

Quiver varieties and Demazure modules*

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Abstract. Using subvarieties, which we call Demazure quiver varieties, of the quiver varieties of Nakajima, we give a geometric realization of Demazure modules of Kac-Moody algebras with symmetric Cartan data. We give a natural geometric characterization of the extremal weights of a representation and show that Lusztig’s semicanonical basis is compatible with the filtration of a representation by Demazure modules. For the case of $\widehat{\mathfrak{sl}}_2$, we give a characterization of the Demazure quiver variety in terms of a nilpotency condition on quiver representations and an explicit combinatorial description of the Demazure crystal in terms of Young pyramids.

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1. Introduction

Let \mathfrak{g} be a symmetrizable Kac-Moody Lie algebra. Let W be its Weyl group with Bruhat order $<$ and $\{r_i\}_{i \in I}$ the set of simple reflections. Let $U(\mathfrak{g})$ be the universal enveloping algebra of \mathfrak{g} and $U^+(\mathfrak{g})$ the subalgebra generated by the e_i ’s. For a dominant integral weight λ , let $V(\lambda)$ be the irreducible highest weight \mathfrak{g} -module with highest weight λ . For $w \in W$, it is known that the extremal weight space $V(\lambda)_{w\lambda}$ is one dimensional. Here V_μ denotes the μ -weight space of a \mathfrak{g} -module V . Let $V_w(\lambda)$ denote the $U^+(\mathfrak{g})$ -module generated by $V(\lambda)_{w\lambda}$. The modules $V_w(\lambda)$, $w \in W$, are called *Demazure modules*. They are finite dimensional subspaces which form a filtration of $V(\lambda)$ that is compatible with the Bruhat order of W . That is, $V_w(\lambda) \subset V_{w'}(\lambda)$ whenever $w < w'$ and $\bigcup_{w \in W} V_w(\lambda) = V(\lambda)$.

An interesting property of Demazure modules is their relation to integrable models in statistical mechanics. The so-called “one-point functions” in exactly solvable two-dimensional lattices can be evaluated using Baxter’s corner transfer matrix method. This method reduces the computation of one-point functions to a weighted sum over combinatorial objects called paths which are one-dimensional configurations defined on the half line. The generating function of all paths turns

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out to be the character of a highest weight module of an affine Lie algebra. The same physical objects have been evaluated in a completely different way, using Bethe Ansatz methods. Equating the two expressions for the same object, one obtains Rogers-Ramanujan-type identities between q -series. When attempting to provide rigorous proofs of these identities, it is often necessary to work at the level of the half-line of finite length L and then take the limit $L \rightarrow \infty$. A natural question then arises as to the representation theoretic meaning of this finite length. In [1], it was shown that the L -restricted generating functions are closely related to the characters of Demazure modules.

Another interesting property of Demazure modules is their compatibility with the global basis and crystal structure. Let $V^q(\lambda)$ and $V_w^q(\lambda)$ be the q -analogues of $V(\lambda)$ and $V_w(\lambda)$ respectively (i.e. simply replace all objects in the above definition with their q -analogues). Let $(L(\lambda), B(\lambda))$ be the crystal base of $V^q(\lambda)$. Kashiwara showed in [7] that for each $w \in W$, there exists a subset $B_w(\lambda)$ of $B(\lambda)$ such that

$$\frac{V_w^q(\lambda) \cap L(\lambda)}{V_w^q(\lambda) \cap qL(\lambda)} = \sum_{b \in B_w(\lambda)} \mathbb{Q}b. \quad (1.1)$$

Kashiwara also proves that

$$V_w^q(\lambda) = \bigoplus_{b \in B_w(\lambda)} \mathbb{Q}(q)G_\lambda(b),$$

where $\{G_\lambda(b)\}_{b \in B(\lambda)}$ is the lower global basis. In addition, $B_w(\lambda)$ has the following recursive property: If $r_i w \succ w$, then $B_{r_i w}(\lambda) = \bigcup_{n \geq 0} \tilde{f}_i^n B_w(\lambda) \setminus \{0\}$. In particular, if $w = r_{i_1} \cdots r_{i_l}$ is a reduced expression of w , then

$$B_w(\lambda) = \{\tilde{f}_{i_1}^{n_1} \cdots \tilde{f}_{i_l}^{n_l} b_\lambda \mid n_1, \dots, n_l \in \mathbb{Z}_{\geq 0}\} \setminus \{0\}. \quad (1.2)$$

Lusztig [9] and Nakajima [12, 13] have developed a geometric realization of universal enveloping algebras of Kac-Moody algebras with symmetric Cartan data and their integrable highest-weight representations. The algebraic objects are realized as a set of perverse sheaves or constructible functions on varieties attached to quivers or in the homology of these varieties. One actually considers a union of many quiver varieties, each one corresponding to a given weight space of the algebra or representation. The crystal structure of the algebras and their representations has also been realized on the set of irreducible components of these varieties (cf. [8, 14]). It is known (cf. [4]) that the lower global basis defined by Kashiwara coincides with Lusztig's canonical basis, defined geometrically by considering perverse sheaves on quiver varieties. Given these developments, it is natural to ask for an interpretation of Demazure modules in terms of quiver varieties. This is the theme of the current paper.

We find that the Demazure modules are very natural in this geometric setting. In particular, it turns out that the quiver varieties corresponding to a given

representation of highest weight λ are single points exactly for the varieties corresponding to the extremal weights $w\lambda$, $w \in W$. These points, which we denote $[x_w, t_w]$, are orbits of representations of the quiver. If one then restricts the quiver varieties by only considering subrepresentations of the representative (x_w, t_w) of $[x_w, t_w]$ (we call the resulting variety the *Demazure quiver variety*), we obtain a geometric realization of the Demazure modules.

In the geometric realization of representations using constructible functions, there is a distinguished function attached to each irreducible component of the quiver varieties. These functions form a special basis of the representation, called the semicanonical basis. This basis has many nice properties such as compatibility with various filtrations and with canonical antiautomorphisms of the universal enveloping algebras (cf. [11]). In this paper, we add another property to this list. Namely, we show that the semicanonical basis is compatible with the filtration by Demazure modules. More specifically, let B be the set of irreducible components of the quiver varieties corresponding to a representation of highest weight λ and B_w the subset corresponding to the Demazure crystal for some $w \in W$ (we show that this is exactly the set of irreducible components completely contained in the Demazure quiver variety). We then show that

$$V_w(\lambda) \cong \bigoplus_{X \in B_w} \mathbb{C}g_X,$$

where the g_X is the semicanonical basis element corresponding to the irreducible component X .

Finally, we consider the case of $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$ in more detail. We show that in this case, the Demazure quiver variety is simply given by imposing nilpotency of a certain order on the quiver representations. We also give an explicit description of the Demazure crystal in terms of the Young pyramids defined in [17]. These objects were inspired by the geometric realization of crystals.

The organization of the paper is as follows. In Sections 2, 3 and 4 we review the geometric construction of representations of Kac-Moody algebras using quiver varieties. In Section 5 we give a geometric characterization of the extremal weights of these representations. We define the Demazure quiver variety and examine the geometric Demazure crystal in Section 6 and then prove the compatibility of the semicanonical basis with the Demazure filtration in Section 7. Finally, in Sections 8 and 9 we consider the special case of $\widehat{\mathfrak{sl}}_2$, describing the Demazure crystals in terms of Young pyramids and the Demazure quiver varieties in terms of the nilpotency of quiver representations respectively.

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in [2] I. B. Frenkel, M. Khovanov and O. Schiffmann hypothesized the existence of a simple geometric description of Demazure modules using quiver varieties.

2. Lusztig's quiver variety

In this section, we will recount the description given in [9] of Lusztig's quiver variety. See this reference for details, including proofs.

Let I be the set of vertices of the Dynkin graph of a symmetric Kac-Moody Lie algebra \mathfrak{g} and let H be the set of pairs consisting of an edge together with an orientation of it. For $h \in H$, let $\text{in}(h)$ (resp. $\text{out}(h)$) be the incoming (resp. outgoing) vertex of h . We define the involution $\bar{\cdot} : H \rightarrow H$ to be the function which takes $h \in H$ to the element of H consisting of the same edge with opposite orientation. An *orientation* of our graph is a choice of a subset $\Omega \subset H$ such that $\Omega \cup \bar{\Omega} = H$ and $\Omega \cap \bar{\Omega} = \emptyset$.

Let \mathcal{V} be the category of finite-dimensional I -graded vector spaces $\mathbf{V} = \bigoplus_{i \in I} \mathbf{V}_i$ over \mathbb{C} with morphisms being linear maps respecting the grading. Then $\mathbf{V} \in \mathcal{V}$ shall denote that \mathbf{V} is an object of \mathcal{V} . We associate to the graded dimension $\mathbf{v} = (\mathbf{v}_i)_{i \in I}$ of \mathbf{V} the element $\alpha_{\mathbf{v}} = \sum_{i \in I} \mathbf{v}_i \alpha_i$ of the root lattice of \mathfrak{g} . Here the α_i are the simple roots corresponding to the vertices of our quiver (graph with orientation), whose underlying graph is the Dynkin graph of \mathfrak{g} .

Given $\mathbf{V} \in \mathcal{V}$, let

$$\mathbf{E}_{\mathbf{V}} = \bigoplus_{h \in H} \text{Hom}(\mathbf{V}_{\text{out}(h)}, \mathbf{V}_{\text{in}(h)}).$$

For any subset $H' \subset H$, let $\mathbf{E}_{\mathbf{V}, H'}$ be the subspace of $\mathbf{E}_{\mathbf{V}}$ consisting of all vectors $x = (x_h)$ such that $x_h = 0$ whenever $h \notin H'$. The algebraic group $G_{\mathbf{V}} = \prod_i \text{Aut}(\mathbf{V}_i)$ acts on $\mathbf{E}_{\mathbf{V}}$ and $\mathbf{E}_{\mathbf{V}, H'}$ by

$$(g, x) = ((g_i), (x_h)) \mapsto (g_{\text{in}(h)} x_h g_{\text{out}(h)}^{-1}).$$

Define the function $\varepsilon : H \rightarrow \{-1, 1\}$ by $\varepsilon(h) = 1$ for all $h \in \Omega$ and $\varepsilon(h) = -1$ for all $h \in \bar{\Omega}$. The Lie algebra of $G_{\mathbf{V}}$ is $\mathfrak{gl}_{\mathbf{V}} = \prod_i \text{End}(\mathbf{V}_i)$ and it acts on $\mathbf{E}_{\mathbf{V}}$ by

$$(a, x) = ((a_i), (x_h)) \mapsto [a, x] = (x'_h) = (a_{\text{in}(h)} x_h - x_h a_{\text{out}(h)}).$$

Let $\langle \cdot, \cdot \rangle$ be the nondegenerate, $G_{\mathbf{V}}$ -invariant, symplectic form on $\mathbf{E}_{\mathbf{V}}$ with values in \mathbb{C} defined by

$$\langle x, y \rangle = \sum_{h \in H} \varepsilon(h) \text{tr}(x_h y_{\bar{h}}).$$

Note that $\mathbf{E}_{\mathbf{V}}$ can be considered as the cotangent space of $\mathbf{E}_{\mathbf{V}, \Omega}$ under this form.

The moment map associated to the $G_{\mathbf{V}}$ -action on the symplectic vector space $\mathbf{E}_{\mathbf{V}}$ is the map $\psi : \mathbf{E}_{\mathbf{V}} \rightarrow \mathfrak{gl}_{\mathbf{V}}$ with i -component $\psi_i : \mathbf{E}_{\mathbf{V}} \rightarrow \text{End}(\mathbf{V}_i)$ given by

$$\psi_i(x) = \sum_{h \in H, \text{in}(h)=i} \varepsilon(h)x_h x_{\bar{h}}.$$

Definition 2.1 ([9]). *An element $x \in \mathbf{E}_{\mathbf{V}}$ is said to be nilpotent if there exists an $N \geq 1$ such that for any sequence h_1, h_2, \dots, h_N in H satisfying $\text{in}(h_1) = \text{out}(h_2)$, $\text{in}(h_2) = \text{out}(h_3)$, \dots , $\text{in}(h_{N-1}) = \text{out}(h_N)$, the composition $x_{h_N} \cdots x_{h_2} x_{h_1} : \mathbf{V}_{\text{out}(h_1)} \rightarrow \mathbf{V}_{\text{in}(h_N)}$ is zero.*

Definition 2.2 ([9]). *Let $\Lambda_{\mathbf{V}}$ be the set of all nilpotent elements $x \in \mathbf{E}_{\mathbf{V}}$ such that $\psi_i(x) = 0$ for all $i \in I$. We call $\Lambda_{\mathbf{V}}$ Lusztig’s quiver variety.*

It is known that the irreducible components of $\Lambda_{\mathbf{V}}$ are in one-to-one correspondence with a basis of the $\alpha_{\mathbf{V}}$ -weight space of $U^+(\mathfrak{g})$.

3. Nakajima’s quiver variety

We introduce here a description of the quiver varieties first presented in [12]. See [12] and [13] for details.

Definition 3.1 ([12]). *For $\mathbf{v}, \mathbf{d} \in (\mathbb{Z}_{\geq 0})^I$, choose I -graded vector spaces \mathbf{V} and \mathbf{D} of graded dimensions \mathbf{v} and \mathbf{d} respectively. We associate to $\mathbf{d} = (\mathbf{d}_i)_{i \in I}$ the element $\lambda_{\mathbf{d}} = \sum_i \mathbf{d}_i \omega_i$ of the root lattice of \mathfrak{g} , where the ω_i are the fundamental weights of \mathfrak{g} . Recall that we associated to \mathbf{v} the weight $\alpha_{\mathbf{V}} = \sum_i \mathbf{v}_i \alpha_i$. Then define*

$$\Lambda \equiv \Lambda(\mathbf{v}, \mathbf{d}) = \Lambda_{\mathbf{V}} \times \sum_{i \in I} \text{Hom}(\mathbf{V}_i, \mathbf{D}_i).$$

Now, suppose that \mathbf{S} is an I -graded subspace of \mathbf{V} . For $x \in \Lambda_{\mathbf{V}}$ we say that \mathbf{S} is x -stable if $x(\mathbf{S}) \subset \mathbf{S}$.

Definition 3.2 ([12]). *Let $\Lambda^{st} = \Lambda(\mathbf{v}, \mathbf{d})^{st}$ be the set of all $(x, t) \in \Lambda(\mathbf{v}, \mathbf{d})$ satisfying the following condition: If $\mathbf{S} = (\mathbf{S}_i)$ with $\mathbf{S}_i \subset \mathbf{V}_i$ is x -stable and $t_i(\mathbf{S}_i) = 0$ for $i \in I$, then $\mathbf{S}_i = 0$ for $i \in I$.*

The group $G_{\mathbf{V}}$ acts on $\Lambda(\mathbf{v}, \mathbf{d})$ via

$$(g, (x, t)) = ((g_i), ((x_h), (t_i))) \mapsto ((g_{\text{in}(h)} x_h g_{\text{out}(h)}^{-1}), (t_i g_i^{-1})).$$

and the stabilizer of any point of $\Lambda(\mathbf{v}, \mathbf{d})^{st}$ in $G_{\mathbf{V}}$ is trivial (see [13, Lemma 3.10]). We then make the following definition.

Definition 3.3 ([12]). *Let $\mathcal{L} \equiv \mathcal{L}(\mathbf{v}, \mathbf{d}) = \Lambda(\mathbf{v}, \mathbf{d})^{st} / G_{\mathbf{V}}$. We call this Nakajima’s quiver variety.*

Lemma 3.4 ([13]). *We have*

$$\dim_{\mathbb{C}} \mathcal{L}(\mathbf{v}, \mathbf{d}) = \frac{1}{2} \mathbf{v}^t (2\mathbf{d} - C\mathbf{v}),$$

where C is the Cartan matrix of \mathfrak{g} .

4. The Lie algebra action

We summarize here some results from [12] that will be needed in the sequel. See this reference for more details, including proofs. We keep the notation of Sections 2 and 3.

Let $\mathbf{d}, \mathbf{v}, \mathbf{v}', \mathbf{v}'' \in \mathbb{Z}_{\geq 0}^I$ be such that $\mathbf{v} = \mathbf{v}' + \mathbf{v}''$. Consider the maps

$$\Lambda(\mathbf{v}'', \mathbf{0}) \times \Lambda(\mathbf{v}', \mathbf{d}) \xleftarrow{p_1} \tilde{\mathbf{F}}(\mathbf{v}, \mathbf{d}; \mathbf{v}'') \xrightarrow{p_2} \mathbf{F}(\mathbf{v}, \mathbf{d}; \mathbf{v}'') \xrightarrow{p_3} \Lambda(\mathbf{v}, \mathbf{d}), \quad (4.1)$$

where the notation is as follows. A point of $\mathbf{F}(\mathbf{v}, \mathbf{d}; \mathbf{v}'')$ is a point $(x, t) \in \Lambda(\mathbf{v}, \mathbf{d})$ together with an I -graded, x -stable subspace \mathbf{S} of \mathbf{V} such that $\dim \mathbf{S} = \mathbf{v}' = \mathbf{v} - \mathbf{v}''$. A point of $\tilde{\mathbf{F}}(\mathbf{v}, \mathbf{d}; \mathbf{v}'')$ is a point (x, t, \mathbf{S}) of $\mathbf{F}(\mathbf{v}, \mathbf{d}; \mathbf{v}'')$ together with a collection of isomorphisms $R'_i : \mathbf{V}'_i \cong \mathbf{S}_i$ and $R''_i : \mathbf{V}''_i \cong \mathbf{V}_i/\mathbf{S}_i$ for each $i \in I$. Then we define $p_2(x, t, \mathbf{S}, R', R'') = (x, t, \mathbf{S})$, $p_3(x, t, \mathbf{S}) = (x, t)$ and $p_1(x, t, \mathbf{S}, R', R'') = (x'', x', t')$ where x'', x', t' are determined by

$$\begin{aligned} R'_{\text{in}(h)} x'_h &= x_h R'_{\text{out}(h)} : \mathbf{V}'_{\text{out}(h)} \rightarrow \mathbf{S}_{\text{in}(h)}, \\ t'_i &= t_i R'_i : \mathbf{V}'_i \rightarrow \mathbf{D}_i \\ R''_{\text{in}(h)} x''_h &= x_h R''_{\text{out}(h)} : \mathbf{V}''_{\text{out}(h)} \rightarrow \mathbf{V}_{\text{in}(h)}/\mathbf{S}_{\text{in}(h)}. \end{aligned}$$

It follows that x' and x'' are nilpotent.

Lemma 4.1 ([12, Lemma 10.3]). *One has*

$$(p_3 \circ p_2)^{-1}(\Lambda(\mathbf{v}, \mathbf{d})^{\text{st}}) \subset p_1^{-1}(\Lambda(\mathbf{v}'', \mathbf{0}) \times \Lambda(\mathbf{v}', \mathbf{d})^{\text{st}}).$$

Thus, we can restrict (4.1) to Λ^{st} , forget the $\Lambda(\mathbf{v}'', \mathbf{0})$ -factor and consider the quotient by $G_{\mathbf{V}}, G_{\mathbf{V}'}$. This yields the diagram

$$\mathcal{L}(\mathbf{v}', \mathbf{d}) \xleftarrow{\pi_1} \tilde{\mathcal{F}}(\mathbf{v}, \mathbf{d}; \mathbf{v} - \mathbf{v}') \xrightarrow{\pi_2} \mathcal{L}(\mathbf{v}, \mathbf{d}), \quad (4.2)$$

where

$$\tilde{\mathcal{F}}(\mathbf{v}, \mathbf{d}, \mathbf{v} - \mathbf{v}') \stackrel{\text{def}}{=} \{(x, t, \mathbf{S}) \in \mathbf{F}(\mathbf{v}, \mathbf{d}; \mathbf{v} - \mathbf{v}') \mid (x, t) \in \Lambda(\mathbf{v}, \mathbf{d})^{\text{st}}\} / G_{\mathbf{V}}.$$

Let $M(\mathcal{L}(\mathbf{v}, \mathbf{d}))$ be the vector space of all constructible functions on $\mathcal{L}(\mathbf{v}, \mathbf{d})$. For a subvariety Y of a variety A , let $\mathbf{1}_Y$ denote the function on A which takes the value 1 on Y and 0 elsewhere. Let $\chi(Y)$ denote the Euler characteristic of the algebraic variety Y . Then for a map π between algebraic varieties A and B , let $\pi_!$

denote the map between the abelian groups of constructible functions on A and B given by

$$\pi_!(\mathbf{1}_Y)(y) = \chi(\pi^{-1}(y) \cap Y), \quad Y \subset A,$$

and let π^* be the pullback map from functions on B to functions on A acting as $\pi^* f(y) = f(\pi(y))$. Then define

$$\begin{aligned} H_i &: M(\mathcal{L}(\mathbf{v}, \mathbf{d})) \rightarrow M(\mathcal{L}(\mathbf{v}, \mathbf{d})); & H_i f &= u_i f, \\ E_i &: M(\mathcal{L}(\mathbf{v}, \mathbf{d})) \rightarrow M(\mathcal{L}(\mathbf{v} - \mathbf{e}^i, \mathbf{d})); & E_i f &= (\pi_1)_!(\pi_2^* f), \\ F_i &: M(\mathcal{L}(\mathbf{v} - \mathbf{e}^i, \mathbf{d})) \rightarrow M(\mathcal{L}(\mathbf{v}, \mathbf{d})); & F_i g &= (\pi_2)_!(\pi_1^* g). \end{aligned}$$

Here

$$\mathbf{u} = {}^t(u_0, \dots, u_n) = \mathbf{d} - C\mathbf{v}$$

where C is the Cartan matrix of \mathfrak{g} and we are using diagram (4.2) with $\mathbf{v}' = \mathbf{v} - \mathbf{e}^i$ where \mathbf{e}^i is the vector whose components are given by $e_j^i = \delta_{ij}$.

Now let φ be the constant function on $\mathcal{L}(\mathbf{0}, \mathbf{d})$ with value 1. Let $L(\mathbf{d})$ be the vector space of functions generated by acting on φ with all possible combinations of the operators F_i . Then let $L(\mathbf{v}, \mathbf{d}) = M(\mathcal{L}(\mathbf{v}, \mathbf{d})) \cap L(\mathbf{d})$.

Proposition 4.2 ([12, Thm 10.14]). *The operators E_i, F_i, H_i on $L(\mathbf{d})$ provide the structure of the irreducible highest weight integrable representation of \mathfrak{g} with highest weight $\lambda_{\mathbf{d}}$. Each summand of the decomposition $L(\mathbf{d}) = \bigoplus_{\mathbf{v}} L(\mathbf{v}, \mathbf{d})$ is a weight space with weight $\lambda_{\mathbf{d}} - \alpha_{\mathbf{v}}$.*

Let $X \in \text{Irr } \mathcal{L}(\mathbf{v}, \mathbf{d})$, the set of irreducible components of $\mathcal{L}(\mathbf{v}, \mathbf{d})$, and define a linear map $T_X : L(\mathbf{v}, \mathbf{d}) \rightarrow \mathbb{C}$ as in [10, §3.8]. The map T_X associates to a constructible function $f \in L(\mathbf{v}, \mathbf{d})$ the (constant) value of f on a suitable open dense subset of X . The fact that $L(\mathbf{v}, \mathbf{d})$ is finite-dimensional allows us to take such an open set on which any $f \in L(\mathbf{v}, \mathbf{d})$ is constant. So we have a linear map

$$\Phi : L(\mathbf{v}, \mathbf{d}) \rightarrow \mathbb{C}^{\text{Irr } \mathcal{L}(\mathbf{v}, \mathbf{d})}.$$

The following proposition is proved in [10, 4.16] (slightly generalized in [12, Proposition 10.15]).

Proposition 4.3. *The map Φ is an isomorphism; for any $X \in \text{Irr } \mathcal{L}(\mathbf{v}, \mathbf{d})$, there is a function $g_X \in L(\mathbf{v}, \mathbf{d})$ such that for some open dense subset O of X we have $g_X|_O = 1$ and for some closed $G_{\mathbf{v}}$ -invariant subset $K \subset \mathcal{L}(\mathbf{v}, \mathbf{d})$ of dimension $< \dim \mathcal{L}(\mathbf{v}, \mathbf{d})$ we have $g_X = 0$ outside $X \cup K$. The functions g_X for $X \in \text{Irr } \mathcal{L}(\mathbf{v}, \mathbf{d})$ form a basis of $L(\mathbf{v}, \mathbf{d})$.*

We note that in [12] it was only asserted that the map Φ is an isomorphism for \mathfrak{g} of type A , D , E or of affine type. The extension to arbitrary (symmetric) type follows from the results of [8, 14, 11]. The basis given by the g_X is called the semicanonical basis and has some very nice properties (see [11]).

The set of irreducible components of Nakajima's quiver variety can be endowed with the structure of a crystal. We denote this crystal by $B(\mathbf{d})$. It is isomorphic to the crystal $B(\lambda_{\mathbf{d}})$ of the module $V(\lambda_{\mathbf{d}})$. We refer the reader to [8, 14] for the details of this construction.

5. Quiver varieties and extremal weights

We give here a natural geometric characterization of the extremal weights $w\lambda$, $w \in W$, of the module $V(\lambda)$.

Proposition 5.1. *The quiver variety $\mathfrak{L}(\mathbf{v}, \mathbf{d})$ is a point if and only if $\lambda_{\mathbf{d}} - \alpha_{\mathbf{v}} = w\lambda_{\mathbf{d}}$ for some $w \in W$.*

Proof. We first prove the “if” part of the statement. Since it follows from the definition of Nakajima's quiver varieties that $L(0, \mathbf{d})$ is a point, it suffices to show that if $\mathfrak{L}(\mathbf{v}, \mathbf{d})$ is a point where $\lambda_{\mathbf{d}} - \alpha_{\mathbf{v}} = w\lambda_{\mathbf{d}}$ for some $w \in W$ then $\mathfrak{L}(\mathbf{v}', \mathbf{d})$ is a point where $\lambda_{\mathbf{d}} - \alpha_{\mathbf{v}'} = r_i w\lambda_{\mathbf{d}}$ for an arbitrary simple reflection r_i . Let $\mu = \lambda_{\mathbf{d}} - \alpha_{\mathbf{v}}$. Then

$$r_i \mu = \mu - \langle \mu, \alpha_i \rangle \alpha_i = \sum_j \mathbf{d}_j \omega_j - \sum_j \mathbf{v}_j \alpha_j - \langle \mu, \alpha_i \rangle \alpha_i.$$

So

$$\begin{aligned} \mathbf{v}'_j &= \mathbf{v}_j \text{ for } j \neq i, \\ \mathbf{v}'_i &= \mathbf{v}_i + \langle \mu, \alpha_i \rangle \\ &= \mathbf{v}_i + \mathbf{d}_i - \sum_j \mathbf{v}_j \langle \alpha_j, \alpha_i \rangle \\ &= \mathbf{v}_i + \mathbf{d}_i - \sum_j \mathbf{v}_j C_{ji} \\ &= \mathbf{v}_i + \mathbf{d}_i - \mathbf{v}'^t C \mathbf{e}^i, \end{aligned}$$

where C is the Cartan matrix of \mathfrak{g} . Let $\delta = \mathbf{v}'^t C \mathbf{e}^i$. Note that since C is symmetric, we also have $\delta = (\mathbf{e}^i)^t C \mathbf{v}$. Then by Lemma 3.4,

$$\begin{aligned} 2 \dim_{\mathbb{C}} \mathfrak{L}(\mathbf{v}', \mathbf{d}) &= (\mathbf{v}')^t (2\mathbf{d} - C\mathbf{v}') \\ &= (\mathbf{v} + (\mathbf{d}_i - \delta)\mathbf{e}^i)^t (2\mathbf{d} - C(\mathbf{v} + (\mathbf{d}_i - \delta)\mathbf{e}^i)) \\ &= \mathbf{v}^t (2\mathbf{d} - C\mathbf{v}) - (\mathbf{d}_i - \delta)\mathbf{v}'^t C \mathbf{e}^i + 2(\mathbf{d}_i - \delta)(\mathbf{e}^i)^t \mathbf{d} \\ &\quad - (\mathbf{d}_i - \delta)(\mathbf{e}^i)^t C \mathbf{v} - (\mathbf{d}_i - \delta)^2 (\mathbf{e}^i)^t C \mathbf{e}^i. \end{aligned}$$

Since $\mathbf{v}^t(2\mathbf{d} - C\mathbf{v}) = 2 \dim_{\mathbb{C}} \mathfrak{L}(\mathbf{v}, \mathbf{d}) = 0$, we have

$$\begin{aligned} 2 \dim_{\mathbb{C}} \mathfrak{L}(\mathbf{v}', \mathbf{d}) &= -2(\mathbf{d}_i - \delta)\delta + 2(\mathbf{d}_i - \delta)\mathbf{d}_i - 2(\mathbf{d}_i - \delta)^2 \\ &= 0. \end{aligned}$$

Since it is known that all $\mathfrak{L}(\mathbf{v}, \mathbf{d})$ are connected, it follows that $\mathfrak{L}(\mathbf{v}', \mathbf{d})$ is a point.

Now, to prove the “only if” part of the proposition, we use the crystal structure on the set of irreducible components of Nakajima’s quiver variety. Let X be an irreducible component of $\mathfrak{L}(\mathbf{v}, \mathbf{d})$ of dimension zero (i.e. X is a point). Since we know that $\mathfrak{L}(0, \mathbf{d})$ is a point, we can assume $\mathbf{v} \neq 0$. Choose an $i \in I$ such that $\tilde{e}_i X \neq 0$ (since the crystal for $V(\lambda_{\mathbf{d}})$ is connected, we can always find such an i). Let c be the maximum exponent such that $\tilde{e}_i^c X \neq 0$. Let $\tilde{e}_i^c X$ be an irreducible component of $\mathfrak{L}(\mathbf{v}', \mathbf{d})$. That is,

$$\lambda_{\mathbf{d}} - \alpha_{\mathbf{v}'} = \lambda_{\mathbf{d}} - \alpha_{\mathbf{v}} + c\alpha_i. \quad (5.1)$$

It follows from [14, Lemma 4.2.2] that $\dim \tilde{e}_i^c X \leq \dim X = 0$. So $\dim \tilde{e}_i^c X = 0$. Thus, it also follows from [14, Lemma 4.2.2] that

$$c = \langle h_i, \lambda_{\mathbf{d}} - \alpha_{\mathbf{v}'} \rangle = \langle \alpha_i, \lambda_{\mathbf{d}} - \alpha_{\mathbf{v}'} \rangle$$

Substituting into (5.1), we see that

$$\lambda_{\mathbf{d}} - \alpha_{\mathbf{v}} = r_i(\lambda_{\mathbf{d}} - \alpha_{\mathbf{v}'}).$$

The result then follows by induction on the length of w or, equivalently, on $|\mathbf{v}| = \sum_i \mathbf{v}_i$. \square

6. The geometric Demazure crystal and the Demazure quiver variety

In [8, 14], Kashiwara and Saito defined a crystal structure on the set of irreducible components $B(\mathbf{d})$ of $\sqcup_{\mathbf{v}} \mathfrak{L}(\mathbf{v}, \mathbf{d})$, endowing it with the structure of the crystal of $V(\lambda_{\mathbf{d}})$. As mentioned in Section 1, there is a Demazure crystal $B_w(\mathbf{d})$ corresponding to the Demazure module $V_w(\lambda_{\mathbf{d}})$ for $w \in W$. Note that $B_w(\mathbf{d})$ is isomorphic to $B_w(\lambda_{\mathbf{d}})$. For a fixed \mathbf{d} , let $\mathbf{v}_w, w \in W$ be defined by

$$\lambda_{\mathbf{d}} - \alpha_{\mathbf{v}_w} = w\lambda_{\mathbf{d}}.$$

By Proposition 5.1, $\mathfrak{L}(\mathbf{v}_w, \mathbf{d})$ is a point for all $w \in W$. Denote this point by $X_w = [x_w, t_w]$, the $G_{\mathbf{V}}$ -orbit of the representation (x_w, t_w) of the quiver.

Proposition 6.1. *For $X \in B(\mathbf{d})$, we have that $X \in B_w(\mathbf{d})$ if and only if all points of X are (orbits of) subrepresentations (up to isomorphism) of (x_w, t_w) .*

Proof. Let $w = r_{i_l} \cdots r_{i_1}$ be a reduced expression. From (1.2), we see that if $X \in B_w(\mathbf{d})$ then

$$X = \tilde{e}_{i_1}^{n_1} \cdots \tilde{e}_{i_l}^{n_l} X_w$$

for some $n_1, \dots, n_l \in \mathbb{Z}_{\geq 0}$. It then follows from the definition of the geometric crystal operators that an open dense subset of the irreducible component X consists of points which are the orbits of subrepresentations of (x_w, t_w) . Thus, to prove that X consists entirely of orbits of subrepresentations of (x_w, t_w) , it suffices to prove that the set of such orbits is closed. This set is precisely $\pi_1(\pi_2^{-1}(\mathcal{L}(\mathbf{v}_w, \mathbf{d}))) = \pi_1(\pi_2^{-1}([x_w, t_w]))$ where we have used diagram (4.2) with

$$\mathbf{v}' = \mathbf{v} - \sum_{j=1}^l n_j \mathbf{e}^{i_j}.$$

Since the map π_2 is proper (see [12, 13]) and the point $[x_w, t_w]$ is obviously compact, we have that $\pi_2^{-1}([x_w, t_w])$ is compact and thus $\pi_1(\pi_2^{-1}([x_w, t_w]))$ is closed.

Now suppose that every point of X is an orbit of a subrepresentation of (x_w, t_w) and that

$$X_w = \tilde{f}_{i_1}^{m_1} \cdots \tilde{f}_{i_l}^{m_l} X_{\lambda_{\mathbf{d}}}, \quad (6.1)$$

Where $X_{\lambda_{\mathbf{d}}} = \mathcal{L}(0, \mathbf{d})$. If $l = 0$ or, equivalently, $w = \text{id}$ then the result follows. So we assume $l > 0$. From the proof of Proposition 5.1, we see that

$$\tilde{e}_{i_l}^{m_l} X_w = X_{r_{i_l} w}.$$

Note that $r_{i_l} w = r_{i_{l-1}} \cdots r_{i_1}$ is a reduced expression.

Let \mathbf{V}^w be the space corresponding to the representation whose orbit is X_w . We will use the notation \mathbf{V} to denote the subspace of \mathbf{V}^w corresponding to a representation whose orbit is a point of X . Note that this subspace depends on the point of X in question. From (6.1) and the definition of the crystal operators, we have an x_w -stable flag $0 \subset \mathbf{V}^1 \subset \cdots \subset \mathbf{V}^l = \mathbf{V}^w$ such that $\dim \mathbf{V}^j / \mathbf{V}^{j-1} = m_j \mathbf{e}^{i_j}$.

By the definition of the operator $\tilde{e}_{i_1}^{m_1}$ given in [14], $\tilde{e}_{i_1}^{m_1} X_w$ consists of the point $[x, t] = [x_{r_{i_1} w}, t_{r_{i_1} w}]$ where (x, t) is the restriction of (x_w, t_w) to the space \mathbf{V}^{l-1} . We have $\mathbf{V}_i^{l-1} = \mathbf{V}_i^l = \mathbf{V}_i^w$ for $i \neq i_l$ and

$$\mathbf{V}_i^{l-1} = \text{Im} \left(\bigoplus_{\text{in}(h)=i_l} \mathbf{V}_{\text{out}(h)}^w \xrightarrow{x_w} \mathbf{V}_{i_l}^w \right).$$

Now, let c be the maximum integer such that $\tilde{e}_{i_l}^c X \neq 0$. By the definition of the operator $\tilde{e}_{i_l}^c$ given in [14], an open dense subset of $\tilde{e}_{i_l}^c X$ consists of points $[x, t]$ where (x, t) is the restriction of (x', t') , for some $[x', t'] \in X$, to the space

$$\text{Im} \left(\bigoplus_{\text{in}(h)=i_l} \mathbf{V}_{\text{out}(h)}^w \xrightarrow{x_w} \mathbf{V}_{i_l}^w \right)$$

at the i_l th vertex. Since $\mathbf{V}_{\text{out}(h)} \subset \mathbf{V}_{\text{out}(h)}^w$ for any $h \in H$, the above is a subspace of $\mathbf{V}_{i_l}^{l-1}$. Thus we have that (a representative of) every point of $\tilde{e}_{i_l}^c X$ is a subrepresentation of (a representative of) the unique point of $\tilde{e}_{i_l}^{m_l} X_w = X_{r_{i_l} w}$. The result then follows by induction on the length of w . \square

Definition 6.2. For $w \in W$, we define $\mathfrak{L}_w(\mathbf{v}, \mathbf{d})$ to be the set of all $[x, t] \in \mathfrak{L}(\mathbf{v}, \mathbf{d})$ such that (x, t) is isomorphic to a subrepresentation of (x_w, t_w) . We call $\mathfrak{L}_w(\mathbf{v}, \mathbf{d})$ the Demazure quiver variety.

7. Demazure modules and the semicanonical basis

In [7], Kashiwara proves that

$$V_w^q(\lambda) = \bigoplus_{b \in B_w(\lambda)} \mathbb{Q}(q)G_\lambda(b), \tag{7.1}$$

where $\{G_\lambda(b)\}_{b \in B(\lambda)}$ is the lower global basis. Thus the lower global basis behaves very nicely with respect to the filtration of representations given by Demazure modules. It is known (cf. [4]) that the lower global basis coincides with Lusztig’s canonical basis, defined geometrically by considering perverse sheaves on quiver varieties. Using the geometric description of the Demazure modules given above, one could produce a geometric proof of (7.1).

In this section, we prove that the semicanonical basis also behaves nicely with respect to the filtration by Demazure modules. Let

$$L_w(\mathbf{d}) = \bigoplus_{X \in B_w(\mathbf{d})} \mathbb{C}g_X. \tag{7.2}$$

Theorem 7.1. *We have*

$$V_w(\lambda_{\mathbf{d}}) \cong L_w(\mathbf{d}).$$

Proof. We know that the Demazure module $V_w(\lambda_{\mathbf{d}})$ is isomorphic to the subset of constructible functions on $\bigsqcup_{\mathbf{v}} \mathfrak{L}(\mathbf{v}, \mathbf{d})$ spanned by the result of acting on constant functions on the point X_w by all combinations of the E_i . We also denote this space of functions by $V_w(\lambda_{\mathbf{d}})$. Let f be a function in $V_w(\lambda_{\mathbf{d}})$. By the definition of the action of the E_i ’s (see Section 4), the support of f consists of points $[x, t]$ where (x, t) is a subrepresentation of (x_w, t_w) . Thus f must be a linear combination of g_X with $X \in B_w(\mathbf{d})$. This is because for $X \notin B_w(\mathbf{d})$, g_X is non-zero on an open dense subset of X and thus by Proposition 6.1 contains a point $[x, t]$ in its support such that (x, t) is not isomorphic to a subrepresentation of (x_w, t_w) . So we have that $V_w(\lambda_{\mathbf{d}}) \subset L_w(\mathbf{d})$. However, by (1.1) or (7.1), the number of elements in the Demazure crystal $B_w(\mathbf{d})$ is equal to the dimension of the Demazure module. Thus the result follows by dimension considerations. \square

So we see that the semicanonical basis has a compatibility with the filtration by Demazure modules analogous to that of the canonical basis. Note, however, that it was shown in [3] that the semicanonical and the specialization at $q = 1$ of the canonical bases do indeed differ in general.

8. Young pyramids and Demazure crystals for $\widehat{\mathfrak{sl}}_2$

In [17], using the geometric realization of crystals using quiver varieties, the author developed a combinatorial realization of the crystals of integrable highest weight modules of type $A_n^{(1)}$ using objects called Young pyramids. These can be considered a higher level analogue of the Young wall realization of level one modules developed by Kang (cf. [5]). We note that Kang and Lee have also developed a higher level generalization of Young walls, called Young slices (cf. [6]). In this section, we consider the case $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$ and give an explicit description of the Demazure crystals in the language of Young pyramids.

Let α_0 and α_1 be the two simple roots of \mathfrak{g} and let r_i be the reflection with respect to α_i . The Weyl group W is generated by the r_i and for every $n > 0$, W contains two elements of length n . They are

$$\begin{aligned} w_n^+ &= r_{i_n} \cdots r_{i_2} r_{i_1}, & i_j &\equiv j + 1 \pmod{2}, \\ w_n^- &= r_{i_n} \cdots r_{i_2} r_{i_1}, & i_j &\equiv j \pmod{2}. \end{aligned}$$

Let $\lambda = s\omega_0 + t\omega_1$. Then the ground state pyramid P_λ looks as in Figure 1. Let $\mathcal{P}(\lambda)$ be the set of all 1-reduced Young pyramids built on the ground state pyramid P_λ . Recall that Young pyramids are built by placing blocks of color 0 or 1 on the ground state pyramid such that any block placed in an empty slot matches the color of that slot and any block placed on top of another block must be of opposite color. The collection of blocks placed in one slot is called a *stack* and the words *row* and *column* refer to positions in the ground state pyramid. We

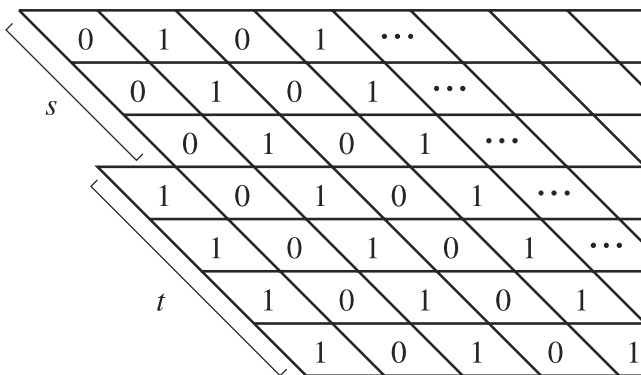


Fig. 1. The ground state pyramid P_λ of highest weight $\lambda = s\omega_0 + t\omega_1$

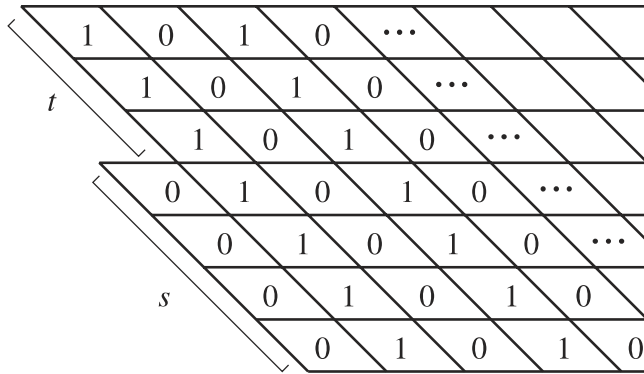


Fig. 2. The modified ground state pyramid of highest weight $\lambda = s\omega_0 + t\omega_1$ for describing the Demazure crystal $\mathcal{P}_{w_n^+}$

use the compass directions to refer to the relative positions of slots in the ground state pyramid (or the stacks placed on them). We number the columns, starting at zero, from left to right. We adopt the convention that the westernmost column is the column containing the t slots of color 1. If $t = 0$ then the westernmost column, containing s slots of color 0 is column one (and not zero).

Since the weight space of an extremal weight is one-dimensional, we know that there is exactly one proper, 1-reduced Young pyramid in $\mathcal{P}(\lambda)$ of weight $w\lambda$ which we will denote by P_λ^w . It is a straightforward calculation based on weights to see that $P_\lambda^{w_n^-}$ is the Young pyramid where the height of all stacks in the i th column is $n - i$ for $0 \leq i \leq n$ and there are no other non-empty stacks. For the case of w_n^+ , we write the ground state pyramid for $\lambda = s\omega_0 + t\omega_1$ as in Figure 2. The properness condition on Young pyramids ensures that the crystal we obtain is the same with this modified ground state pyramid. Then the unique pyramid $P_\lambda^{w_n^+}$ of weight $w_n^+\lambda$ is again the pyramid with stacks in the i th column of height $n - i$ for $0 \leq i \leq n$ and no other stacks. Let $\mathcal{P}_w(\lambda)$ be the subcrystal of $\mathcal{P}(\lambda)$ corresponding to the Demazure crystal $B_w(\lambda)$. We also call $\mathcal{P}_w(\lambda)$ the Demazure crystal.

Theorem 8.1. *The Demazure crystal $\mathcal{P}_{w_n^\pm}(\lambda)$ is the subcrystal of $\mathcal{P}(\lambda)$ consisting of those Young pyramids which are subpyramids of $P_\lambda^{w_n^\pm}$.*

Proof. We prove the case of w_n^- . The case of w_n^+ is analogous. Since all elements of the Demazure crystal are obtained from the extremal element by applying the crystal operators \tilde{e}_i , which correspond to removing blocks of a Young pyramid, it follows that all elements of $\mathcal{P}_{w_n^-}(\lambda)$ are subpyramids of $P_\lambda^{w_n^-}$.

Now let P be a subpyramid of $P_\lambda^{w_n^-}$. Then the maximum possible height of a stack is n . Suppose the maximum height is m . Since P is 1-reduced, the columns

of height m must all be of the same color. Let this color be i . Recall (see [17]) that in defining the action of \tilde{e}_i on a Young pyramid, we compute the i -signature of P . This is done by arranging the stacks from tallest to shortest and assigning a $-$ to i -removable stacks and a $+$ to i -admissible stacks. We arrange these $+$ and $-$ signs from left to right (in the order of the columns to which they correspond) and cancel $(+, -)$ pairs until we obtain a sequence of $-$'s followed by $+$'s. Then \tilde{e}_i acts by removing a block from the stack corresponding to the rightmost $-$ sign (and $\tilde{e}_i P = 0$ if no such sign exists). Now, the columns of height m , being of color i , cannot be i -admissible. Also, being the columns of maximal height, their $-$ signs occur to the left of all the others and thus cannot be cancelled in $(+, -)$ pairs. Thus, after applying \tilde{e}_i as many times as possible (without obtaining zero), we see that the height of all stacks is $\leq m - 1$. We use here the fact that the only way a stack of height m can fail to be i -removable is if the column immediately to the east is one block shorter. Thus its top block would also be of color i . Continuing in this manner, moving east, we see that we will eventually have an i -removable block which cannot be cancelled in a $(+, -)$ pair. When this block is removed, the stack to the west becomes i -removable. Continuing, the block on top of the stack of height m will eventually be removed after enough applications of the operator \tilde{e}_i .

Now, the top of stacks which are now of the highest height must be of color $(1 - i)$ or else we could have applied \tilde{e}_i more times. So we repeat the above process, applying \tilde{e}_{1-i} the maximum number of times. Since after each step we have reduced the maximum height of the columns by at least one, we see that

$$\tilde{e}_{i_k}^{c_k} \cdots \tilde{e}_{i_1}^{c_1} P = P_\lambda,$$

where $k \leq n$ and the indices i_1 alternate between 0's and 1's. Therefore, by the description (1.2) of the Demazure crystal, we see that $P \in \mathcal{P}_{w_n^-}(\lambda)$ which completes the proof. \square

For $\lambda = s\omega_0 + t\omega_1$ (corresponding to $\mathbf{d} = (\mathbf{d}_0, \mathbf{d}_1) = (s, t)$) and $w \in W$, let

$$m_{\mathbf{d}, w} = \begin{cases} n & \text{if } w = w_n^-, t \neq 0 \\ n - 1 & \text{if } w = w_n^-, t = 0 \\ n & \text{if } w = w_n^+, s \neq 0 \\ n - 1 & \text{if } w = w_n^+, s = 0 \end{cases}.$$

Corollary 8.2. $\mathcal{P}_w(\lambda_{\mathbf{d}})$ is the subcrystal of $\mathcal{P}(\lambda_{\mathbf{d}})$ consisting of Young pyramids whose stacks are all of height $\leq m_{\mathbf{d}, w}$.

Given the above descriptions of the Demazure crystals, the character of the Demazure modules is given by

$$\dim V_w(\lambda) = \sum_{P \in \mathcal{P}_w(\lambda)} e^{\text{wt}(P)}. \quad (8.1)$$

Explicit expressions for these characters were given in [1] by characterizing the crystal bases in terms of “paths”. Certain specialized characters, obtained by applying the map $e^\mu \mapsto q^{l(\mu)}$ where l is some integral linear function on the weight lattice, were obtained in [16]. The dimensions of the Demazure modules for $\widehat{\mathfrak{sl}}_2$ were also computed in [15]. As an application of our description in terms of Young pyramids, one can reproduce the dimension formulas of [15] by a simple counting argument. Namely, we can show by counting the Young pyramids of $\mathcal{P}_w(\lambda)$ that the dimensions of the Demazure modules of $\widehat{\mathfrak{sl}}_2$ are given by

$$\dim V_w(s\omega_0 + t\omega_1) = \begin{cases} (s + 1)(s + t + 1)^{n-1} & \text{if } w = w_n^+ \text{ for some } n \geq 1, \\ (t + 1)(s + t + 1)^{n-1} & \text{if } w = w_n^- \text{ for some } n \geq 1. \end{cases} \tag{8.2}$$

9. The Demazure quiver variety for $\widehat{\mathfrak{sl}}_2$

In this section, we examine the Demazure quiver variety defined in Section 6 in the special case where $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$.

For $\mathbf{V} \in \mathcal{V}$ and $x = (x_h)_{h \in H} \in \mathbf{E}_{\mathbf{V}}$, we say that $x^n = 0$ if for any sequence $h_1, \dots, h_n \in H$ such that $\text{in}(h_i) = \text{out}(h_{i+1})$, we have $x_{h_n} \cdots x_{h_1} = 0$.

Theorem 9.1. *When $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$,*

$$\mathcal{L}_w(\mathbf{v}, \mathbf{d}) = \{[x, t] \in \mathcal{L}(\mathbf{v}, \mathbf{d}) \mid x^{m_{\mathbf{d}, w}} = 0\}.$$

Thus to obtain the Demazure quiver variety from the usual quiver variety, we simply have to impose the condition that the quiver representation is nilpotent of the given order.

Proof. Let $m = m_{\mathbf{d}, w}$. It is easy to see from the description of the extremal Young pyramid P_w and the association of this pyramid to the point of the quiver variety as described in [17] that $x_w^m = 0$. Then the fact that all the points $[x, t]$ of the Demazure quiver variety $\mathcal{L}_w(\mathbf{v}, \mathbf{d})$ satisfy $x^m = 0$ follows from the fact that (x, t) is a subrepresentation of (x_w, t_w) .

Now, let $[x, t] \in \mathcal{L}(\mathbf{v}, \mathbf{d})$ with $x = (x_h)_{h \in H}$ and $x^m = 0$. Then x is in the conormal bundle to the $G_{\mathbf{V}}$ -orbit through $x_\Omega = (x_h)_{h \in \Omega}$, which corresponds to a Young pyramid (see [17]). Since $x^m = 0$, each stack in the Young pyramid must have height at most m . Now, by Corollary 8.2, $\mathcal{P}_w(\lambda)$ is precisely the set of Young pyramids with stacks of height $\leq m$. Thus, since the irreducible components of $\mathcal{L}(\mathbf{v}, \mathbf{d})$ correspond to the closures of the conormal bundles to orbits through the x_Ω corresponding to Young pyramids, we see that $[x, t]$ is a point of an irreducible component in $B_w(\mathbf{d})$. Then by Proposition 6.1, we know that (x, t) is isomorphic to a subrepresentation of (x_w, t_w) and therefore is contained in $\mathcal{L}_w(\mathbf{v}, \mathbf{d})$. \square

Corollary 9.2. *Let irr_X be the irreducible component of $\mathfrak{L}_w(\mathbf{v}, \mathbf{d})$ corresponding to $X \in B_w(\mathbf{d})$. Then for the Lie algebra $\widehat{\mathfrak{sl}}_2$, we have*

$$\mathfrak{L}_w(\mathbf{v}, \mathbf{d}) = \bigcup_{X \in B_w(\mathbf{d})} \text{irr}_X.$$

Proof. This follows from the proof of Theorem 9.1. □

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