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Branching Rules and Quiver Varieties

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The Lie Algebra $\mathfrak{gl}_n$

$$\mathfrak{gl}_n = n \times n \text{ matrices}$$

$\{E_{ij}\}_{i,j=1}^n$ is the standard basis of $\mathfrak{gl}_n$.

$$E_i = E_{i,i+1}, \quad F_i = E_{i+1,i}$$

$\{E_i, F_i\}_{i=1}^{n-1}$ are *Chevalley generators* (of $\mathfrak{sl}_n \subset \mathfrak{gl}_n$).

We have

$$\mathfrak{gl}_{n-1} \subset \mathfrak{gl}_n$$

$$\mathfrak{gl}_{n-1} = \text{Span}\{E_{ij} \mid 1 \le i, j \le n-1\}.$$

Representations of $\mathfrak{gl}_n$

F.d. irred. (h.w.) reps of $\mathfrak{gl}_n \stackrel{1-1}{\longleftrightarrow}$ partitions

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$$

Let $L_n(\lambda)$ be the rep. corr. to λ.

A vector $v \in L_n(\lambda)$ has weight $\mu = (\mu_i)_{i=1}^n$ if

$$E_{ii} v = \mu_i v.$$

We have the weight space decomposition

$$L_n(\lambda) = \bigoplus_{\mu} L_n(\lambda)_\mu$$

where $L_n(\lambda)_\mu$ is the space of vectors of weight μ.

Branching Rules for $\mathfrak{gl}_n$

We can restrict a rep of $\mathfrak{gl}_n$ to a rep of $\mathfrak{gl}_{n-1}$.

$$L_n(\lambda)|_{\mathfrak{gl}_{n-1}} \simeq \bigoplus_{\mu} L_{n-1}(\mu)$$

where the sum is over all partitions μ such that

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n.$$

This is called the *branching rule*.

NOTE: The restriction is *multiplicity free*.

Continuing restrictions gives natural *Gelfand-Tsetlin* basis.

Geometric Construction of $L_n(\lambda)$
(Beilinson, Lusztig, MacPherson, Ginzburg)

Assume $\lambda_n = 0$. Let $d = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.

Let $x \in \mathrm{End}(\mathbb{C}^d)$ be nilpotent with $\lambda_i - \lambda_{i+1}$ Jordan blocks of size $i \times i$.

Equivalently,

$$\lambda_i = \sum_{j \geq i} \# \ j \times j \text{ Jordan blocks in } x.$$

Let $\mathcal{F}_x^n$ be the Spaltenstein variety

$$\{(0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n = \mathbb{C}^d) \mid x(F_i) \subset F_{i-1}\}$$

$\mathcal{F}_x^n$ has connected components

$$\mathcal{F}_{\mathbf{d},x}^n = \{(F_i) \in \mathcal{F}_x^n \mid \dim F_i/F_{i-1} = d_i\}.$$

Geometric Construction of $L_n(\lambda)$

We are going to use $\mathcal{F}_x^n$ to construct $L_n(\lambda)$.

$$\mathcal{F}_{\mathbf{d},x}^n \longleftrightarrow L_n(\lambda)_\mu$$

$$\mu_i = d_i - d_{i+1}$$

To construct E_i, F_i define

$$^i\mathcal{F}_{\mathbf{d},x}^n = \{(F, F') \in F_{\mathbf{d},x}^n \times F_{\mathbf{d}+\mathbf{e}^i,x}^n \mid F_j = F_j', j \neq i, \\ F_i \subset F_i',\ \dim F_i'/F_i = 1\}.$$

Have natural projections

$$\mathcal{F}_{\mathbf{d},x}^n \xleftarrow{\pi_1} {}^i\mathcal{F}_{\mathbf{d},x}^n \xrightarrow{\pi_2} \mathcal{F}_{\mathbf{d}+\mathbf{e}^i,x}^n.$$

Geometric Construction of $L_n(\lambda)$

$M(\mathcal{F}_x^n) =$ space of "constructible" functions on $\mathcal{F}_x^n$.

Define action of Chevalley generators E_i and F_i on $M(\mathcal{F}_x^n)$ by

$$E_i f = (\pi_2)_! \pi_1^* f, \; F_i f = (\pi_1)_! \pi_2^* f,$$
$$\pi_k^* = \text{ pullback},$$
$$(\pi_k)_! = \text{ "push-forward"}.$$

$$\mathcal{F}^n_{\mathbf{d},x} \xleftarrow{\pi_1} {}^i\mathcal{F}^n_{\mathbf{d},x} \xrightarrow{\pi_2} \mathcal{F}^n_{\mathbf{d}+\mathbf{e}^i,x}.$$

Geometric Construction of $L_n(\lambda)$

Highest weight space of $L_n(\lambda)$ corresponds to constant functions on the point $\mathcal{F}^n_{\mathbf{d}^{max},x} = \{(F_i^{max})\}$, where

$$F_i^{max} = \ker x^i.$$

Let $\widetilde{M}(\mathcal{F}^n_x)$ be the functions in $M(\mathcal{F}^n_x)$ generated by action of F_i on constant functions on $\mathcal{F}^n_{\mathbf{d}^{max},x}$.

Theorem:

$$\widetilde{M}(\mathcal{F}^n_x) \simeq L_n(\lambda)$$

as $\mathfrak{sl}_n$-modules.

Geometric Construction of Branching

Want to create a natural map

$$\widetilde{M}(\mathcal{F}_x^n) \longrightarrow \bigoplus_{x'} \widetilde{M}(\mathcal{F}_{x'}^{n-1})$$

which is an isom of $\mathfrak{gl}_{n-1}$-modules.

Basic idea is

- restrict flags to the subspace F_{n-1}
- set $x' = x|_{F_{n-1}}$

What are the possible Jordan normal forms of $x|_{F_{n-1}}$?

Geometric Construction of Branching

Consider an $i \times i$ Jordan block of x

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$x(\mathbb{C}^d) = x(F_n) \subset F_{n-1} \Rightarrow F_{n-1}$ contains at least first $i-1$ of these basis vectors.

Each Jordan block of x can be reduced in size by 0 or 1 after restriction to F_{n-1}.

Geometric Construction of Branching

Recall

$$\lambda_i = \sum_{j \geq i} \# \ j \times j \text{ Jordan blocks in } x.$$

Let

$$\mu_i = \sum_{j \geq i} \# \ j \times j \text{ Jordan blocks in } x|_{F_{n-1}}.$$

Thus

$$\lambda_i \geq \mu_i \geq \lambda_{i+1}$$

This is precisely the branching rule.

Further Directions

Geometric construction of representations can be extended to arbitrary symmetric Kac-Moody algebras using *quiver varieties* (Lusztig, Nakajima).

In type D case, the natural restriction of reps is *not* multiplicity free.

However, there are different ways to construct same rep using geometry (in above, for $\mathfrak{sl}_n$, can add $n \times n$ Jordan blocks to x).

Above construction, suitably generalized, may yield only one copy of each realization, thus giving a natural way to deal with multiplicities and define Gelfand-Tsetlin type bases in other types.