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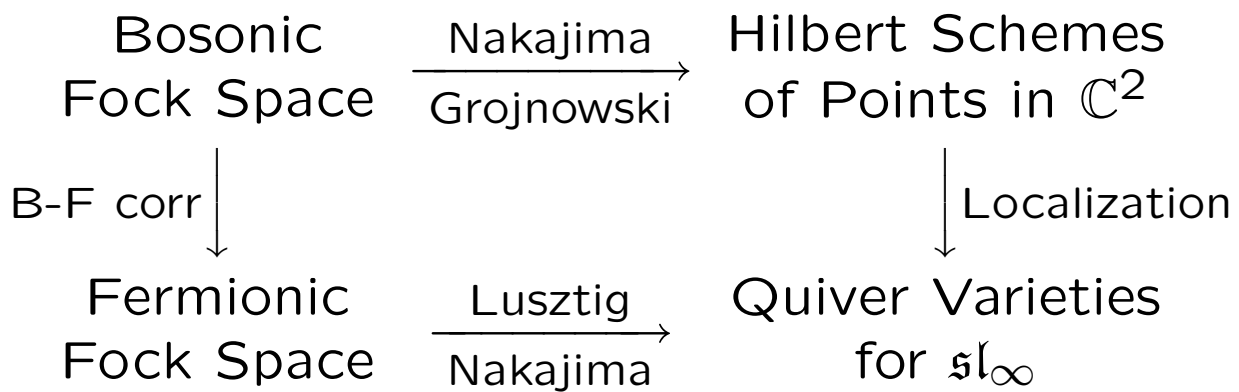
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A Geometric Boson-Fermion Correspondence

Slides available at www.math.toronto.edu/alistair

Preprint: math.RT/0508438

Overview



- Geometric realizations give us nice bases with remarkable properties.
- Rep theory organizes cohomological information

Fermionic Fock Space

$$\begin{aligned}
 F &= \text{Span}_{\mathbb{C}}\{\underline{i}_0 \wedge \underline{i}_1 \wedge \underline{i}_2 \wedge \dots \mid i_k \in \mathbb{Z}, \\
 &\quad i_0 > i_1 > \dots, i_k = -k \text{ for } k \gg 0\} \\
 &= \text{“infinite wedge space”}
 \end{aligned}$$

Part of a larger Fock space with action of an infinite Clifford algebra

\mathfrak{gl}_{∞} (and \mathfrak{sl}_{∞}) act on F by derivations:

For $a \in \mathfrak{gl}_{\infty}$,

$$a(\underline{i}_0 \wedge \underline{i}_1 \wedge \dots) = (a \cdot \underline{i}_0) \wedge \underline{i}_1 \wedge \dots + \underline{i}_0 \wedge (a \cdot \underline{i}_1) \wedge \dots + \dots$$

Bosonic Fock Space

Infinite-dim Heisenberg algebra:

$$\mathfrak{s} = \bigoplus_{m \in \mathbb{Z} \setminus \{0\}} \mathbb{C}s_m \oplus \mathbb{C}K$$

$$[\mathfrak{s}, K] = 0, \quad [s_m, s_n] = m\delta_{m,-n}K$$

Bosonic Fock Space:

$$B = \mathbb{C}[p_1, p_2, \dots]$$

Action of \mathfrak{s} on B by

$$s_m \mapsto m \frac{\partial}{\partial p_m}, \quad s_{-m} \mapsto p_m, \quad m > 0$$

$$K \mapsto \text{Id}$$

Boson-Fermion Correspondence

Action of \mathfrak{s} on F : “Bosonization”

$$sm \mapsto \sum_{j \in \mathbb{Z}} E_{j, j+m}, \quad m \in \mathbb{Z} \setminus \{0\},$$

$$K \mapsto \text{Id}$$

$$F \cong B \text{ as } \mathfrak{s}\text{-modules}$$

Also have action of Clifford algebra on B (generating functions, vertex operator algebras):

“Fermionization”

$$F \cong B \text{ as Clifford algebra modules}$$

Geometric Bosonic Fock Space

$$\mathbb{H}_n^B = H_T^{2n}(X_n), \quad \mathbb{H}^B = \bigoplus_{n=0}^{\infty} \mathbb{H}_n^B$$

Natural projections

$$X_{n+k} \xleftarrow{\pi_1} X_{n+k} \times X_n \xrightarrow{\pi_2} X_n$$

Define $\mathfrak{p}_{-k} \in \text{End } \mathbb{H}^B$, $k > 0$, by

$$\mathfrak{p}_{-k}(\alpha) = (\pi_1)_!(\pi_2^* \alpha \cup [\Sigma_{n,k}]), \quad \alpha \in \mathbb{H}_n^B$$

Here

$$\begin{aligned} \Sigma_{n,k} &\subset X_{n+k} \times X_n \\ &\leftrightarrow \text{“adding } k \text{ points” at } z \in \mathbb{C} \times \{0\} \subset \mathbb{C}^2 \end{aligned}$$

Geometric Bosonic Fock Space

Define adjoint operator \mathfrak{p}_k , $k > 0$

Prop (Nakajima, Vasserot):

$$[\mathfrak{p}_k, \mathfrak{p}_l] = k\delta_{k,-l}\text{Id}$$

So

$$s_k \mapsto \mathfrak{p}_k, \quad K \mapsto \text{Id}$$

defines action of \mathfrak{s} on \mathbb{H}^B and

$$\mathbb{H}^B \cong B \text{ as } \mathfrak{s}\text{-modules}$$

Quiver Varieties

$V =$ f.d. \mathbb{Z} -graded complex vector space

$$v = \dim V = (\dim V_k)_{k \in \mathbb{Z}}, \quad |v| = \sum v_k$$

$$\mathcal{M}(v) = \{(B_1, B_2, i) \mid [B_1, B_2] = 0, (B_1, B_2, i) \text{ stable}\} \\ / \prod_k GL(V_k)$$

$$B_1 \in \text{End } V, \quad \deg B = 1 \quad (\text{i.e. } B(V_k) \subseteq V_{k+1})$$

$$B_2 \in \text{End } V, \quad \deg B = -1 \quad (\text{i.e. } B(V_k) \subseteq V_{k-1})$$

$$i \in \text{Hom}(\mathbb{C}, V_0)$$

Geometric Fermionic Fock Space

$$\mathbb{H}^F = \bigoplus_v H_T^{2|v|}(\mathcal{M}(v)) \quad (\text{trivial } T\text{-action})$$

Have natural projections

$$\mathcal{M}(v + e_k) \longleftarrow \mathcal{M}(v + e_k) \times \mathcal{M}(v) \longrightarrow \mathcal{M}(v)$$

Define geometric action of \mathfrak{sl}_∞ (similar to Hilbert scheme picture) and

$$\mathbb{H}^F \cong F \text{ as } \mathfrak{sl}_\infty\text{-modules}$$

Torus Fixed Points

At a T -fixed point of X_n , weight decomposition of V gives grading

$$V = \bigoplus_{m \in \mathbb{Z}} V_m$$

Can show that

- B_1, B_2 have degrees 1 and -1 resp.
- $\text{im } i \subseteq V_0$

Thus,

T -fixed points of H.S. = quiver varieties

Localization and the Boson-Fermion Correspondence

X smooth with a T -action

$X^T = T$ -fixed points of X

Localization theorem states

$$H_T^*(X) \otimes \mathbb{C}(t) \cong H_T^*(X^T) \otimes \mathbb{C}(t)$$

Thus (after a few technical steps),

$$\mathbb{H}^B = \bigoplus_n H_T^*(X_n) \cong \bigoplus_n H_T^*(X_n^T) = \mathbb{H}^F$$

A geometric boson-fermion correspondence!

Further Directions

Fixed points of $\mathbb{Z}/n\mathbb{Z} \subset T$ yield quiver varieties for $\widehat{\mathfrak{sl}}_n$

Vertex operator construction of basic rep of $\widehat{\mathfrak{sl}}_n$ should fit into this geometric picture

Should help give algebraic description of nice geometric bases