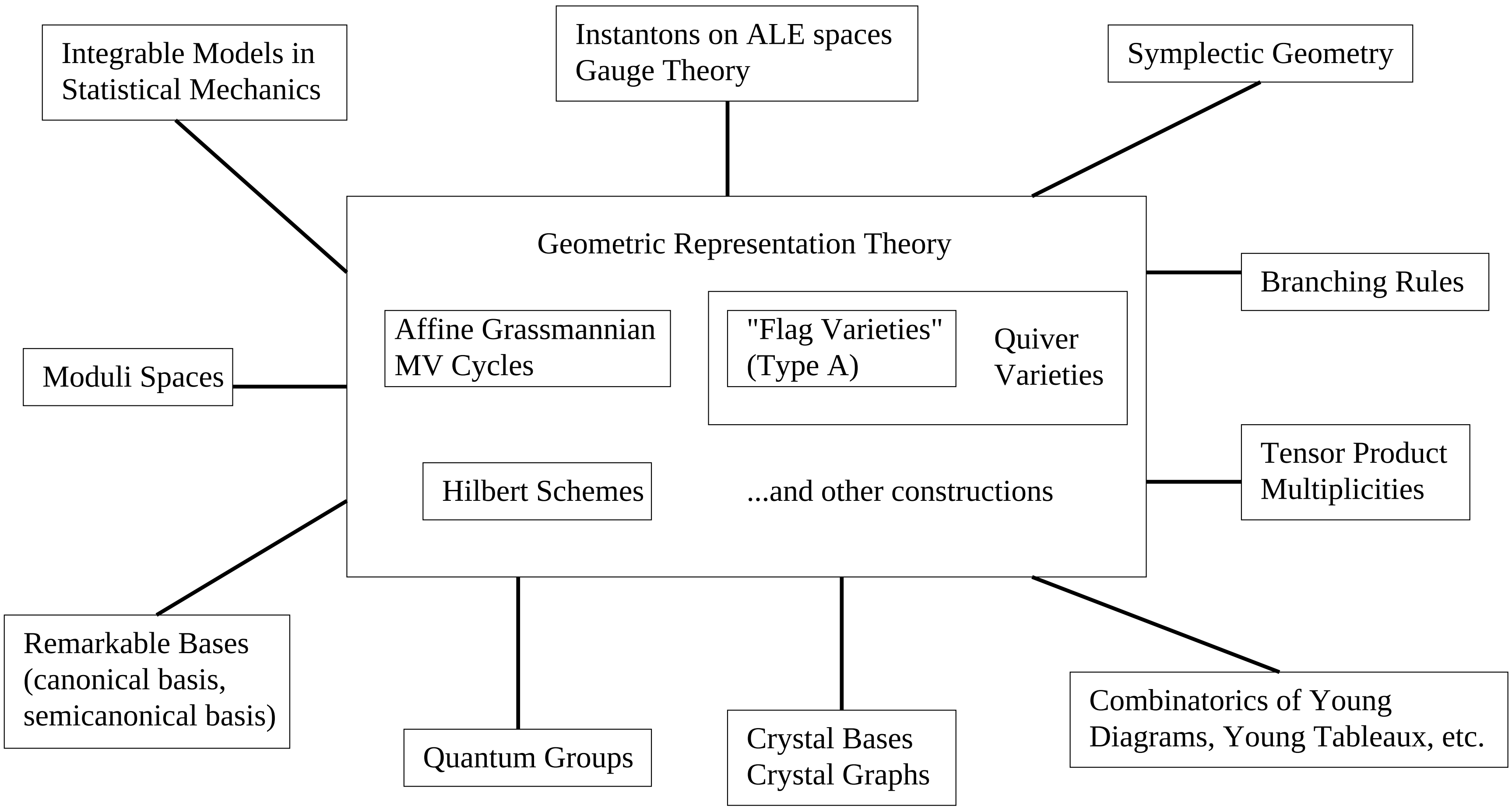
Quiver Varieties
and
Geometric Representation Theory

Geometric Representation Theory



A Sample Lie Algebra: $\mathfrak{sl}_n$

$\mathfrak{sl}_n = n \times n$ traceless matrices

$$[A, B] = AB - BA$$

Presentation in terms of *Chevalley generators*

$$e_i = E_{i,i+1}, \; f_i = E_{i+1,i}, \; i = 1, \dots, n-1$$

and certain relations.

A *representation* of $\mathfrak{sl}_n$ is

- A complex vector space V
- A map $\mathfrak{sl}_n \to \mathrm{End}(V)$ such that

$$[A, B](v) = A(B(v)) - B(A(v))$$

Dictionary

simply-laced Kac-Moody algebra $\leftrightarrow \mathfrak{sl}_n$

irreducible integrable highest weight rep $\leftrightarrow$ f.d. rep

weight space of a rep $\leftrightarrow$ eigenspace of diagonal matrices

Quiver Variety Approach

$\mathfrak{g}$ = simply-laced Kac-Moody Lie algebra (e.g. $\mathfrak{g} = \mathfrak{sl}_n$)

L = irreducible integrable highest weight rep of $\mathfrak{g}$

L decomposes into weight spaces:

$$L = \bigoplus_{\lambda} L_\lambda$$

$\lambda \longleftrightarrow$ quiver variety $QV(\lambda)$.

Quiver Variety Approach

Weight space $L_\lambda \longleftrightarrow$ "Homology" of $QV(\lambda)$

$\dim L_\lambda \longleftrightarrow$ # irr. comps. of $QV(\lambda)$

Action of $\mathfrak{g} \longleftrightarrow$ Correspondences

Correspondences:

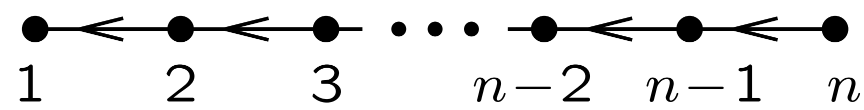
$QV(\lambda_1) \leftarrow$ "Intermediate Variety" $\rightarrow QV(\lambda_2)$

Quivers

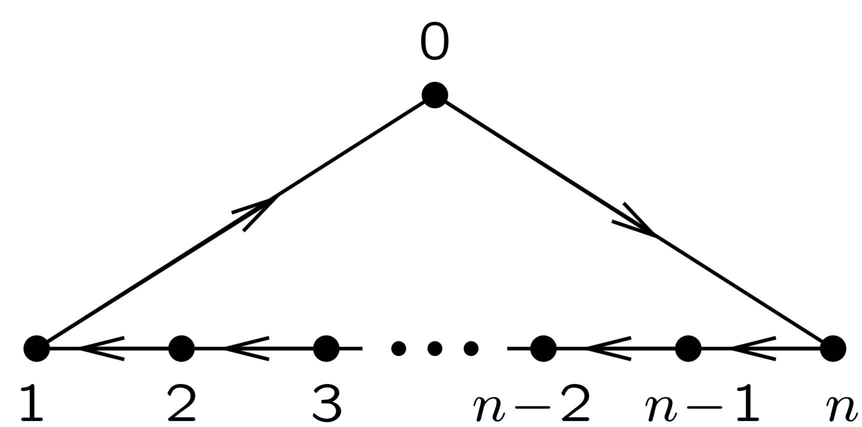
quiver = oriented graph

Examples:

1. Quiver of type A_n



2. Quiver of type $A_n^{(1)}$



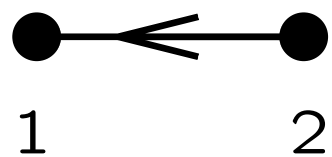
Representations of Quivers

Representation of a quiver:

vertex $\longrightarrow$ f.d. vector space

arrow $\longrightarrow$ linear map

Example: A representation of the quiver



consists of

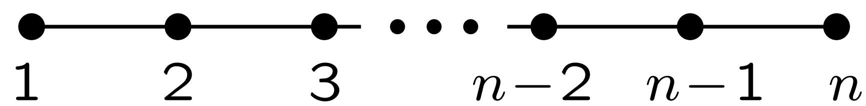
V_1, V_2 - f.d. $\mathbb{C}$-vector spaces

$x \in \mathrm{Hom}(V_2, V_1)$

Quiver Varieties

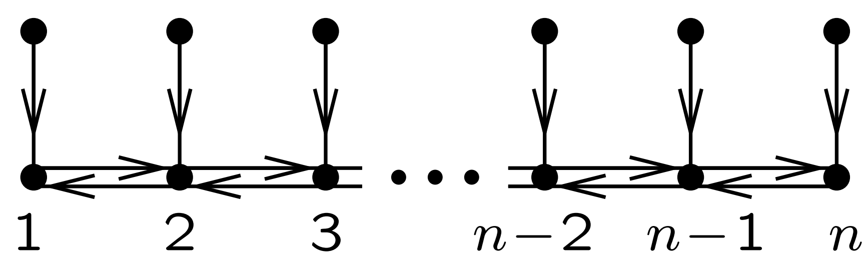
$\mathfrak{g}$ = simply-laced Kac-Moody Lie algebra (e.g. $\mathfrak{g} = \mathfrak{sl}_{n+1}$)

Consider the Dynkin graph of $\mathfrak{g}$



Take *both* orientations of each edge

Add in *shadow vertices*



Call this quiver $Q(\mathfrak{g})$

Quiver Varieties

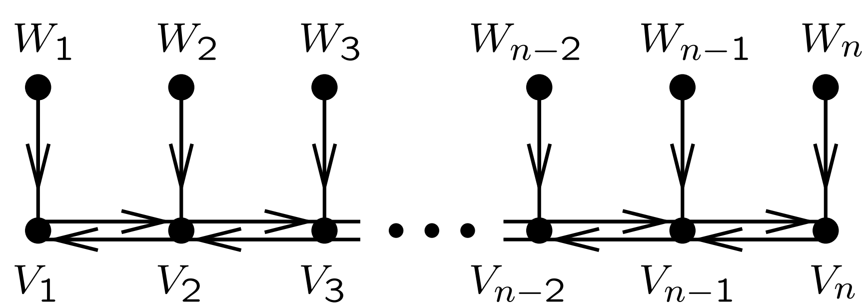
I = set of vertices of Dynkin graph of $\mathfrak{g}$

Fix collections of v.s. $\mathbf{V} = (V_k)_{k \in I}$, $\mathbf{W} = (W_k)_{k \in I}$

Let $\mathbf{v} = (\dim V_k)_{k \in I}$, $\mathbf{w} = (\dim W_k)_{k \in I}$

Define (Nakajima)

$$\mathcal{L}(\mathbf{v}, \mathbf{w}) = \{\text{reps of } Q(\mathfrak{g}) \text{ with v.s. } \mathbf{V} \text{ and } \mathbf{W} + \text{conditions}\} / \prod_{k \in I} GL(V_k)$$



Alternative Description

$\mathbf{M}(\mathbf{v},\mathbf{w})$ = space of reps of quiver with fixed vector spaces

$$\begin{array}{ccl} \mathbf{M}(\mathbf{v},\mathbf{w})//_{\xi_1}\prod_k GL(V_k) & \longleftarrow & \text{hyper-Kähler quotient (smooth)} \\ \pi\Big\downarrow & & \\ \mathbf{M}(\mathbf{v},\mathbf{w})//_{\xi_2}\prod_k GL(V_k) & \longleftarrow & \text{hyper-Kähler quotient (singular)} \end{array}$$

$\mathcal{L}(\mathbf{v},\mathbf{w}) = \pi^{-1}(0)$

$\mathcal{L}(\mathbf{v},\mathbf{w})$ is a Lagrangian subvariety and a deformation retract of

$$\mathcal{M}(\mathbf{v},\mathbf{w}) \overset{\text{def}}{=} \mathbf{M}(\mathbf{v},\mathbf{w})//_{\xi_1}\prod_k GL(V_k)$$

Weights

Want to construct irred. integ. highest weight rep of $\mathfrak{g}$.

In $\mathcal{L}(\mathbf{v}, \mathbf{w})$,

$$\mathbf{w} \longleftrightarrow \text{highest weight of rep}$$

$$\mathbf{w} \longleftrightarrow \sum_{k \in I} w_k \Lambda_k$$

Λ_k – fundamental weights

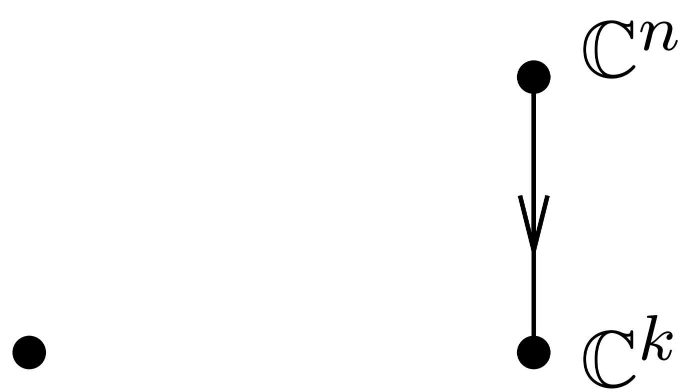
$$\mathbf{v} \longleftrightarrow \text{weight space}$$

$$\mathbf{v} \longleftrightarrow \sum_{k \in I} w_k \Lambda_k - \sum_{k \in I} v_k \alpha_k$$

α_k – simple roots

Example: $\mathfrak{sl}_2$

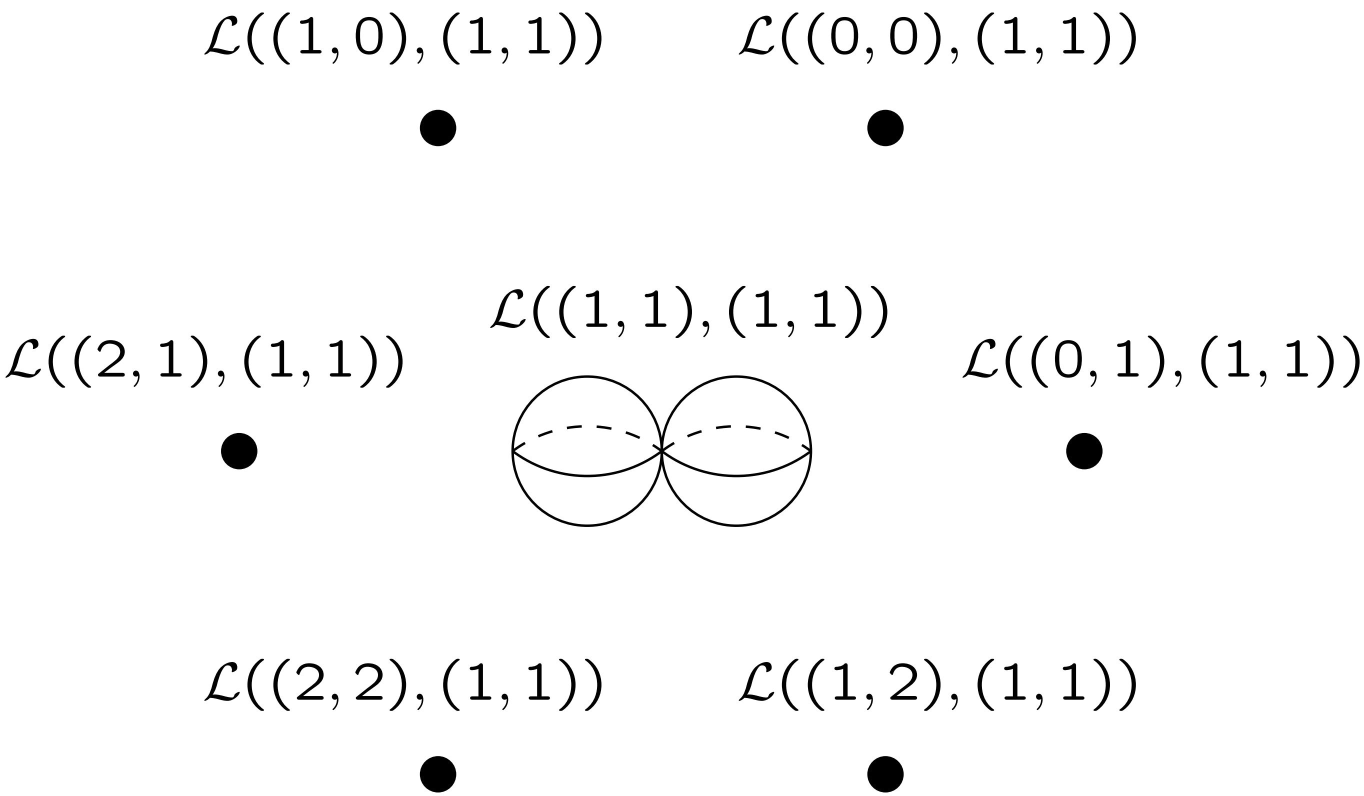
Let $\mathfrak{g} = \mathfrak{sl}_2$.



$$\begin{aligned}\mathcal{L}(k,n) &= \{B \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^k) \mid B \text{ surjective}\}/GL(\mathbb{C}^k) \\ &\cong Gr(n-k,n)\end{aligned}$$

Example: Adjoint Representation of $\mathfrak{sl}_3$

Adjoint rep of $\mathfrak{sl}_3$ has h.w. $\Lambda_1 + \Lambda_2 \Rightarrow \mathbf{w} = (1,1)$.



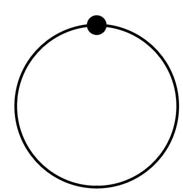
Other Examples of Quiver Varieties

- $\mathfrak{g} = \mathfrak{sl}_n$, highest weight $a\Lambda_1$
  $\longrightarrow$ partial flag varieties

- $\mathfrak{g}$ affine, $\mathbf{w} = 0$, $\mathbf{v} \leftrightarrow$ imaginary root
  $\longrightarrow$ ALE spaces (resolutions of simple singularities)

- $\mathfrak{g}$ affine
  $\longrightarrow$ moduli spaces of instantons on ALE spaces

- Jordan quiver



$\longrightarrow$ Hilbert schemes of points in $\mathbb{C}^2$

Benefits of Geometric Approach

- Alternative (often simpler) geometric proofs of algebraic facts
- Rep theory organizes homological information
- Connection to crystal graphs ($q \to 0$ limit of quantum groups)
- Geometrically defined bases with remarkable properties

Geometrically Defined Bases

Recall,

$$\text{weight space} \longleftrightarrow \text{``homology'' of QV}$$
$$\text{dimension of weight space} \longleftrightarrow \#\ \text{irr. comps. of QV}$$

Classes of irr. comps. of QV yield basis of representation

Homology Theory	**Basis**
Constructible functions	Semicanonical basis
Top dim Borel-Moore homology	Semicanonical basis?
Perverse sheaves	Canonical basis

Nice positivity, integrality, and compatibility properties

Nice Properties of Geometric Bases

Positivity & Integrality:

$$f_k \cdot b = \sum c_j b_j, \quad c_j \in \mathbb{Z}_{\geq 0}$$

Compatibility:

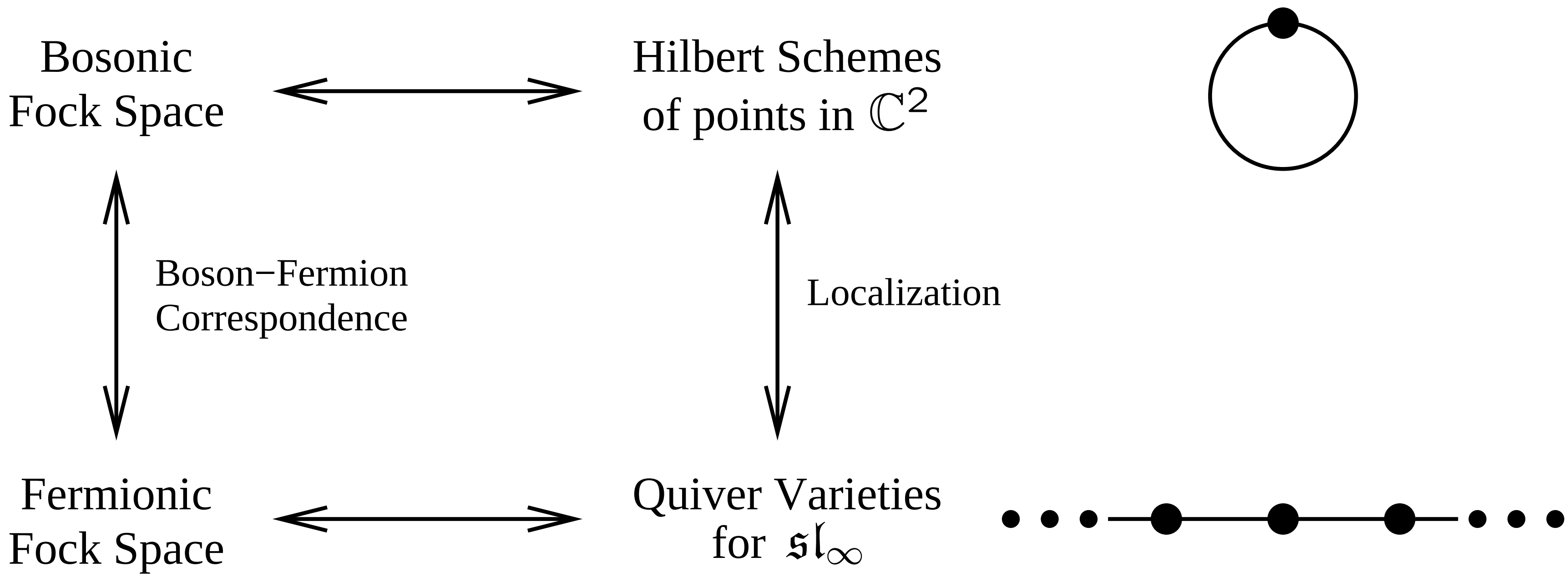
Have geometric basis B of $U^-(\mathfrak{g})$

For *any* irred hw rep L of $\mathfrak{g}$ with hw vec v,

$$\{b \cdot v \mid b \in B,\ b \cdot v \neq 0\}$$

is a basis of L.

Geometric Boson-Fermion Correspondence



Reference: Savage, arXiv:math.RT/0508438

Bosonic Fock Space

Infinite-dim Heisenberg algebra:

$$\mathfrak{s} = \bigoplus_{m \in \mathbb{Z} \setminus \{0\}} \mathbb{C} s_m \oplus \mathbb{C} K$$

$$[\mathfrak{s}, K] = 0, \qquad [s_m, s_n] = m \delta_{m,-n} K$$

Bosonic Fock Space:

$$B = \mathbb{C}[p_1, p_2, \ldots]$$

Action of $\mathfrak{s}$ on B by

$$s_m \mapsto m \frac{\partial}{\partial p_m}, \quad s_{-m} \mapsto p_m, \quad m > 0$$

$$K \mapsto \mathrm{Id}$$

Fermionic Fock Space

$F = \mathrm{Span}_{\mathbb{C}}\{\underline{i_0} \wedge \underline{i_1} \wedge \underline{i_2} \wedge \dots \mid i_k \in \mathbb{Z},\ i_0 > i_1 > \dots,\ i_k = -k \text{ for } k \gg 0\}$
$\quad = $ "infinite wedge space"

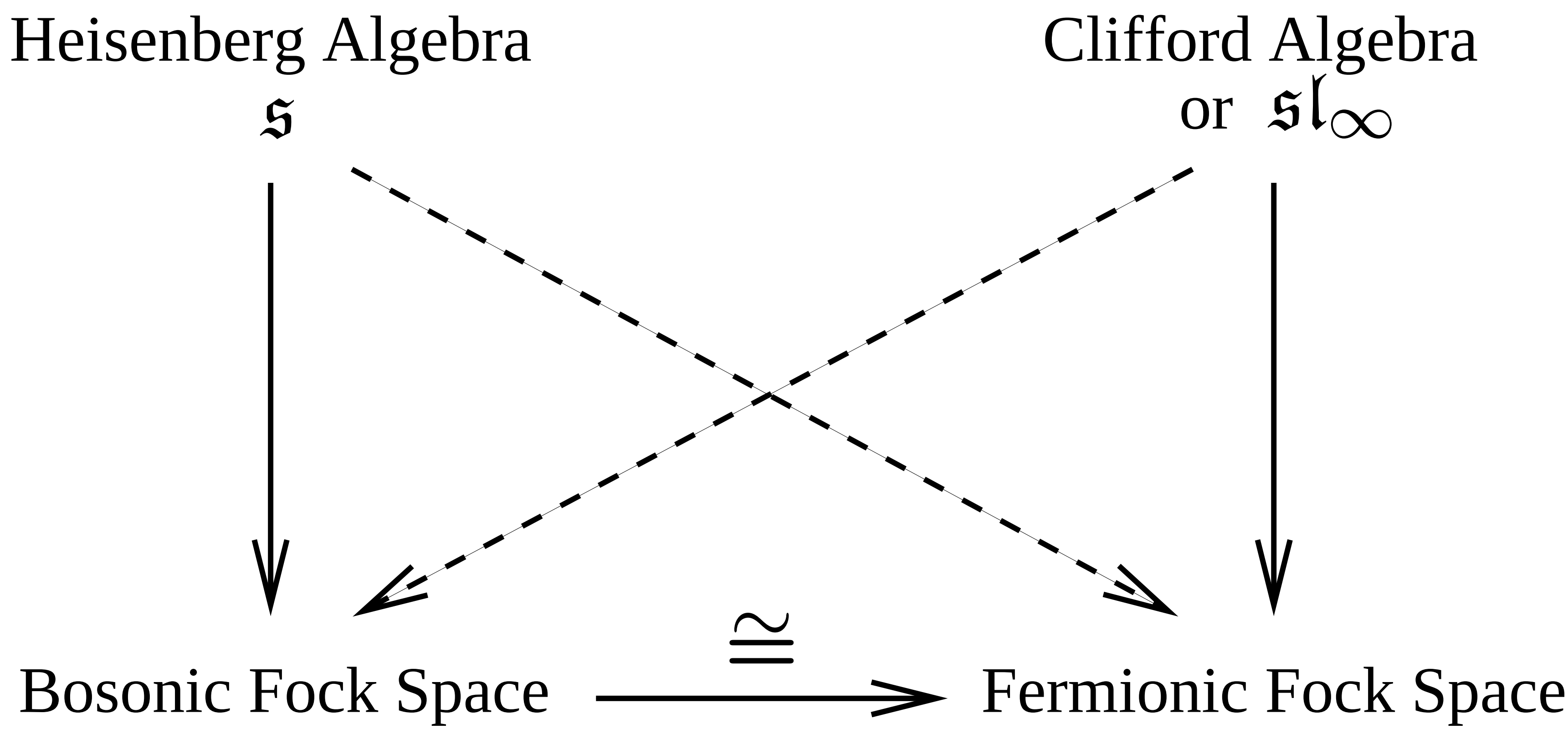
Part of a larger Fock space with action of an infinite Clifford algebra

$\mathfrak{gl}_\infty$ (and $\mathfrak{sl}_\infty$) act on F by derivations:

For $a \in \mathfrak{gl}_\infty$,

$$a(\underline{i_0} \wedge \underline{i_1} \wedge \dots) = (a \cdot \underline{i_0}) \wedge \underline{i_1} \wedge \cdots + \underline{i_0} \wedge (a \cdot \underline{i_1}) \wedge \cdots + \dots$$

Boson-Fermion Correspondence



Boson-Fermion Correspondence

Action of $\mathfrak{s}$ on F: "Bosonization"

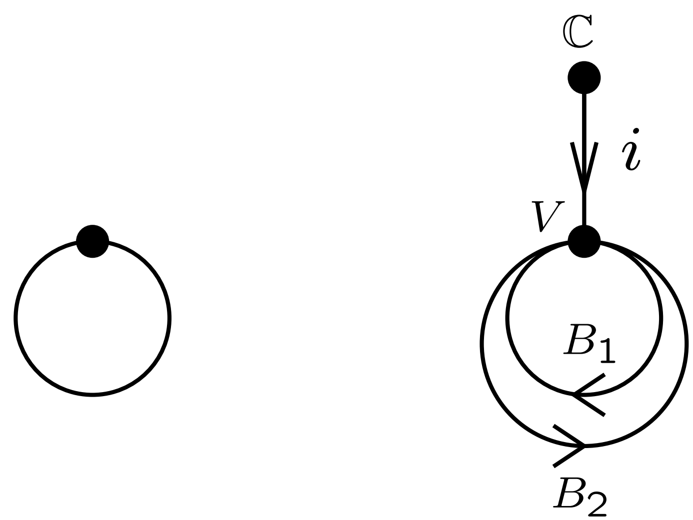
$$s_m \mapsto \sum_{j \in \mathbb{Z}} E_{j,j+m}, \;\; m \in \mathbb{Z} \backslash \{0\},$$
$$K \mapsto \mathrm{Id}$$

$F \cong B$ as $\mathfrak{s}$-modules

Action of Clifford algebra on B (generating functions, vertex operator algebras): "Fermionization"

$F \cong B$ as Clifford algebra modules

Hilbert Schemes



$X_n = \{(B_1, B_2, i) \mid B_j \in \text{End}\, V,\ i \in \text{Hom}(\mathbb{C}, V),\ + \text{ conditions}\}/GL(V)$

Here, $V \cong \mathbb{C}^n$.

Conditions:

- $B_1 B_2 = B_2 B_1$
- $\text{im}\, i$ generates V under B_1, B_2

Hilbert Schemes

$X_n \cong$ Hilbert scheme of n points in $\mathbb{C}^2$

Hilbert scheme is a resolution of singularities:

$$X_n \to (\mathbb{C}^2)^n / S_n$$

Have $T = \mathbb{C}^*$ action on X_n:

$$z \cdot (GL(V) \cdot (B_1, B_2, i)) = GL(V) \cdot (zB_1, z^{-1}B_2, i)$$

Geometric Bosonic Fock Space

$$\mathbb{H}_n^B = H_T^{2n}(X_n), \quad \mathbb{H}^B = \bigoplus_{n=0}^{\infty} \mathbb{H}_n^B$$

Correspondences: Natural projections

$$X_{n+k} \xleftarrow{\pi_1} X_{n+k} \times X_n \xrightarrow{\pi_2} X_n$$

Define $\mathfrak{p}_{-k} \in \operatorname{End} \mathbb{H}^B$, $k \geq 0$, by

$$\mathfrak{p}_{-k}(\alpha) = (\pi_1)_!(\pi_2^* \alpha \cup [\Sigma_{n,k}]), \ \alpha \in \mathbb{H}_n^B$$

Here

$$\Sigma_{n,k} \subset X_{n+k} \times X_n$$
$$\leftrightarrow \text{“adding } k \text{ points'' at } z \in \mathbb{C} \times \{0\} \subset \mathbb{C}^2$$

Geometric Bosonic Fock Space

Define adjoint operator $\mathfrak{p}_k$, $k > 0$

Prop (Nakajima, Vasserot):

$$[\mathfrak{p}_k, \mathfrak{p}_l] = k\delta_{k,-l}\mathrm{Id}$$

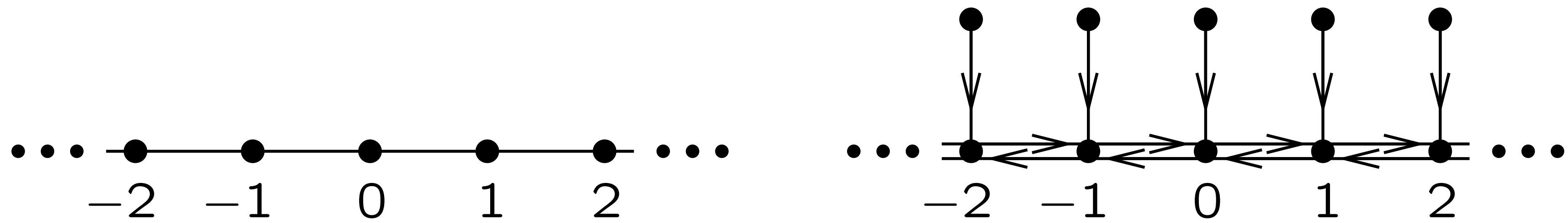
So

$$s_k \mapsto \mathfrak{p}_k, \quad K \mapsto \mathrm{Id}$$

defines action of $\mathfrak{s}$ on $\mathbb{H}^B$ and

$$\mathbb{H}^B \cong B \text{ as } \mathfrak{s}\text{-modules}$$

Quiver Varieties for $\mathfrak{sl}_\infty$



$\mathbf{V}$ = f.d. $\mathbb{Z}$-graded complex vector space

$\mathbf{v} = \dim \mathbf{V} = (\dim V_k)_{k \in I}$
$\mathbf{w} = \mathbf{e}_0$

$$\mathcal{M}(\mathbf{v}, \mathbf{e}_0) = \{(B_1, B_2, i) \mid B_j \in \operatorname{End} \mathbf{V}, i \in \operatorname{Hom}(\mathbb{C}, V_0) + \text{conditions}\} / \prod_k GL(V_k)$$

Conditions:

- $B_1 \in \operatorname{End} \mathbf{V}$, $\deg B_1 = 1$ (i.e. $B(V_k) \subseteq V_{k+1}$)
- $B_2 \in \operatorname{End} \mathbf{V}$, $\deg B_2 = -1$ (i.e. $B(V_k) \subseteq V_{k-1}$)
- $B_1 B_2 = B_2 B_1$
- $\operatorname{im} i$ generates $\mathbf{V}$ under B_1, B_2

Geometric Fermionic Fock Space

$$\mathbb{H}^F = \bigoplus_{\mathbf{v}} H_T^{2|\mathbf{v}|}(\mathcal{M}(\mathbf{v}, \mathbf{e}_0)) \quad \text{(trivial } T\text{-action)}$$

Correspondences: Have natural projections

$$\mathcal{M}(\mathbf{v} + \mathbf{e}_k, \mathbf{e}_0) \longleftarrow \mathcal{M}(\mathbf{v} + \mathbf{e}_k, \mathbf{e}_0) \times \mathcal{M}(\mathbf{v}, \mathbf{e}_0) \longrightarrow \mathcal{M}(\mathbf{v}, \mathbf{e}_0)$$

Define geometric action of $\mathfrak{sl}_\infty$ (similar to Hilbert scheme picture) and

$$\mathbb{H}^F \cong F \text{ as } \mathfrak{sl}_\infty\text{-modules}$$

Torus Fixed Points of Hilbert Schemes

At a T-fixed point of X_n, weight decomposition of V gives grading

$$V = \bigoplus_{m \in \mathbb{Z}} V_m$$

Can show that

- B_1, B_2 have degrees 1 and -1 resp.
- $\operatorname{im} i \subseteq V_0$

Thus,

$$T\text{-fixed points of H.S.} = \text{quiver varieties for } \mathfrak{sl}_\infty$$

Localization and the Boson-Fermion Correspondence

X smooth with a T-action
X^T = T-fixed points of X

Localization theorem states

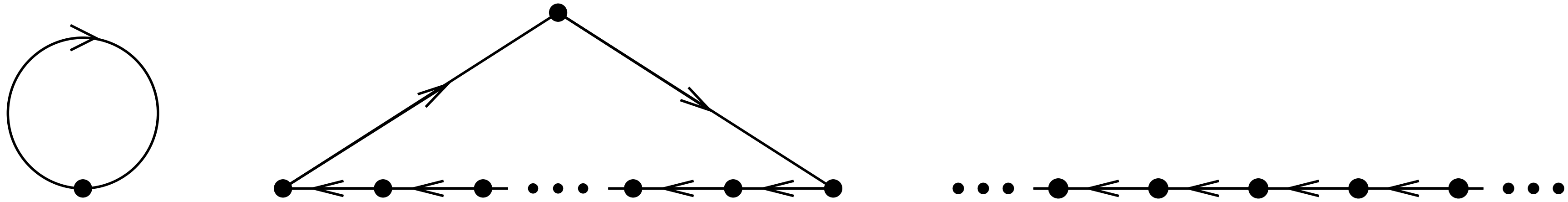
$$H_T^*(X) \otimes \mathbb{C}(t) \cong H_T^*(X^T) \otimes \mathbb{C}(t)$$

Thus (after a few technical steps),

$$\mathbb{H}^B = \bigoplus_n H_T^{2n}(X_n) \cong \bigoplus_n H_T^{2n}(X_n^T) = \mathbb{H}^F$$

A geometric boson-fermion correspondence!

Further Directions



Fixed points of $\mathbb{Z}/n\mathbb{Z} \subset T$ yield quiver varieties for $\widehat{\mathfrak{sl}}_n$

Vertex operator construction of basic rep of $\widehat{\mathfrak{sl}}_n$ should fit into this geometric picture

Should help give algebraic description of nice geometric bases

Crystal Graphs

Quantum group $U_q(\mathfrak{g})$ is a q-deformation of $U(\mathfrak{g})$

Representations of $\mathfrak{g}$ (or $U(\mathfrak{g})$) have a q-deformation

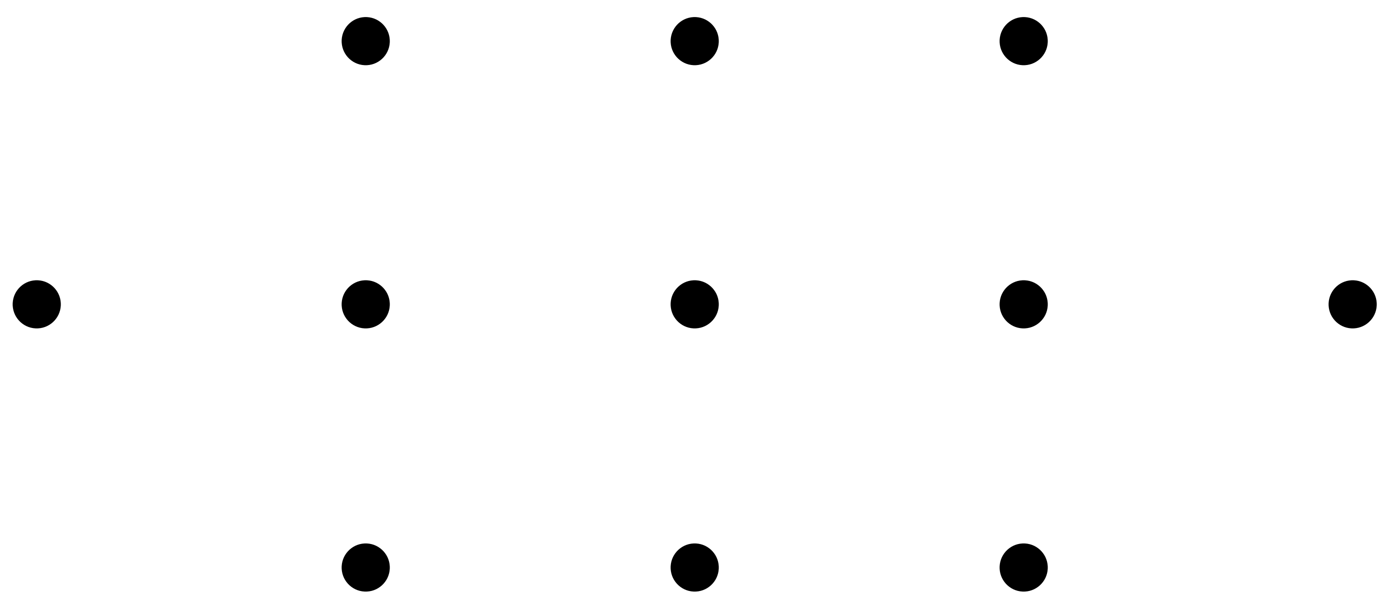
$$\text{usual rep theory} \xleftarrow{q \to 1} \text{quantum groups} \xrightarrow{q \to 0} \text{crystals}$$

In $q \to 0$ limit, rep theory becomes combinatorics

Crystal graphs

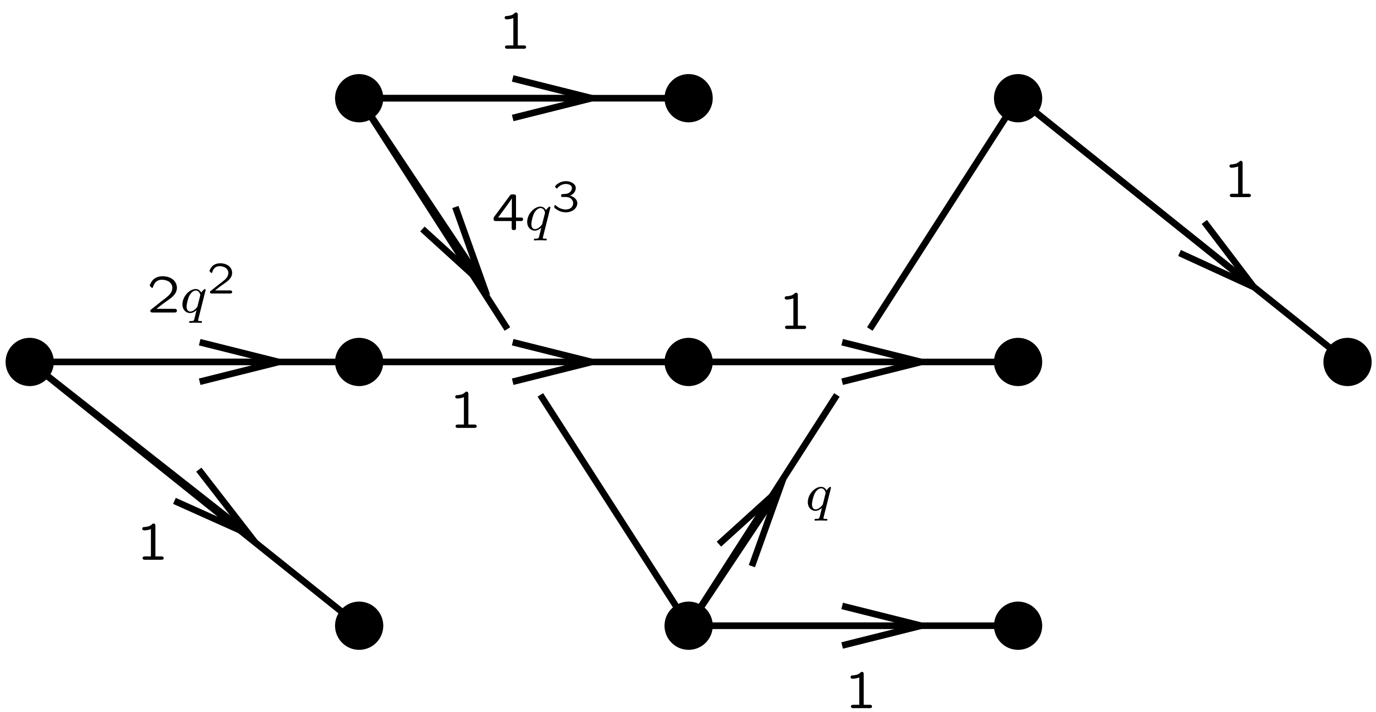
$L =$ rep of $U_q(\mathfrak{g})$

Fix a basis of L and depict elements of the basis by vertices:



Crystal graphs

Consider the action of a Chevalley generator, say f_2:



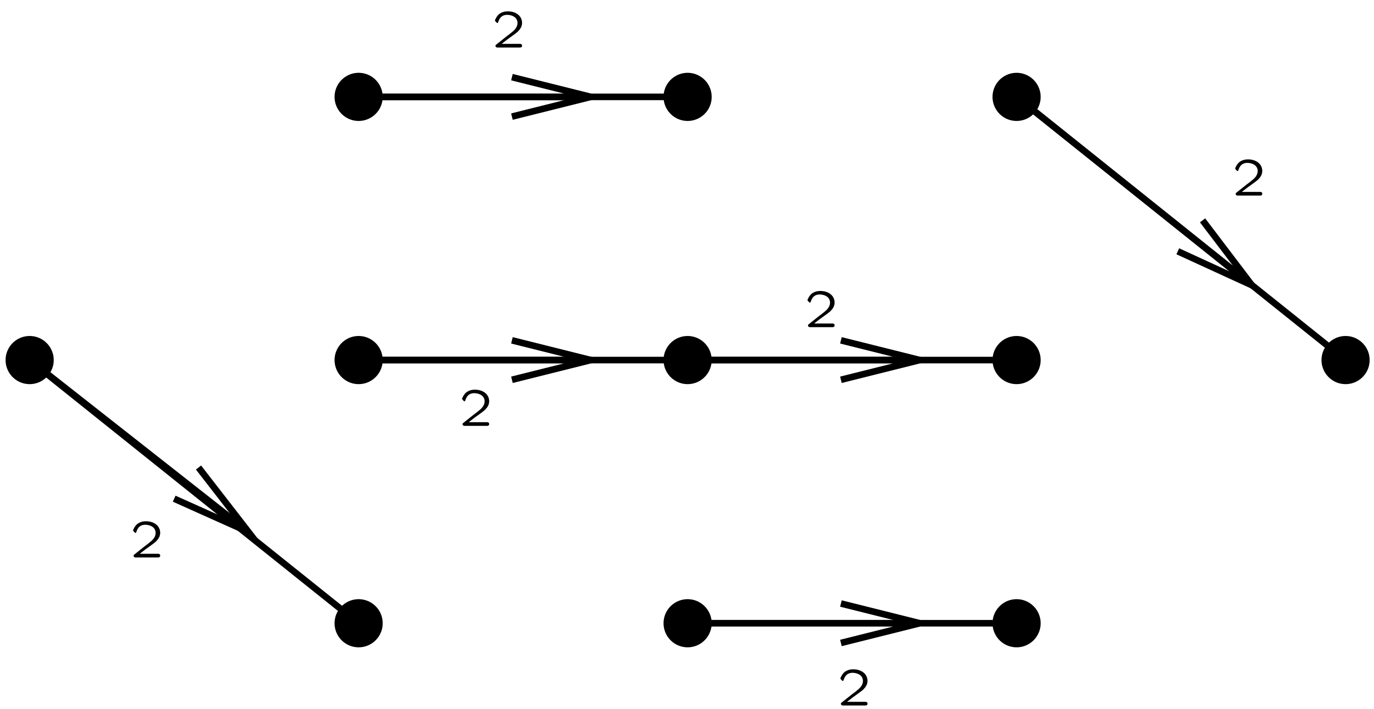
"Nice" basis:

- At most one coeff of 1 in each expression
- All other coeffs have positive power of q

Crystal graphs

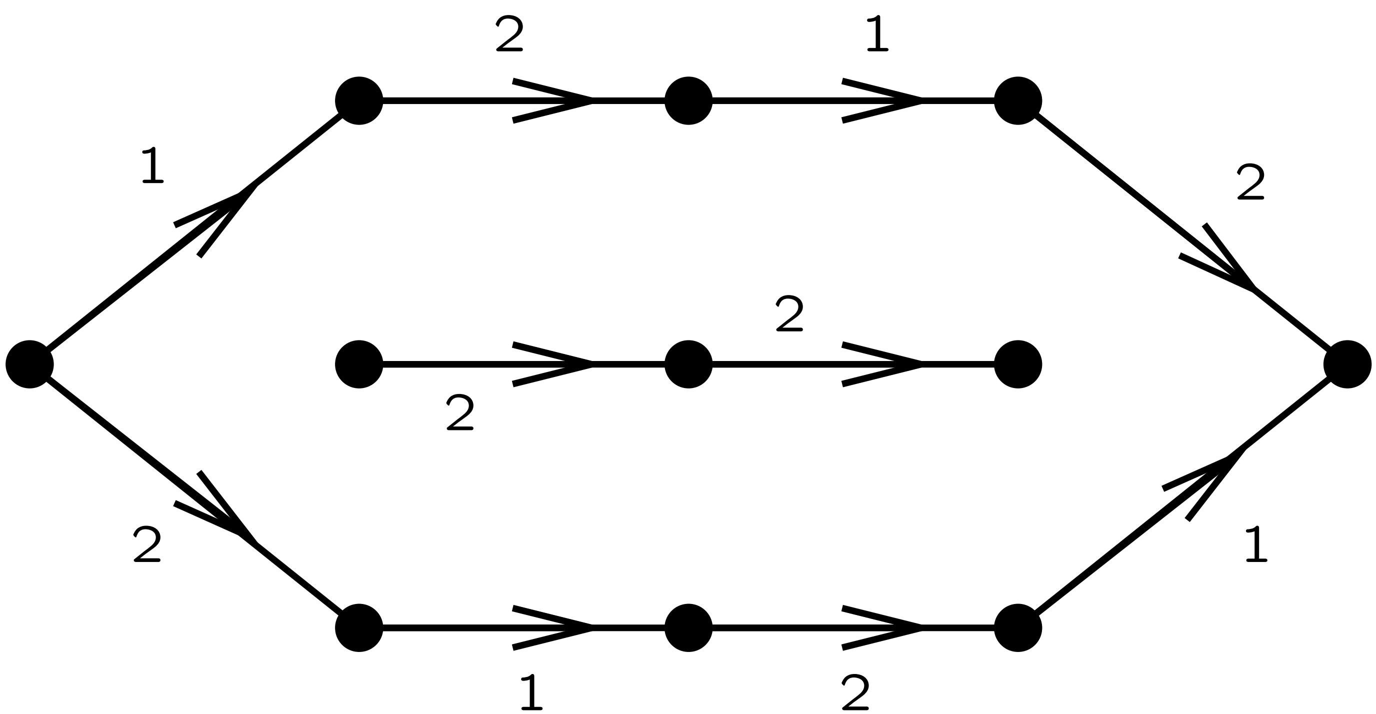
Take $q \to 0$ limit

Label remaining edges by Chevalley generator index:



Crystal graphs

Repeat for remaining Chevalley generators:



Obtain the *crystal graph* of rep L

Crystal Graphs

- Connected graph $\Rightarrow$ irreducible rep

- Can compute characters by counting vertices of fixed weight

- Tensor product rule

“Nice” basis = canonical basis !!

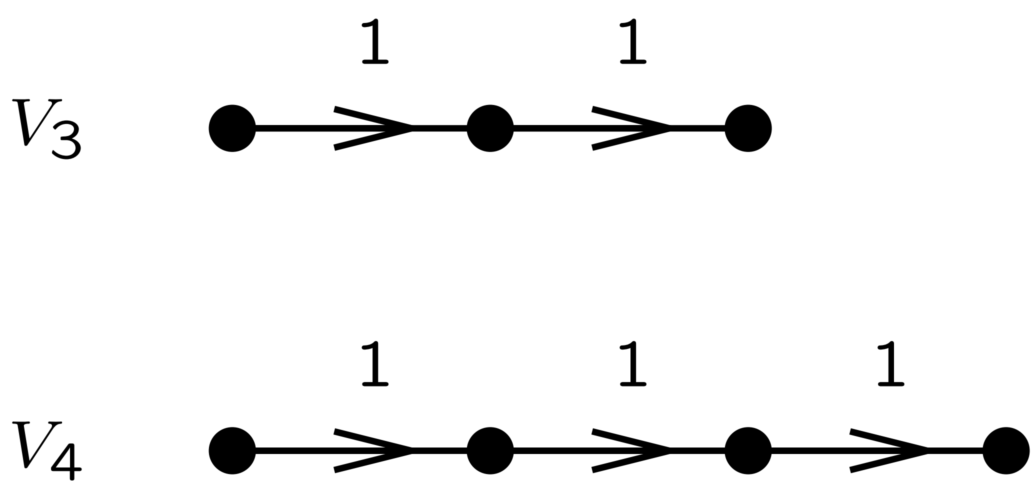
Problem: How do we “thaw” a crystal?

Tensor Product Rule

Let $\mathfrak{g} = \mathfrak{sl}_2$.

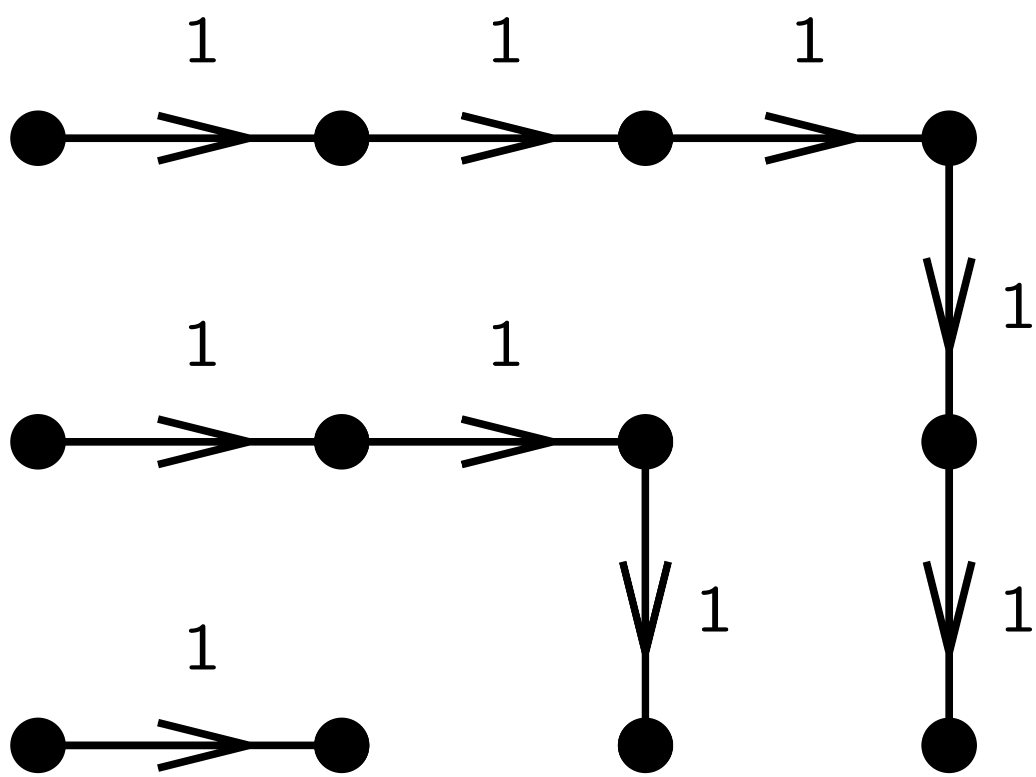
Consider $V_3 \otimes V_4$.

Crystal graphs are



Tensor Product Rule

Tensor product rule yields



Thus

$$V_3 \otimes V_4 = V_2 \oplus V_4 \oplus V_6$$

Realizations of Crystal Graphs

Combinatorial Realizations:

- Young tableaux (classical Lie algebras)
- Young walls (affine Lie algebras)
- Kyoto path model (affine Lie algebras)
- Littelmann path model

Geometric Realization:

Vertices of crystal graph = irred. comps. of QVs

Crystal operators (edges) defined geometrically

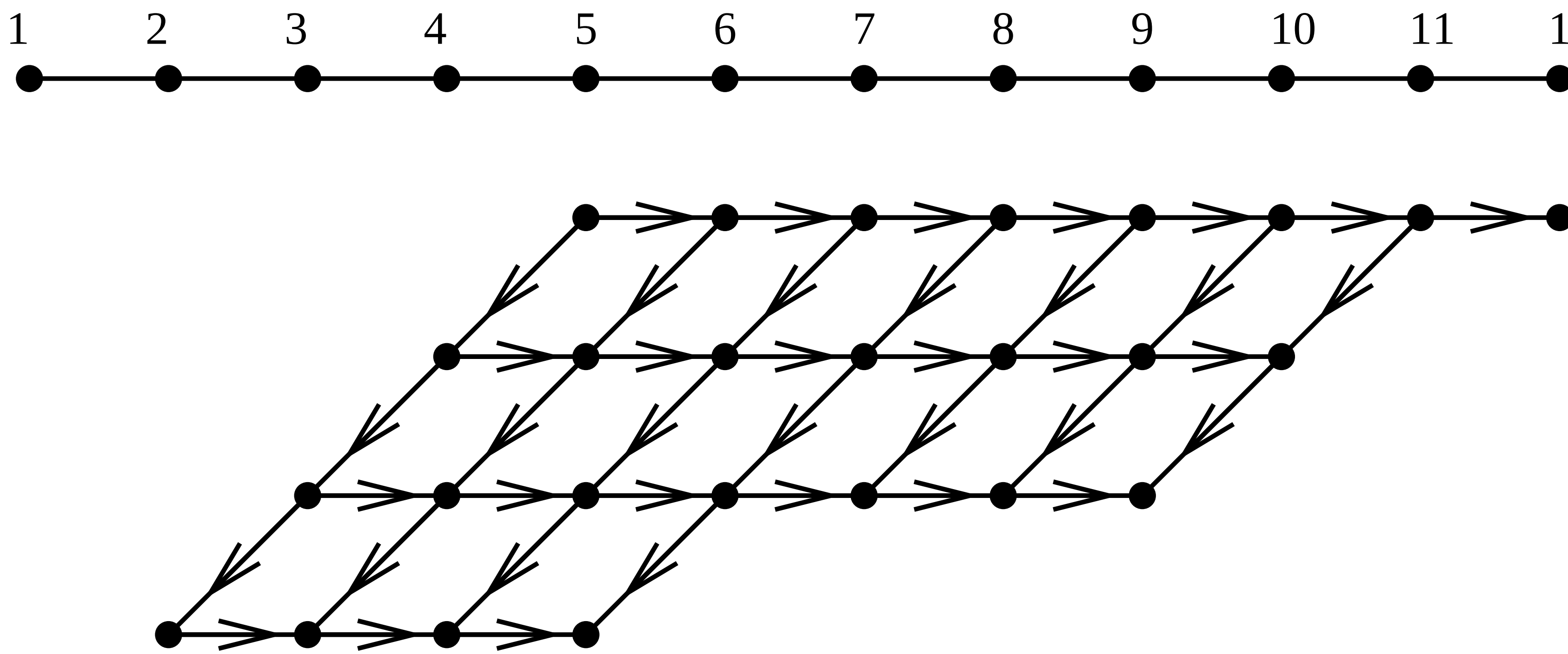
Connections Between Realizations

Finite and affine types A and D: Explicit isomorphism between Young tableaux/wall realizations and geometric realization (S)

Advantages:

- Explicit description of irr comps of QVs
- Gives geometric interpretation and suggests extensions of combinatorial constructions
- General QV theory gives universal method to "thaw" crystals

Connections Between Realizations



Other Constructions

Extension to non-simply laced case:

- Crystal structure – done (S)
- "Full" structure – open

Other constructions:

- Tensor products (Nakajima, Malkin)
- Fusion products (Schiffmann-S)
- Demazure modules (S)
- Spin representations, Clifford algebras (S)
- Virasoro algebra (Lehn)
- Others? Lie superalgebras? Jordan (super)algebras?

Connections to Affine Grassmannian Approach