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Quivers and the Euclidean group

Alistair Savage

Abstract. We show that the category of representations of the Euclidean group of orientation-preserving isometries of two-dimensional Euclidean space is equivalent to the category of representations of the preprojective algebra of type A_∞. We also consider the moduli space of representations of the Euclidean group along with a set of generators. We show that these moduli spaces are quiver varieties of the type considered by Nakajima. Using these identifications, we prove various results about the representation theory of the Euclidean group. In particular, we prove it is of wild representation type but that if we impose certain restrictions on weight decompositions, we obtain only a finite number of indecomposable representations.

1. Introduction

The Euclidean group $E(n) = \mathbb{R}^n \rtimes SO(n)$ is the group of orientation-preserving isometries of n-dimensional Euclidean space. The study of these objects, at least for $n = 2, 3$, predates even the concept of a group. In this paper we will focus on the Euclidean group $E(2)$. Even in this case, much is still unknown about the representation theory.

Since $E(2)$ is solvable, all its finite-dimensional irreducible representations are one-dimensional. The finite-dimensional unitary representations, which are of interest in quantum mechanics, are completely reducible and thus isomorphic to direct sums of such one-dimensional representations. The infinite-dimensional unitary irreducible representations have received considerable attention (see [**1, 3, 4**]). There also exist finite-dimensional nonunitary indecomposable representations (which are not irreducible) and much less is known about these. However, they play an important role in mathematical physics and the representation theory of the Poincaré group. The Poincaré group is the group of isometries of Minkowski spacetime. It is the semidirect product of the translations of $\mathbb{R}^3$ and the Lorentz transformations. In 1939, Wigner [**25**] studied the subgroups of the Lorentz group leaving invariant the four-momentum of a given free particle. The maximal such subgroup is called the *little group*. The little group governs the internal space-time symmetries of the relativistic particle in question. The little groups of massive particles are locally

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isomorphic to the group $O(3)$ while the little groups of massless particles are locally isomorphic to $E(2)$. That is, their Lie algebras are isomorphic to those of $O(3)$ and $E(2)$ respectively. We refer the reader to [**2, 5, 16, 20**] for further details.

The group $E(2)$ also appears in the Chern-Simons formulation of Einstein gravity in $2+1$ dimensions. In the case when the space-time has Euclidean signature and the cosmological constant vanishes, the phase space of gravity is the moduli space of flat $E(2)$-connections.

In the current paper, we relate the representation theory of the Euclidean group $E(2)$ to the representation theory of preprojective algebras of quivers of type A_∞. In fact, we show that the categories of representations of the two are equivalent. To prove this, we introduce a modified enveloping algebra of the Lie algebra of $E(2)$ and show that it is isomorphic to the preprojective algebra of type A_∞. Furthermore, we consider the moduli space of representations of $E(2)$ along with a set of generators. We show that these moduli spaces are quiver varieties of the type considered by Nakajima in [**21, 22**]. These identifications allow us to draw on known results about preprojective algebras and quiver varieties to prove various statements about representations of $E(2)$. In particular, we show that $E(2)$ is of wild representation type but that if we impose certain restrictions on the weight decomposition of a representation, we obtain only a finite number of indecomposable representations. We conclude with some potential directions for future investigation.

2. The Euclidean algebra

Let $E(2) = \mathbb{R}^2 \rtimes SO(2)$ be the Euclidean group of motions in the plane and let $\mathfrak{e}(2)$ be the complexification of its Lie algebra. We call $\mathfrak{e}(2)$ the (three-dimensional) Euclidean algebra. It has basis $\{p_+, p_-, l\}$ and commutation relations

$$[p_+, p_-] = 0, \quad [l, p_\pm] = \pm p_\pm. \tag{2.1}$$

Since $SO(2)$ is compact, the category of finite-dimensional $E(2)$-modules is equivalent to the category of finite-dimensional $\mathfrak{e}(2)$-modules in which l acts semisimply with integer eigenvalues. Will will use the term $\mathfrak{e}(2)$-module to refer only to such modules. For $k \in \mathbb{Z}$, we shall write V_k to indicate the eigenspace of l with eigenvalue k (the *k-weight space*). Thus, for an $\mathfrak{e}(2)$-module V, we have the weight space decomposition

$$V = \bigoplus_k V_k, \quad V_k = \{v \in V \mid l \cdot v = kv\}, \quad k \in \mathbb{Z},$$

and

$$p_+ V_k \subseteq V_{k+1}, \quad p_- V_k \subseteq V_{k-1}.$$

We may form the tensor product of any representation V with the character χ_n for $n \in \mathbb{Z}$. Here χ_n is the one-dimensional module $\mathbb{C}$ on which $p_\pm$ act by zero and l acts by multiplication by n. Then a weight space V_k of weight k becomes a weight space $V_k \otimes \chi_n$ of weight $k+n$. In this way, we may "shift weights" as we please.

For $k \in \mathbb{Z}$, let $\mathbf{e}^k$ be the element of $(\mathbb{Z}_{\geq 0})^{\mathbb{Z}}$ with kth component equal to one and all others equal to zero. For an $\mathfrak{e}(2)$-module V we define

$$\mathbf{dim}\, V = \sum_{k \in \mathbb{Z}} (\dim V_k) \mathbf{e}^k.$$

Let U be the universal enveloping algebra of $\mathfrak{e}(2)$ and let U^+, U^- and U^0 be the subalgebras generated by p_+, p_- and l respectively. Then we have the triangular decomposition

$$U \cong U^+ \otimes U^0 \otimes U^- \quad \text{(as vector spaces)}.$$

Note that the category of representations of U is equivalent to the category of representations of $\mathfrak{e}(2)$. In [**18**, Chapter 23], Lusztig introduced the modified quantized enveloping algebra of a Kac-Moody algebra. Following this idea, we introduce the *modified enveloping algebra* $\tilde{U}$ by replacing U^0 with a sum of 1-dimensional algebras

$$\tilde{U} = U^+ \otimes \left(\bigoplus_{k \in \mathbb{Z}} \mathbb{C} a_k \right) \otimes U^-.$$

Multiplication is given by

$$\begin{gathered} a_k a_l = \delta_{kl} a_k, \\ p_+ a_k = a_{k+1} p_+, \quad p_- a_k = a_{k-1} p_-, \\ p_+ p_- a_k = p_- p_+ a_k. \end{gathered}$$

One can think of a_k as projection onto the kth weight space. Note that $\tilde{U}$ is an algebra without unit. We say a $\tilde{U}$-module V is *unital* if

(1) for any $v \in V$, we have $a_k v = 0$ for all but finitely many $k \in Z$, and
(2) for any $v \in V$, we have $\sum_{k \in \mathbb{Z}} a_k v = v$.

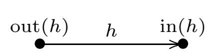
A unital $\tilde{U}$-module can be thought of as a U-module with weight decomposition. Thus we have the following proposition.

Proposition 2.1. *The category of unital $\tilde{U}$-modules is equivalent to the category of U-modules and hence the category of $\mathfrak{e}(2)$-modules.*

3. Preprojective algebras

In this section, we review some basic results about preprojective algebras. The reader is referred to [**12**] for further details.

A *quiver* is a 4-tuple $(I, H, \text{out}, \text{in})$ where I and H are disjoint sets and out and in are functions from H to I. The sets I and H are called the *vertex set* and *arrow set* respectively. We think of an element $h \in H$ as an arrow from the vertex $\text{out}(h)$ to the vertex $\text{in}(h)$.



An arrow $h \in H$ is called a *loop* if $\text{out}(h) = \text{in}(h)$. A quiver is said to be *finite* if both its vertex and arrow sets are finite.

We shall be especially concerned with the following quivers. For $a, b \in \mathbb{Z}$ with $a \leq b$, let $Q_{a,b}$ be the quiver with vertex set $I = \{k \in \mathbb{Z} \mid a \leq k \leq b\}$ and arrows $H = \{h_i \mid a \leq i \leq b-1\}$ with $\text{out}(h_i) = i$ and $\text{in}(h_i) = i+1$. We say that $Q_{a,b}$ is a quiver of type A_{b-a+1} since this is the type of its underlying graph. The quiver Q_∞ has vertex set $I = \mathbb{Z}$ and arrows $H = \{h_i \mid i \in \mathbb{Z}\}$ with $\text{out}(h_i) = i$ and $\text{in}(h_i) = i+1$. We say that the quiver Q_∞ is of type A_∞. Note that the quivers $Q_{a,b}$ are finite while the quiver Q_∞ is not.

Let $Q = (I, H, \text{out}, \text{in})$ be a quiver without loops and let $Q^* = (I, H^*, \text{out}^*, \text{in}^*)$ be the *double quiver* of Q. By definition,

$$H^* = \{h \mid h \in H\} \cup \{\bar{h} \mid h \in H\},$$
$$\text{out}^*(h) = \text{out}(h), \quad \text{in}^*(h) = \text{in}(h), \quad \text{out}^*(\bar{h}) = \text{in}(h), \quad \text{in}^*(\bar{h}) = \text{out}(h).$$

From now on, we will write in and out for in^* and out^* respectively. Since $\text{in}^*|_H = \text{in}$ and $\text{out}^*|_H = \text{out}$, this should cause no confusion.

A *path* in a quiver Q is a sequence $p = h_n h_{n-1} \cdots h_1$ of arrows such that $\text{in}(h_i) = \text{out}(h_{i+1})$ for $1 \le i \le n-1$. We call the integer n the *length* of p and define $\text{out}(p) = \text{out}(h_1)$ and $\text{in}(p) = \text{in}(h_n)$. The *path algebra* $\mathbb{C}Q$ is the algebra spanned by the paths in Q with multiplication given by

$$p \cdot p' = \begin{cases} pp' & \text{if } \text{in}(p') = \text{out}(p) \\ 0 & \text{otherwise} \end{cases}$$

and extended by linearity. We note that there is a trivial path ϵ_i starting and ending at i for each $i \in I$. The path algebra $\mathbb{C}Q$ has a unit (namely $\sum_{i \in I} \epsilon_i$) if and only if the quiver Q is finite.

A *relation* in a quiver Q is a sum of the form $\sum_{j=1}^k a_j p_j$, $a_j \in \mathbb{C}$, p_j a path for $1 \le j \le k$. For $i \in I$ let

$$r_i = \sum_{h \in H,\, \text{out}(h) = i} \bar{h}h - \sum_{h \in H,\, \text{in}(h) = i} h\bar{h}$$

be the *Gelfand-Ponomarev relation* in Q^* associated to i. The *preprojective algebra* $P(Q)$ corresponding to Q is defined to be

$$P(Q) = \mathbb{C}Q^*/J$$

where J is the two-sided ideal generated by the relations r_i for $i \in I$.

Let $\mathcal{V}(I)$ denote the category of finite-dimensional I-graded vector spaces with morphisms being linear maps respecting the grading. For $V \in \mathcal{V}(I)$, we let $\mathbf{dim}\, V = (\dim V_i)_{i \in I}$ be the I-graded dimension of V. A *representation* of the quiver Q^* is an element $V \in \mathcal{V}(I)$ along with a linear map $x_h : V_{\text{out}(h)} \to V_{\text{in}(h)}$ for each $h \in H^*$. We let

$$\text{rep}(Q^*, V) = \bigoplus_{h \in H^*} \text{Hom}_{\mathbb{C}}(V_{\text{out}(h)}, V_{\text{in}(h)})$$

be the affine variety consisting of representations of Q^* with underlying graded vector space V. A representation of a quiver can be naturally interpreted as a $\mathbb{C}Q^*$-module structure on V. For a path $p = h_n h_{n-1} \ldots h_1$ in Q^*, we let

$$x_p = x_{h_n} x_{h_{n-1}} \cdots x_{h_1}.$$

We say a representation $x \in \text{rep}(Q^*, V)$ *satisfies the relation* $\sum_{j=1}^k a_j p_j$, if

$$\sum_{j=1}^k a_j x_{p_j} = 0.$$

If R is a set of relations, we denote by $\text{rep}(Q^*, R, V)$ the set of all representations in $\text{rep}(Q^*, V)$ satisfying all relations in R. This is a closed subvariety of $\text{rep}(Q^*, V)$. Every element of $\text{rep}(Q^*, J, V)$ can be naturally interpreted as a $P(Q)$-module structure on V and so we also write

$$\text{mod}(P(Q), V) = \text{rep}(Q^*, J, V)$$

for the affine variety of $P(Q)$-modules with underlying vector space V.

The algebraic group $G_V = \prod_{i\in I} GL(V_i)$ acts on $\operatorname{mod}(P(Q),V)$ by

$$g \cdot x = (g_i)_{i\in I} \cdot (x_h)_{h\in H^*} = (g_{\mathrm{in}(h)} x_h g^{-1}_{\mathrm{out}(h)})_{h\in H^*}.$$

Two $P(Q)$-modules are isomorphic if and only if they lie in the same orbit. For a dimension vector $\mathbf{v} \in (\mathbb{Z}_{\geq 0})^I$, let

$$V^{\mathbf{v}} = \bigoplus_{i\in I} \mathbb{C}^{\mathbf{v}_i}, \quad \operatorname{mod}(P(Q),\mathbf{v}) = \operatorname{mod}(P(Q),V^{\mathbf{v}}), \quad G_{\mathbf{v}} = G_{V^{\mathbf{v}}}.$$

Then we have that $\operatorname{mod}(P(Q),V) \cong \operatorname{mod}(P(Q),\operatorname{\mathbf{dim}} V)$ for all $V \in \mathcal{V}(I)$. Therefore, we will blur the distinction between $\operatorname{mod}(P(Q),V)$ and $\operatorname{mod}(P(Q),\operatorname{\mathbf{dim}} V)$.

We say an element $x \in \operatorname{mod}(P(Q),V)$ is *nilpotent* if there exists an $N \in \mathbb{Z}_{>0}$ such that for any path p of length greater than N, we have $x_p = 0$. Denote the closed subset of nilpotent elements of $\operatorname{mod}(P(Q),V)$ by $\Lambda_{V,Q}$ and let $\Lambda_{\mathbf{v},Q} = \Lambda_{V^{\mathbf{v}},Q}$. The varieties $\Lambda_{V,Q}$ are called *nilpotent varieties* or *Lusztig quiver varieties.* Lusztig [**17**, Theorem 12.3] has shown that the $\Lambda_{V,Q}$ have pure dimesion $\dim(\operatorname{rep}(Q,V))$.

Proposition 3.1. *For a quiver Q, the following are equivalent:*

(1) *$P(Q)$ is finite-dimensional,*
(2) *$\Lambda_{V,Q} = \operatorname{mod}(P(Q),V)$ for all $V \in \mathcal{V}(I)$,*
(3) *Q is a Dynkin quiver (i.e. its underlying graph is of ADE type).*

Proof. The equivalence of (1) and (3) is well-known (see for example [**23**]). That (2) implies (3) was proven by Crawley-Boevey [**6**] and the converse was proven by Lusztig [**17**, Proposition 14.2]. □

Thus, for a Dynkin quiver Q, nilpotency holds automatically and $\Lambda_{V,Q}$ is just the variety of representations of the preprojective algebra $P(Q)$ with underlying vector space V.

The representation type of the preprojective algebras is known.

Proposition 3.2 ([**7, 13**]). *Let Q be a finite quiver. Then the following hold:*

(1) *$P(Q)$ is of finite representation type if and only if Q is of Dynkin type A_n, $n \leq 4$, and*
(2) *$P(Q)$ is of tame representation type if and only if Q is of Dynkin type A_5 or D_4.*

Thus $P(Q)$ is of wild representation type if Q is not of Dynkin type A_n, $n \leq 5$, or D_4.

In the sequel, we will refer to the preprojective algebra $P(Q_\infty)$. While Q_∞ is not a finite quiver, any finite-dimensional representation is supported on finitely many vertices and thus is a representation of a quiver of type A_n for sufficiently large n. Thus we deduce the following.

Corollary 3.3. *All finite-dimensional representations of $P(Q_\infty)$ are nilpotent and $P(Q_\infty)$ is of wild representation type.*

For a finite quiver Q, let $\mathfrak{g}_Q$ denote the Kac-Moody algebra whose Dynkin graph is the underlying graph of Q and let $U(\mathfrak{g}_Q)^-$ denote the lower half of its universal enveloping algebra. It turns out that Lusztig quiver varieties are intimately related to $U(\mathfrak{g}_Q)^-$. Namely, Lusztig [**17**] has shown that there is a space of constructible functions on the varieties $\Lambda_{\mathbf{v},Q}$, $\mathbf{v} \in (\mathbb{Z}_{\geq 0})^I$, and a natural convolution product such that this space of functions is isomorphic as an algebra to $U(\mathfrak{g}_Q)^-$. The functions on an individual $\Lambda_{\mathbf{v},Q}$ correspond to the weight space of weight $-\sum_{i\in I}\mathbf{v}_i\alpha_i$, where the α_i are the simple roots of $\mathfrak{g}_Q$. Furthermore, the irreducible components of $\Lambda_{\mathbf{v},Q}$ are in one-to-one correspondence with a basis of this weight space. Under this correspondence, each irreducible component is associated to the unique function equal to one on an open dense subset of that component and equal to zero on an open dense subset of all other components. The set of these functions yields a basis of $U(\mathfrak{g}_Q)^-$, called the *semicanonical basis*, with very nice integrality and positivity properties (see [**19**]). If instead of constructible functions one works with the Grothendieck group of a certain class of perverse sheaves, a similar construction yields a realization of (the lower half of) the quantum group $U_q(\mathfrak{g}_Q)^-$ and the *canonical basis* (see [**17**]).

4. Representations of the Euclidean algebra and preprojective algebras

In this section we examine the close relationship between representations of the Euclidean algebra $\mathfrak{e}(2)$ and the preprojective algebras of types A_n and A_∞.

Theorem 4.1. *The modified universal enveloping algebra $\tilde{U}$ is isomorphic to the preprojective algebra $P(Q_\infty)$.*

Proof. Define a map $\psi : \mathbb{C}Q_\infty^* \to \tilde{U}$ by

$$\psi(\epsilon_i) = a_i, \quad \psi(h_i) = p_+a_i = a_{i+1}p_+, \quad \psi(\bar{h}_i) = a_ip_- = p_-a_{i+1}, \quad i \in I.$$

It is easily verified that this extends to a surjective map of algebras with kernel J and thus the result follows. □

Let $\mathbf{Mod}\,\mathfrak{e}(2)$ be the category of $\mathfrak{e}(2)$-modules. For $a \le b$, let $\mathbf{Mod}_{a,b}\,\mathfrak{e}(2)$ be the full subcategory consisting of representations V such that $V_k = 0$ for $k < a$ or $k > b$. For $\mathbf{v} \in (\mathbb{Z}_{\ge 0})^{\mathbb{Z}}$, we also define $\mathbf{Mod}^{\mathbf{v}}_{a,b}\,\mathfrak{e}(2)$ and $\mathbf{Mod}^{\mathbf{v}}\,\mathfrak{e}(2)$ to be the full subcategories of $\mathbf{Mod}_{a,b}\,\mathfrak{e}(2)$ and $\mathbf{Mod}\,\mathfrak{e}(2)$ consisting of representations V such that $\mathbf{dim}\,V = \mathbf{v}$.

Let $\mathbf{Mod}\,P(Q)$ be the category of finite-dimensional $P(Q)$-modules and for $\mathbf{v} \in (\mathbb{Z}_{\ge 0})^I$, let $\mathbf{Mod}^{\mathbf{v}}\,P(Q)$ be the full subcategory consisting of modules of graded dimension $\mathbf{v}$.

Corollary 4.2. *We have the following equivalences of categories.*

1. $\mathbf{Mod}^{\mathbf{v}}\,\mathfrak{e}(2) \cong \mathbf{Mod}^{\mathbf{v}}\,P(Q_\infty)$, $\mathbf{Mod}\,\mathfrak{e}(2) \cong \mathbf{Mod}\,P(Q_\infty)$,
2. $\mathbf{Mod}^{\mathbf{v}}_{a,b}\,\mathfrak{e}(2) \cong \mathbf{Mod}^{\mathbf{v}}\,P(Q_{a,b})$, $\mathbf{Mod}_{a,b}\,\mathfrak{e}(2) \cong \mathbf{Mod}\,P(Q_{a,b})$.

Proof. Statement (1) follows from Theorem 4.1 and Proposition 2.1. Statement (2) is obtained by restricting weights to lie between a and b. □

Theorem 4.3. *The following statements hold.*

1. *The Euclidean algebra $\mathfrak{e}(2)$, and hence the Euclidean group $E(2)$, have wild representation type, and*
2. *for $a, b \in \mathbb{Z}$ with $0 \le b - a \le 3$, there are a finite number of isomorphism classes of indecomposable $\mathfrak{e}(2)$-modules V whose weights lie between a and b; that is, such that $V_k = 0$ for $k < a$ or $k > b$.*

Proof. These statements follow immediately from Corollary 4.2, Proposition 3.2 and Corollary 3.3. □

Corollary 4.4. *Let A be a finite subset of $\mathbb{Z}$ with the property that A does not contain any five consecutive integers. Then there are a finite number of isomorphism classes of indecomposable $\mathfrak{e}(2)$-modules V with the property that $V_k = 0$ if $k \notin A$.*

Proof. Partition A into subsets $A_1, \dots, A_n$ such that $A_j = \{a_j, a_j+1, \dots, a_j + m_j\}$ and $|a-b| > 1$ for $a \in A_i$, $b \in A_j$ with $i \neq j$. By hypothesis, we have $m_j \leq 3$ for $1 \leq j \leq n$. Let V be an $\mathfrak{e}(2)$-module such that $V_k = 0$ if $k \notin A$. Then V decomposes as a direct sum of modules $V = \bigoplus_{j=1}^n V^j$ where $V_k^j = 0$ if $k < a_j$ or $k > a_j + m_k$. Thus, if V is indecomposable, we must have $V = V^j$ for some j. But there are a finite number of such V^j, up to isomorphism, by Theorem 4.3. The result follows. □

For $a \in \mathbb{Z}$, we say an $\mathfrak{e}(2)$-module V has *lowest weight a* if $V_a \neq 0$ and $V_k = 0$ for $k < a$.

Corollary 4.5. *For all $a \in \mathbb{Z}$, there are a finite number of isomorphism classes of indecomposable $\mathfrak{e}(2)$-modules with lowest weight a and dimension less than or equal to five.*

Proof. By tensoring with the character χ_{-a} we may assume that $a = 0$. In order for an $\mathfrak{e}(2)$-module to be indecomposable, its set of weights must be a set of consecutive integers. By Corollary 4.4, it suffices to consider the modules of dimension 5. Again, by Corollary 4.4, we need only consider the case when $\dim V_k = 1$ for $0 \leq k \leq 4$. We consider the equivalent problem of classifying the G_V-orbits of indecomposable elements $x \in \Lambda_{V,Q_{0,4}}$ where $V_k = \mathbb{C}$ for $0 \leq k \leq 4$. Fixing the standard basis in each V_k, we can view the maps x_h, $h \in H^*$, as complex numbers. Considering the Gelfand-Ponomarev relation r_0, we see that $x_{\bar{h}_0} x_{h_0} = 0$. Then the relation r_1 implies $x_{\bar{h}_1} x_{h_1} = 0$. Continuing in this manner, we see that $x_{\bar{h}_i} x_{h_i} = 0$ for $0 \leq i \leq 3$. Thus $x_{h_i} = 0$ or $x_{\bar{h}_i} = 0$ for $0 \leq i \leq 3$. Since x is indecomposable, we cannot have both $x_{h_i} = 0$ and $x_{\bar{h}_i} = 0$ for any i. Thus, there are precisely $2^4 = 16$ G_V-orbits in $\Lambda_{V,Q_{0,4}}$. Representatives for these orbits correspond to setting one of x_{h_i} or $x_{\bar{h}_i}$ equal to one and the other to zero for each $0 \leq i \leq 3$. □

We note that Douglas [**9**] has shown that there are finitely many indecomposable $\mathfrak{e}(2)$-modules (up to isomorphism) of dimensions five and six. The proof of Corollary 4.5 shows how Corollary 4.4 can simply such proofs. We also point out that the graphs appearing in [**9**] roughly correspond, under the equivalence of categories in Corollary 4.2, to the diagrams appearing in the enumeration of irreducible components of quiver varieties given in [**11**].

Remark 4.6. *As noted at the end of Section 3, the Lusztig quiver varieties $\Lambda_{\mathbf{v},Q}$ are closely related to the Kac-Moody algebra $\mathfrak{g}_Q$. Thus, the results of this section show that there is a relationship between the representation theory of the Euclidean group $E(2)$ and the Lie algebra $\mathfrak{sl}_\infty$ (or the Lie groups $SL(n)$).*

5. Nakajima quiver varieties

In this section we briefly review the quiver varieties introduced by Nakajima [**21, 22**]. We restrict our attention to the case when the quiver involved is of type A.

Let Q be the quiver Q_∞ or $Q_{a,b}$ for some $a \le b$. For $V, W \in \mathcal{V}(I)$ define

$$L_Q(V,W) = \Lambda_{V,Q} \oplus \bigoplus_{i \in I} \mathrm{Hom}_{\mathbb{C}}(W_i, V_i).$$

We denote points of $L_Q(V,W)$ by (x,s) where $x = (x_h)_{h \in H^*} \in \Lambda_{V,Q}$ and $s = (s_i)_{i \in I} \in \bigoplus_{i \in I} \mathrm{Hom}_{\mathbb{C}}(W_i, V_i)$. We say an I-graded subspace S of V is *x-invariant* if $x_h(S_{\mathrm{out}(h)}) \subseteq S_{\mathrm{in}(h)}$ for all $h \in H^*$. We say a point $(x,s) \in L_Q(V,W)$ is *stable* if the following property holds: If S is an I-graded x-invariant subspace of V containing $\mathrm{im}\, s$, then $S = V$. We denote by $L_Q(V,W)^{\mathrm{st}}$ the set of stable points.

The group G_V acts on $L_Q(V,W)$ by

$$g \cdot (x,s) = (g_i)_{i \in I} \cdot ((x_h)_{h \in H^*}, (s_i)_{i \in I}) = ((g_{\mathrm{in}(h)} x_h g_{\mathrm{out}(h)}^{-1})_{h \in H^*}, (g_i s_i)_{i \in I}).$$

The action of G_V preserves the stability condition and the stabilizer in G_V of a stable point is trivial. We form the quotient

$$\mathcal{L}_Q(V,W) = L_Q(V,W)^{\mathrm{st}} / G_V.$$

The $\mathcal{L}_Q(V,W)$ are called *Nakajima quiver varieties.* For $\mathbf{v}, \mathbf{w} \in (\mathbb{Z}_{\ge 0})^I$, we set

$$L_Q(\mathbf{v},\mathbf{w}) = L_Q(V^{\mathbf{v}}, V^{\mathbf{w}}), \quad L_Q(\mathbf{v},\mathbf{w})^{\mathrm{st}} = L_Q(V^{\mathbf{v}}, V^{\mathbf{w}})^{\mathrm{st}}, \quad \mathcal{L}_Q(\mathbf{v},\mathbf{w}) = \mathcal{L}_Q(V^{\mathbf{v}}, V^{\mathbf{w}}).$$

We then have

$$L_Q(V,W) \cong L_Q(\mathbf{dim}\, V, \mathbf{dim}\, W), \quad L_Q(V,W)^{\mathrm{st}} \cong L_Q(\mathbf{dim}\, V, \mathbf{dim}\, W)^{\mathrm{st}},$$
$$\mathcal{L}_Q(V,W) \cong \mathcal{L}_Q(\mathbf{dim}\, V, \mathbf{dim}\, W),$$

and so we often blur the distinction between these pairs of isomorphic varieties.

Let $\mathrm{Irr}\, \Lambda_{V,Q}$ (resp. $\mathrm{Irr}\, \mathcal{L}_Q(V,W)$) denote the set of irreducible components of $\Lambda_{V,Q}$ (resp. $\mathcal{L}_Q(V,W)$). Then $\mathrm{Irr}\, \mathcal{L}_Q(V,W)$ can be identified with

$$\left\{ Y \in \mathrm{Irr}\, \Lambda_{V,Q} \;\middle|\; \left(Y \oplus \bigoplus_{i \in I} \mathrm{Hom}_{\mathbb{C}}(W_i, V_i) \right)^{\mathrm{st}} \neq \emptyset \right\}.$$

Specifically, the irreducible components of $\mathcal{L}_Q(V,W)$ are precisely those

$$\left(\left(Y \oplus \bigoplus_{i \in I} \mathrm{Hom}_{\mathbb{C}}(W_i, V_i) \right)^{\mathrm{st}} \right) / G_V$$

which are nonempty.

Proposition 5.1 ([**22**, Corollary 3.12]). *The dimension of the Nakajima quiver varieties associated to the quiver Q_∞ are given by*

$$\dim_{\mathbb{C}} \mathcal{L}_{Q_\infty}(\mathbf{v},\mathbf{w}) = \sum_{i \in \mathbb{Z}} (\mathbf{v}_i \mathbf{w}_i - \mathbf{v}_i^2 + \mathbf{v}_i \mathbf{v}_{i+1}).$$

In a manner analogous to the way in which Lusztig quiver varieties are related to $U(\mathfrak{g}_Q)^-$ (see Section 3), Nakajima quiver varieties are closely related to the representation theory of $\mathfrak{g}_Q$. In particular, Nakajima [**22**] has shown that $\bigoplus_{\mathbf{v}} H_{\mathrm{top}}(\mathcal{L}_Q(\mathbf{v},\mathbf{w}))$ is isomorphic to the irreducible integrable highest-weight representation of $\mathfrak{g}_Q$ of highest weight $\sum_{i \in I} \mathbf{w}_i \omega_i$ where the ω_i are the fundamental weights of $\mathfrak{g}_Q$. Here H_{top} is top-dimensional Borel-Moore homology. The action of the Chevalley generators of $\mathfrak{g}_Q$ are given by certain convolution operations. The vector space $H_{\mathrm{top}}(\mathcal{L}_Q(\mathbf{v},\mathbf{w}))$ corresponds to the weight space of weight

$\sum_{i \in I}(\mathbf{w}_i\omega_i - \mathbf{v}_i\alpha_i)$. In [**21**], Nakajima gave a similar realization of these representations using a space of constructible functions on the quiver varieties rather than their homology. The irreducible components of Nakajima quiver varieties enumerate a natural basis in the representations of $\mathfrak{g}_Q$. These bases are given by the fundamental classes of the irreducible components in the Borel-Moore homology construction and by functions equal to one on an open dense subset of an irreducible component (and equal to zero on an open dense subset of all other irreducible components) in the constructible function realization.

6. Moduli spaces of representations of the Euclidean algebra

Given that $\mathfrak{e}(2)$ has wild representation type, it is prudent to restrict one's attention to certain subclasses of modules and to attempt a classification of the modules belonging to these classes. One possible approach is to impose a restriction on the number of generators of a representation (see [**8, 9**] for some results in this direction and [**10**] for other classes). In this section we will examine the relationship between moduli spaces of representations of the Euclidean algebra along with a set of generating vectors and Nakajima quiver varieties.

Let V be a finite-dimensional $\mathfrak{e}(2)$-module. For $u_1, u_2, \dots, u_n \in V$, we denote by $\langle u_1, \dots, u_n\rangle$ the submodule of V generated by $\{u_1, \dots, u_n\}$. It is defined to be the smallest submodule of V containing all the u_i. A element $u \in V$ is called a *weight vector* if it lies in some weight space V_k of V. For a weight vector u, we let $\operatorname{wt} u = k$ where $u \in V_k$. We say that $\{u_1, \dots, u_n\}$ is a set of *generators* of V if each u_i is a weight vector and $\langle u_1, \dots, u_n\rangle = V$. For $\mathbf{v} \in (\mathbb{Z}_{\geq 0})^{\mathbb{Z}}$, we let $|\mathbf{v}| = \sum_{k\in\mathbb{Z}} v_k$.

DEFINITION 6.1. *For $\mathbf{v}, \mathbf{w} \in (\mathbb{Z}_{\geq 0})^{\mathbb{Z}}$, let $E(\mathbf{v}, \mathbf{w})$ be the set of all*

$$(V, (u_k^j)_{k\in\mathbb{Z},\, 1\leq j\leq \mathbf{w}_k})$$

where V is a finite-dimensional $\mathfrak{e}(2)$-module with $\mathbf{dim}\, V = \mathbf{v}$ and $\{u_k^j\}_{k\in\mathbb{Z},\, 1\leq j\leq \mathbf{w}_k}$ is a set of generators of V such that $\operatorname{wt} u_k^j = k$. We say that two elements $(V, (u_k^j))$ and $(\tilde{V}, (\tilde{u}_k^j))$ of $E(\mathbf{v}, \mathbf{w})$ are equivalent *if there exists a $\mathfrak{e}(2)$-module isomorphism $\phi : V \to \tilde{V}$ such that $\phi(u_k^j) = \tilde{u}_k^j$. We denote the set of equivalence classes by $\mathcal{E}(\mathbf{v}, \mathbf{w})$.*

THEOREM 6.2. *There is a natural one-to-one correspondence between $\mathcal{E}(\mathbf{v}, \mathbf{w})$ and $\mathcal{L}_{Q_\infty}(\mathbf{v}, \mathbf{w})$.*

PROOF. Let $(V, (u_k^j)_{k\in\mathbb{Z},\, 1\leq j\leq \mathbf{w}_k}) \in E(\mathbf{v}, \mathbf{w})$ and let $V = \bigoplus V_k$ be the weight space decomposition of V. Thus V_k is isomorphic to $\mathbb{C}^{\mathbf{v}_k}$ and we identity the two via this isomorphism. We then define a point $\varphi(V, (u_k^j)) = (x, s) \in L_{Q_\infty}(\mathbf{v}, \mathbf{w})$ by setting

$$x_{h_i} = p_+|_{V_i}, \quad x_{\bar{h}_i} = p_-|_{V_{i+1}}, \quad i \in \mathbb{Z},$$
$$s(w_k^j) = u_k^j, \quad k \in \mathbb{Z}, \quad 1 \leq j \leq \mathbf{w}_k,$$

where $\{w_k^j\}_{1\leq j\leq \mathbf{w}_k}$ is the standard basis of $\mathbb{C}^{\mathbf{w}_k}$ and the map s is extended by linearity. It follows from the results of Section 4 that $x \in \Lambda_{V^{\mathbf{v}},Q}$ and so $(x, s) \in L_{Q_\infty}(\mathbf{v}, \mathbf{w})$. Furthermore, it follows from the fact that (u_k^j) is a set of generators, that (x, s) is a stable point. Thus $\varphi : E(\mathbf{v}, \mathbf{w}) \to L_{Q_\infty}(\mathbf{v}, \mathbf{w})^{\mathrm{st}}$. It is easily verified that two elements $(V, (u_k^j))$ and $(\tilde{V}, (\tilde{u}_k^j))$ are equivalent if and only if $\varphi(V, (u_k^j))$ and $\varphi(\tilde{V}, (\tilde{u}_k^j))$ lie in the same $G_{\mathbf{v}}$-orbit. Thus φ induces a map

$\varphi' : \mathcal{E}(\mathbf{v}, \mathbf{w}) \to \mathcal{L}_{Q_\infty}(\mathbf{v}, \mathbf{w})$ which is independent of the isomorphism $V \cong \mathbb{C}^{\mathbf{v}_k}$ chosen in our construction. It is easily seen that φ' is a bijection. □

As noted in Section 5, the irreducible components of Nakajima quiver varieties can be identified with the irreducible components of Lusztig quiver varieties that are not killed by the stability condition. In the language of $\mathfrak{e}(2)$-modules, passing from Lusztig quiver varieties to Nakajima quiver varieties amounts to imposing the condition that the module be generated by a set of $|\mathbf{w}|$ weight vectors with weights prescribed by $\mathbf{w}$.

A *partition* is a sequence of non-increasing natural numbers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$. The corresponding *Young diagram* is a collection of rows of square boxes which are left justified, with λ_i boxes in the ith row, $1 \le i \le l$. We will identify a partition and its Young diagram and we denote by $\mathcal{Y}$ the set of all partitions (or Young diagrams). If b is a box in a Young diagram λ, we write $x \in \lambda$ and we denote the box in the ith column and jth row of λ by $x_{i,j}$ (if such a box exists). The *residue* of $x_{i,j} \in \lambda$ is defined to be $\operatorname{res} x_{i,j} = i - j$. For $\lambda \in \mathcal{Y}$ and $a \in \mathbb{Z}$, define $\mathbf{v}^{\lambda,a} \in (\mathbb{Z}_{\ge 0})^{\mathbb{Z}}$ by setting $\mathbf{v}^{\lambda,a}_{i+a}$ to be the number of boxes in λ of residue i.

Proposition 6.3. *For $\lambda \in \mathcal{Y}$, there exists a unique $\mathfrak{e}(2)$-module V (up to isomorphism) with a single generator of weight $a \in \mathbb{Z}$ and $\mathbf{dim}\, V = \mathbf{v}^{\lambda,a}$. It is given by*

$$\begin{aligned} V &= \operatorname{Span}_{\mathbb{C}}\{x \mid x \in \lambda\} \\ l(x_{i,j}) &= (a + \operatorname{res} x_{i,j}) x_{i,j} = (a + i - j) x_{i,j} \\ p_+(x_{i,j}) &= x_{i+1,j} \\ p_-(x_{i,j}) &= x_{i,j+1}, \end{aligned}$$

where we set $x_{i,j} = 0$ if there is no box of λ in the ith column and jth row.

For $\mathbf{v} \in (\mathbb{Z}_{\ge 0})^{\mathbb{Z}}$ such that $\mathbf{v} \ne \mathbf{v}^{\lambda,a}$ for all $\lambda \in \mathcal{Y}$ and $a \in \mathbb{Z}$, there are no $\mathfrak{e}(2)$-modules V with a single generator and $\mathbf{dim}\, V = \mathbf{v}$

Proof. By tensoring with an appropriate χ_n, we may assume that the generator of our module has weight zero. It is shown in [**11**, §5.1] that

$$\dim_{\mathbb{C}} \mathcal{L}_{Q_\infty}(\mathbf{v}, \mathbf{w}^0) = \begin{cases} 1 & \text{if } \mathbf{v} = \mathbf{v}^{\lambda,0},\ \lambda \in \mathcal{Y}, \\ 0 & \text{otherwise} \end{cases},$$

where $\mathbf{w}^0_0 = 1$ and $\mathbf{w}^0_i = 0$ for $i \ne 0$ (the first case can be deduced from the dimension formula in Proposition 5.1). It then follows from Theorem 6.2 that if V is an $\mathfrak{e}(2)$-module with a single generator v of weight zero, we must have $\mathbf{dim}\, V = \mathbf{v}^{\lambda,0}$. Furthermore, up to isomorphism, there is only one such pair (V, v) and thus only one such module V. □

Thus $\mathfrak{e}(2)$-modules with a single generator of a fixed weight are determined completely by the dimensions of their weight spaces. This was proven directly by Gruber and Henneberger in [**14**]. As in the proof of Corollary 4.5, we see that our knowledge of the precise relationship between quivers and the Euclidean algebra allows us to use known results about quivers and quiver varieties to simplify such proofs.

Remark 6.4. *As explained at the end of Section 5, the Nakajima quiver varieties $\mathcal{L}_Q(\mathbf{v}, \mathbf{w})$ are closely connected to the representation theory of $\mathfrak{g}_Q$. Therefore,* *the relationship noted in Remark 4.6 between the representation theory of the Euclidean group and the Lie algebra $\mathfrak{sl}_\infty$ (or the Lie groups $SL(n)$) is emphasized further by the above results. Namely, the moduli space of representations of the Euclidean group along with a set of generators is closely related to the representation theory of $\mathfrak{sl}_\infty$ and the Lie groups $SL(n)$.*

REMARK 6.5. *Although Theorem 4.3 tells us that the Euclidean group has wild representation type, the results of this section produce a method of approaching the unwieldy problem of classifying its representations. Namely, if we fix the cardinality and weights of a generating set, the resulting moduli space of representations (along with a set of generators) is enumerated by a countable number of finite-dimensional varieties, one variety for the representations of each graded dimension.*

7. Further directions

The ideas presented in this paper open up some possible avenues of further investigation. We present here two of these.

Consider the Euclidean algebra over a field k of characteristic p instead of over the complex numbers. This algebra is still spanned by $\{p_+, p_-, l\}$ with commutation relations (2.1) but the weights of representations are elements of $\mathbb{Z}/p\mathbb{Z}$ (if we restrict our attention to "integral" weights as usual) instead of $\mathbb{Z}$. One can then show that this category of representations is equivalent to the category of representations of the preprojective algebra of the quiver of affine type $\hat{A}_{p-1}$. In this case, the representations with one generator are, in general, more complicated than in the complex case. We refer the reader to [**11**] for an analysis of the corresponding quiver varieties. There a graphical depiction of the irreducible components of these varieties is developed. These quiver varieties are related to moduli spaces of solutions of anti-self-dual Yang-Mills equations and Hilbert schemes of points in $\mathbb{C}^2$ and it would be interesting to further examine the relationship between these spaces and the Euclidean algebra.

In [**15**] and [**24**], Kashiwara and Saito defined a crystal structure on the sets of irreducible components of Lusztig and Nakajima quiver varieties. Using this structure, each irreducible component can be identified with a sequence of crystal operators acting on the highest weight element of the crystal. Under the identification of quiver varieties with (moduli spaces of) $\mathfrak{e}(2)$-modules, these sequences correspond to the Jordan-Hölder decomposition of $\mathfrak{e}(2)$-modules. It could be fruitful to further examine the implications of this correspondence.

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University of Ottawa, Ottawa, Ontario, Canada
E-mail address: alistair.savage@uottawa.ca