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Vertex operators and the geometry of moduli spaces of framed torsion-free sheaves

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Abstract We define complexes of vector bundles on products of moduli spaces of framed rank r torsion-free sheaves on $\mathbb{P}^2$. The top non-vanishing equivariant Chern classes of the cohomology of these complexes yield actions of the r-colored Heisenberg and Clifford algebras on the equivariant cohomology of the moduli spaces. In this way we obtain a geometric realization of the boson-fermion correspondence and related vertex operators.

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0 Introduction

Vertex operators play a vital role in the representation theory of affine Lie algebras. The central idea is that instead of considering individual generators of certain algebras, one should construct formal generating series of such operators. In this formalism, the necessary relations in various constructions become tractable (for an introduction to these topics, see [4, 12]). One of the fundamental building blocks of the vertex operator calculus is the boson-fermion correspondence (see [5]), which also plays an important role in mathematical physics.

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The relationship between Hilbert schemes of points on surfaces and the representation theory of Heisenberg algebras has been well studied. In particular, Nakajima [19] defined an action of the Heisenberg algebra on the cohomology of such Hilbert schemes, endowing this cohomology with the structure of bosonic Fock space. Relations to vertex operators were demonstrated by Grojnowski [7], Lehn [10], Li et al. [13], and Carlsson and Okounkov [2]. The construction of the Heisenberg algebra was then extended to the case of equivariant cohomology by Vasserot [24]. In this setting, connections to vertex operator constructions appeared in [14]. These relationships suggest that vertex operators themselves should have geometric descriptions in terms of the Hilbert scheme and related moduli spaces. In this paper, we give such a description (see Sect. 6.1).

In [21], the second author pointed out that the $\mathbb{C}^*$-fixed points of the Hilbert schemes of points in $\mathbb{C}^2$ can be naturally identified with quiver varieties of type A_∞. By a general theory of Nakajima [17, 18], the homology (or equivariant cohomology) of these quiver varieties can be given the structure of a level one representation of $\mathfrak{gl}_\infty$. Thus, the localization theorem, which relates the equivariant cohomology of a space with that of its fixed point set, yields a geometric setting for the Fock space realization of the basic representation of $\mathfrak{gl}_\infty$. A similar idea was used by the first author in [15] to construct the action of the Clifford algebra instead of the Lie algebra $\mathfrak{gl}_\infty$.

While these approaches provide a geometric interpretation of the boson-fermion correspondence, the action of the Clifford algebra is via correspondences on a finite fixed point set of a torus action while the action of the Heisenberg algebra is defined via correspondences on the entire Hilbert schemes. It would be interesting to put the action of the Clifford algebra on the same footing as that of the Heisenberg algebra. That is, one would like to have globally defined operators, rather than just operators living on torus fixed point sets. In this way, the boson-fermion correspondence would appear as the action of both the Clifford and Heisenberg algebras on the same space—the equivariant cohomology of Hilbert schemes or their higher rank generalizations. Apart from being satisfying in its own right, this should be an important step in the program of understanding the entire vertex operator calculus in terms of the moduli space geometry.

In the current paper, we accomplish the aforementioned task as well as the generalization to higher rank. Namely, we define actions of the r-colored Clifford and Heisenberg algebras on the equivariant cohomology of moduli spaces of framed rank r torsion-free sheaves on $\mathbb{P}^2$. The action is defined using equivariant Chern classes of vector bundles defined via complexes on products of these moduli spaces, yielding a new and uniform realization of the Heisenberg and Clifford algebra actions. This method of defining operators was inspired by the work of Carlsson and Okounkov [2], who realized a large class of vertex operators as Chern classes of virtual vector bundles on Hilbert schemes on an arbitrary (quasi-)projective surface. Indeed, our vector bundles have a natural description in the language of sheaves, and in the case of framed rank one sheaves on $\mathbb{P}^2$, they are closely related to the virtual vector bundles considered by Carlsson and Okounkov for the surface $\mathbb{C}^2$ (see Sect. 3.4 for details).

Our extension to higher rank is important in that the r-colored Fock spaces play a key role in other constructions in the theory of vertex operators, such as the vertex operator realization of the basic representations of affine Lie algebras. In particular, we predict that complexes similar to those defined in this paper will lead to moduli space constructions of all vertex operator realizations (including the homogeneous and principle realizations) of the basic representations. Since our constructions are defined via complexes of vector bundles, we expect this method to extend naturally to K-theory and even to a full categorification of representations of the Heisenberg algebras, Clifford algebras, and affine Lie algebras. We also anticipate obtaining natural interpretations of level-rank duality as a certain compatibility between different geometrically defined operators on the equivariant cohomology of moduli spaces.

The organization of this paper is as follows. In Sect. 1, we review the r-colored bosonic and fermionic Fock spaces. The moduli spaces of framed rank r torsion-free sheaves and various results about their equivariant cohomology are discussed in Sect. 2. There we also discuss how one can define operators on this equivariant cohomology via equivariant Chern classes of virtual vector bundles. In Sect. 3, we define the geometric Clifford and Heisenberg operators, leaving the proofs of the main theorems to Sects. 4 and 5. We make some concluding remarks as well as indications of direction of further research in Sect. 6.

1 Fock spaces

In this section, we introduce the r-colored bosonic and fermionic Fock spaces. Further exposition can be found in the references mentioned in Sect. 1.3.

1.1 Bosonic Fock space

A *partition* is an infinite sequence

$$\lambda = (\lambda_1, \lambda_2, \ldots)$$

with a finite number of non-zero terms and such that $\lambda_1 \geq \lambda_2 \geq \cdots$. We identify a finite sequence $(\lambda_1, \ldots, \lambda_n)$ with the infinite sequence obtained by setting $\lambda_i = 0$ for $i > n$. Let $\mathcal{P}$ denote the set of all partitions. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, let $\lambda^t = ((\lambda^t)_1, (\lambda^t)_2, \ldots)$ denote the dual (transposed) partition. That is,

$$(\lambda^t)_i = \#\{j \mid \lambda_j \geq i\}.$$

The largest integer i such that $\lambda_i \neq 0$ is called the *length* of λ (if $\lambda_i = 0$ for all i, the length of λ is zero). Denote the *size* of the partition λ by $|\lambda| = \sum_i \lambda_i$. Following [20], we will identify a partition λ with the set

$$\{(i, j) \mid i \geq 1,\ 1 \leq j \leq \lambda_i\} \subset \mathbb{N}_+^2,$$

where $\mathbb{N}_+$ denotes the set of (strictly) positive integers. When we write $s \in \lambda$, we are referring to this identification. Note that this convention differs from the usual English or French notations for Young diagrams.

For $s = (i, j) \in \mathbb{N}_+^2$, we define the *arm* and *leg length of s in* λ by

$$a_\lambda(s) = \lambda_i - j \quad \text{and} \quad l_\lambda(s) = (\lambda^t)_j - i,$$

respectively. Note that these numbers are negative if $s \notin \lambda$.

Given a pair of partitions λ, μ and $s \in \mathbb{N}_+^2$, define the *relative hook length* of s to be

$$h_{\lambda,\mu}(s) = a_\lambda(s) + l_\mu(s) + 1$$

and the *relative hook number*

$$h_{\lambda,\mu} = \prod_{s \in \lambda} h_{\lambda,\mu}(s).$$

The usual *hook length of s in* λ is then

$$h_\lambda(s) = h_{\lambda,\lambda}(s)$$

and the *hook number* h_λ *of* λ is

$$h_\lambda = h_{\lambda,\lambda}.$$

Lemma 1.1 *Suppose $\lambda \neq \mu$ are partitions with*

$$\lambda_k > \mu_k, \quad \text{and} \quad \lambda_i = \mu_i, \quad 1 \leq i < k.$$

Then $h_{\lambda,\mu}((k, \lambda_k)) = 0$.

Proof We have

$$h_{\lambda,\mu}((k, \lambda_k)) = a_\lambda((k, \lambda_k)) + l_\mu((k, \lambda_k)) + 1 = 0 + (-1) + 1 = 0.$$

□

Let Λ denote the ring of symmetric functions with complex coefficients. We let $\{p_n\}_{n\in\mathbb{N}}$ and $\{s_\lambda\}_{\lambda\in\mathcal{P}}$ denote the power sums and Schur functions, respectively. The Schur functions form a basis of Λ and we also have

$$\Lambda = \mathbb{C}[p_1, p_2, \ldots].$$

We fix the bilinear form $\langle \cdot, \cdot \rangle$ on Λ with respect to which the basis of Schur functions is orthonormal.

Fix $r \in \mathbb{N}_+$ and define the *r-colored oscillator algebra* to be the Lie algebra

$$\mathfrak{s} = \bigoplus_{m\in\mathbb{Z},\, l\in\{1,\ldots,r\}} \mathbb{C}p^l(m) \oplus \mathbb{C}K$$

with commutation relations

$$[\mathfrak{s}, K] = [\mathfrak{s}, p^k(0)] = 0, \quad [p^k(m), p^l(n)] = \frac{1}{m}\delta_{m,-n}\delta_{k,l}K, \; m \neq 0.$$

The subalgebra spanned by $p^l(n)$, $l \in \{1, \ldots, r\}$, $n \in \mathbb{Z}\setminus\{0\}$, and K is an r-colored infinite-dimensional Heisenberg algebra.

The oscillator algebra has a natural representation on the *r-colored bosonic Fock space*

$$\mathbf{B} = B^{\otimes r}, \quad B \stackrel{\text{def}}{=} \mathbb{C}[p_1, p_2, \ldots; q, q^{-1}] \cong \Lambda \otimes_{\mathbb{C}} \mathbb{C}[q, q^{-1}]$$

given by:

$$\begin{aligned} p^l(m) &\mapsto \mathrm{id}^{\otimes(l-1)} \otimes \frac{\partial}{\partial p_m} \otimes \mathrm{id}^{\otimes(r-l)}, \quad m > 0, \\ p^l(-m) &\mapsto \mathrm{id}^{\otimes(l-1)} \otimes \frac{1}{m} p_m \otimes \mathrm{id}^{\otimes(r-l)}, \quad m > 0, \\ p^l(0) &\mapsto \mathrm{id}^{\otimes(l-1)} \otimes q\frac{\partial}{\partial q} \otimes \mathrm{id}^{\otimes(r-l)}, \quad K \mapsto \mathrm{id}\,. \end{aligned}$$

Note that

$$\begin{aligned} \mathbf{B} &= \bigoplus_{\mathbf{c}\in\mathbb{Z}^r, n\in\mathbb{N}} \mathbf{B}_n^{\mathbf{c}}, \\ \mathbf{B}_n^{\mathbf{c}} &\stackrel{\text{def}}{=} \left\{ (q^{c^1} f_1, \ldots, q^{c^r} f_r) \;\middle|\; f_\alpha \in \Lambda \; \forall\alpha, \; \sum_{\alpha=1}^{r} \deg f_\alpha = n \right\}. \end{aligned}$$

We fix the bilinear form on $\mathbf{B}$ given by

$$\left\langle (q^{c^1} f_1, \ldots, q^{c^r} f_r), (q^{d^1} g_1, \ldots, q^{d^r} g_r) \right\rangle = \delta_{c^1,d^1} \cdots \delta_{c^r,d^r} \langle f_1, g_1 \rangle \cdots \langle f_r, g_r \rangle$$

for $f_\alpha, g_\alpha \in \Lambda$, $\alpha \in \{1, \ldots, r\}$. It is easily verified that the operators $p^l(m)$ and $p^l(-m)$ are adjoint for $m \in \mathbb{Z}$.

For an r-tuple $\boldsymbol{\lambda} = (\lambda^1, \ldots, \lambda^r)$ of partitions, define $h_{\boldsymbol{\lambda}} = \prod_{\alpha=1}^{r} h_{\lambda^\alpha}$ and $|\boldsymbol{\lambda}| = \sum_{\alpha=1}^{r} |\lambda^\alpha|$. We call r the *rank* of the oscillator algebra and bosonic Fock space.

1.2 Fermionic Fock space

An infinite expression of the form

$$i_1 \wedge i_2 \wedge i_3 \wedge \cdots,$$

where $i_1, i_2, \ldots$ are integers satisfying

$$i_1 > i_2 > i_3 > \cdots, \quad i_n = i_{n-1} - 1 \quad \text{for } n \gg 0,$$

is called a *semi-infinite monomial*. Let F be the complex vector space with basis consisting of all semi-infinite monomials. Then F is called *fermionic Fock space*. Let

$$|c\rangle = c \wedge (c-1) \wedge (c-2) \wedge \cdots$$

be the *vacuum vector of charge* c. We say that a semi-infinite monomial has *charge* c if it differs from $|c\rangle$ at only a finite number of places. Thus $I = i_1 \wedge i_2 \wedge \cdots$ is of charge c if $i_k = c - k + 1$ for $k \gg 0$. Let F^c denote the linear span of all semi-infinite monomials of charge c. Then we have the charge decomposition

$$F = \bigoplus_{c \in \mathbb{Z}} F^c.$$

A semi-infinite monomial $I = i_1 \wedge i_2 \wedge \cdots$ determines a partition $\lambda(I) \in \mathcal{P}$ by

$$i_k = (c(I) - k + 1) + \lambda(I)_k, \quad k \in \mathbb{N}_+.$$

This gives a bijection $I \mapsto (\lambda(I), c(I))$ between the set of all semi-infinite monomials and the set $\mathcal{P} \times \mathbb{Z}$. We define the *energy* of I to be $|I| = |\lambda(I)|$. Let F^c_j denote the linear span of all semi-infinite monomials of charge c and energy j. We then have the energy decomposition

$$F^c = \bigoplus_{j \in \mathbb{N}} F^c_j.$$

For $j \in \mathbb{Z}$, define the *wedging* and *contracting* operators $\psi(j)$ and $\psi(j)^*$ on F by:

$$\psi(j)(i_1 \wedge i_2 \wedge \cdots) = \begin{cases} 0 & \text{if } j = i_s \text{ for some } s, \\ (-1)^s i_1 \wedge \cdots \wedge i_s \wedge j \wedge i_{s+1} \wedge \cdots & \text{if } i_s > j > i_{s+1}, \end{cases}$$
$$\psi(j)^*(i_1 \wedge i_2 \wedge \cdots) = \begin{cases} 0 & \text{if } j \neq i_s \text{ for all } s, \\ (-1)^{s-1} i_1 \wedge \cdots \wedge i_{s-1} \wedge i_{s+1} \wedge \cdots & \text{if } j = i_s. \end{cases}$$

Fix $r \in \mathbb{N}_+$. We define the *r-colored Clifford algebra* Cl to be the algebra with generators $\psi^l(j), \psi^l(j)^*$ $j \in \mathbb{Z}$, $l \in \{1, \ldots, r\}$ and relations

$$\{\psi^l(j), \psi^l(i)^*\} = \delta_{ji}, \quad \{\psi^l(j), \psi^l(i)\} = 0 = \{\psi^l(j)^*, \psi^l(i)^*\},$$
$$[\psi^l(j), \psi^k(i)] = [\psi^l(j), \psi^k(i)^*] = [\psi^l(j)^*, \psi^k(i)^*] = 0, \quad l \neq k,$$

where $\{a, b\} = ab + ba$. Note that we choose the convention that generators of different colors commute since it turns out that this is more natural from the geometric viewpoint. If one prefers to consider a Clifford algebra where generators of different colors anti-commute, it is only necessary to insert an appropriate sign in the geometric operators to be defined. We define *r-colored fermionic Fock space* to be the vector space

$$\mathbf{F} = F^{\otimes r}.$$

We call an r-tuple $\mathbf{I} = (I^1, \ldots, I^r)$ of semi-infinite monomials an *r-colored semi-infinite monomial*. These form a basis of $\mathbf{F}$ and we let $\langle \cdot, \cdot \rangle$ denote the bilinear form on $\mathbf{F}$ for which this basis is orthonormal. For $l \in \{1, 2, \ldots, r\}$ and $j \in \mathbb{Z}$, the maps

$$\psi^l(j) = \mathrm{id}^{\otimes(l-1)} \otimes \psi(j) \otimes \mathrm{id}^{\otimes(r-l)}, \quad \text{and}$$
$$\psi^l(j)^* = \mathrm{id}^{\otimes(l-1)} \otimes \psi(j)^* \otimes \mathrm{id}^{\otimes(r-l)}$$

define a representation of Cl on $\mathbf{F}$. Note that

$$\psi^l(j)(F^{\mathbf{c}}) \subseteq F^{\mathbf{c}+1_l}, \quad \psi^l(j)^*(F^{\mathbf{c}}) \subseteq F^{\mathbf{c}-1_l},$$

where $1_l \in \mathbb{Z}^r$ has a one in the lth position and a zero everywhere else. The operators $\psi^l(j)$ and $\psi^l(j)^*$ are called *free fermions*. One can check directly that $\psi^l(j)$ and $\psi^l(j)^*$ are adjoint with respect to the bilinear form $\langle \cdot, \cdot \rangle$ and that $\mathbf{F}$ is an irreducible Cl-module. We call r the *rank* of the Clifford algebra and fermionic Fock space.

We have the charge decomposition

$$\mathbf{F} = \bigoplus_{\mathbf{c} \in \mathbb{Z}^r} \mathbf{F}^{\mathbf{c}}, \quad \mathbf{F}^{\mathbf{c}} = F^{c^1} \otimes \cdots \otimes F^{c^r},$$

and energy decomposition

$$\mathbf{F}^{\mathbf{c}} = \bigoplus_{n \in \mathbb{N}} \mathbf{F}^{\mathbf{c}}_n, \quad \mathbf{F}^{\mathbf{c}}_n = \bigoplus_{j^1 + \cdots + j^r = n} F^{c^1}_{j^1} \otimes \cdots \otimes F^{c^r}_{j^r}.$$

In order to simplify notation in what follows, for a semi-infinite monomial I, we define

$$a_I(s) = a_{\lambda(I)}(s), \quad l_I(s) = l_{\lambda(I)}(s).$$

We define $h_I(s)$, h_I, $h_{I,J}(s)$ and $h_{I,J}$ similarly. For an r-tuple $\mathbf{I} = (I^1, \ldots, I^r)$ of semi-infinite monomials, we define $\boldsymbol{\lambda}(\mathbf{I}) = (\lambda(I^1), \ldots, \lambda(I^r))$, $h_{\mathbf{I}} = h_{\boldsymbol{\lambda}(\mathbf{I})}$, $|\mathbf{I}| = |\boldsymbol{\lambda}(\mathbf{I})|$, and $\mathbf{c}(\mathbf{I}) = (c(I^1), \ldots, c(I^r))$.

1.3 The boson-fermion correspondence

The boson-fermion correspondence is a precise relationship between bosonic and fermionic Fock space (see [5]). It uses vertex operators to express bosons in terms of fermions (bosonization) and fermions in terms of bosons (fermionization). We do not present the details here but instead refer the reader to the expository presentations of this topic found in [8, Chapter 14] and the introduction to [23]. In Sect. 6.1 we present a geometric version of the boson-fermion correspondence.

2 Moduli spaces of framed torsion-free sheaves on $\mathbb{P}^2$

In this section, we discuss our main object of study: the moduli space of framed torsion-free sheaves on the complex projective plane $\mathbb{P}^2$ and its equivariant cohomology. For background on equivariant cohomology, especially equivariant Chern classes, we refer the reader to [3, Chap. 9].

2.1 The moduli space $\mathcal{M}(r, n)$

Let $\mathcal{M}(r, n)$ be the moduli space of framed torsion-free sheaves on $\mathbb{P}^2$ with rank r and second Chern class $c_2 = n$. More precisely, points of $\mathcal{M}(r, n)$ are isomorphism classes of pairs (E, Φ) where E is a torsion free sheaf with rank $E = r$ and $c_2(E) = n$ which is locally free in a neighborhood of l_∞, and $\Phi : E_{l_\infty} \xrightarrow{\cong} \mathcal{O}^r_{l_\infty}$ is a framing at infinity. Here $l_\infty = \{[0 : z_1 : z_2] \in \mathbb{P}^2\}$ is the line at infinity. The existence of a framing implies $c_1(E) = 0$.

There is another description of $\mathcal{M}(r, n)$ (basically due to Barth [1]) which will be important for us. Fix $r \in \mathbb{N}_+$, $n \in \mathbb{N}$ and let $V = \mathbb{C}^n$, $W = \mathbb{C}^r$.

Proposition 2.1 [19, Theorem 2.1] *There exists an isomorphism of algebraic varieties*

$$\mathcal{M}(r, n) \cong \{(A, B, i, j) \mid [A, B] + ij = 0,\ (A, B, i, j) \text{ is stable}\}/GL(V),$$

where $A, B \in \operatorname{End} V$*,* $i \in \operatorname{Hom}(W, V)$*,* $j \in \operatorname{Hom}(V, W)$ *and the* $GL(V)$*-action is given by*

$$g(A, B, i, j) = (gAg^{-1}, gBg^{-1}, gi, jg^{-1}), \quad g \in GL(V).$$

We say (A, B, i, j) *is* stable *if there exists no proper subspace* $S \subsetneq V$ *such that* $A(S) \subseteq S$*,* $B(S) \subseteq S$ *and* $\operatorname{im} i \subseteq S$.

We call the quadruple (A, B, i, j) in the above proposition the *ADHM data* corresponding to the framed sheaf (E, Φ). We note that $\mathcal{M}(1, n)$ is isomorphic to the Hilbert scheme of n points in $\mathbb{C}^2$. Thus, the moduli space $\mathcal{M}(r, n)$ can be thought of as a higher rank version of this Hilbert scheme.

2.2 Torus actions on $\mathcal{M}(r, n)$

Fix the torus $T = (\mathbb{C}^*)^r \times \mathbb{C}^*$. Elements of $(\mathbb{C}^*)^r$ will be written

$$e = (e_1, \ldots, e_r).$$

We denote the one-dimensional T-modules

$$(e,t)\mapsto e_i \quad \text{and} \quad (e,t)\mapsto t$$

by e_i and t, respectively, and the tensor product of such modules by juxtaposition. If pt is the space consisting of a single point with the trivial T-action, then $H_T^*(\mathrm{pt}) = \mathbb{C}[b_1, b_2, \ldots, b_r, \epsilon]$ where ϵ and b_i, $i \in \{1, 2, \ldots, r\}$, are the second Chern classes of the one-dimensional T-modules t and e_i, respectively. They are elements of degree 2. We fix a decomposition $W = \bigoplus_{\alpha=1}^r W^\alpha$ where $W^\alpha \cong \mathbb{C}$ for all α. Then $(\mathbb{C}^*)^r$ acts on W by

$$e \mapsto e_1\,\mathrm{id}_{W^1} \oplus \cdots \oplus e_r\,\mathrm{id}_{W^r}.$$

For $\mathbf{c} \in \mathbb{Z}^r$, let $\mathcal{M}_{\mathbf{c}}(r,n)$ denote the moduli space with the torus action

$$(e,t)\star_{\mathbf{c}}(A,B,i,j) = (tA, t^{-1}B, ie^{-1}t^{-\mathbf{c}}, et^{\mathbf{c}}j),$$

where

$$t^{\mathbf{c}} = (t^{c^1}, \ldots, t^{c^r}) \in (\mathbb{C}^*)^r.$$

Note that the underlying variety of $\mathcal{M}_{\mathbf{c}}(r,n)$ is independent of $\mathbf{c}$. It is only the T-action that changes.

The T-fixed points $\mathcal{M}_{\mathbf{c}}(r,n)^T$ are in natural bijection with r-tuples of semi-infinite monomials $\mathbf{I} = (I^1, \ldots, I^r)$, with $\mathbf{c}(\mathbf{I}) = \mathbf{c}$ and $|\boldsymbol{\lambda}(\mathbf{I})| = n$ (see [20, Proposition 2.9]). We shall identify T-fixed points with such r-tuples in what follows. Let $\mathbf{I} = (I^1, \ldots, I^r) \in \mathcal{M}_{\mathbf{c}}(r,n)^T$ and let $\mathcal{T}_{\mathbf{I}}$ denote the tangent space to $\mathcal{M}_{\mathbf{c}}(r,n)$ at the point $\mathbf{I}$. Then T acts on $\mathcal{T}_{\mathbf{I}}$, which decomposes into one-dimensional T-modules.

Proposition 2.2 *As a T-module, $\mathcal{T}_{\mathbf{I}}$ is given by*

$$\mathcal{T}_{\mathbf{I}} = \bigoplus_{\alpha,\beta=1}^{r} \left(e_\beta e_\alpha^{-1} t^{c(I^\beta)-c(I^\alpha)} \left(\bigoplus_{s\in\lambda(I^\alpha)} t^{-h_{I^\alpha,I^\beta}(s)} \oplus \bigoplus_{s\in\lambda(I^\beta)} t^{h_{I^\beta,I^\alpha}(s)} \right)\right).$$

Proof This follows from [20, Theorem 2.11] after replacing t_1 by t, t_2 by t^{-1}, and e_α by $e_\alpha t^{c(I^\alpha)}$ everywhere. □

It will be convenient to consider the splitting $\mathcal{T}_{\mathbf{I}} = \mathcal{T}_{\mathbf{I}}^- \oplus \mathcal{T}_{\mathbf{I}}^+$ where

$$\mathcal{T}_{\mathbf{I}}^- = \bigoplus_{\alpha,\beta=1}^{r} \left(e_\beta e_\alpha^{-1} t^{c(I^\beta)-c(I^\alpha)} \bigoplus_{s\in\lambda(I^\alpha)} t^{-h_{I^\alpha,I^\beta}(s)} \right),$$

$$\mathcal{T}_{\mathbf{I}}^+ = \bigoplus_{\alpha,\beta=1}^{r} \left(e_\alpha e_\beta^{-1} t^{c(I^\alpha)-c(I^\beta)} \bigoplus_{s\in\lambda(I^\alpha)} t^{h_{I^\alpha,I^\beta}(s)} \right).$$

Lemma 2.3 *The equivariant Euler classes of $\mathcal{T}_{\mathbf{I}}^-$ and $\mathcal{T}_{\mathbf{I}}^+$ in the T-equivariant cohomology of a point are given by*

$$e_T(\mathcal{T}_{\mathbf{I}}^-) = \prod_{\alpha,\beta=1}^{r} \prod_{s\in\lambda(I^\alpha)} (b_\beta - b_\alpha + (c(I^\beta) - c(I^\alpha) - h_{I^\alpha,I^\beta}(s))\epsilon),$$

$$e_T(\mathcal{T}_{\mathbf{I}}^+) = \prod_{\alpha,\beta=1}^{r} \prod_{s\in\lambda(I^\alpha)} (b_\alpha - b_\beta + (c(I^\alpha) - c(I^\beta) + h_{I^\alpha,I^\beta}(s))\epsilon)$$
$$= (-1)^{r|\mathbf{I}|} e_T(\mathcal{T}_{\mathbf{I}}^-).$$

Proof This follows easily from the definition of $\mathcal{T}_{\mathbf{I}}^-$ and $\mathcal{T}_{\mathbf{I}}^+$. □

2.3 Bilinear form and the space A

Let

$$\mathcal{H}_T^*(\mathcal{M}_{\mathbf{c}}(r,n)) = H_T^*(\mathcal{M}_{\mathbf{c}}(r,n)) \otimes_{\mathbb{C}[b_1,\dots,b_r,\epsilon]} \mathbb{C}(b_1,\dots,b_r,\epsilon)$$

denote the localized equivariant cohomology. Let

$$i : \mathcal{M}_{\mathbf{c}}(r,n)^T \hookrightarrow \mathcal{M}_{\mathbf{c}}(r,n)$$

denote inclusion, and let

$$p : \mathcal{M}_{\mathbf{c}}(r,n)^T \twoheadrightarrow \{\text{pt}\}$$

be the projection to a point. The real dimension of $\mathcal{M}_{\mathbf{c}}(r,n)$ is $4rn$. We define a bilinear form $\langle\cdot,\cdot\rangle_{n,\mathbf{c}}$ on the middle degree localized equivariant cohomology $\mathcal{H}_T^{2rn}(\mathcal{M}_{\mathbf{c}}(r,n))$ by

$$\langle a,b\rangle_{n,\mathbf{c}} = (-1)^{rn} p_*(i_*)^{-1}(a\cup b),$$

where i_* is invertible by the localization theorem. We then extend this to a bilinear form

$$\langle\cdot,\cdot\rangle = \bigoplus_{n,\mathbf{c}} \langle\cdot,\cdot\rangle_{n,\mathbf{c}}$$

on $\bigoplus_{n,\mathbf{c}} \mathcal{H}_T^{2rn}(\mathcal{M}_{\mathbf{c}}(r,n))$ by declaring classes from different summands to be orthogonal. For $\mathbf{I} \in \mathcal{M}_{\mathbf{c}}(r,n)^T$, let

$$[\mathbf{I}] = \frac{i_*(1_{\mathbf{I}})}{e_T(\mathcal{T}_{\mathbf{I}}^-)} \in \mathcal{H}_T^{2r|\mathbf{I}|}(\mathcal{M}_{\mathbf{c}(\mathbf{I})}(r,|\mathbf{I}|)).$$

Here $1_{\mathbf{I}}$ denotes the unit in the T-equivariant cohomology ring of a point and the denominator $e_T(\mathcal{T}_{\mathbf{I}}^-)$ is to be interpreted as an element in this ring. By the localization theorem in equivariant cohomology, the elements $[\mathbf{I}]$ form a $\mathbb{C}(b_1, b_2, \ldots, b_r, \epsilon)$-basis of $\bigoplus_{n,\mathbf{c}} \mathcal{H}_T^*(\mathcal{M}_{\mathbf{c}}(r, n))$.

Proposition 2.4 *The classes* $\{[\mathbf{I}]\}$ *are orthonormal.*

Proof For $\mathbf{I}, \mathbf{J} \in \mathcal{M}_{\mathbf{c}}(r, n)^T$ we compute

$$\langle [\mathbf{I}], [\mathbf{J}] \rangle = (-1)^{rn} p_*(i_*)^{-1}([\mathbf{I}] \cup [\mathbf{J}]) = (-1)^{rn} p_*(i_*)^{-1}\left(\frac{i_*(1_{\mathbf{I}})}{e_T(\mathcal{T}_{\mathbf{I}}^-)} \cup \frac{i_*(1_{\mathbf{J}})}{e_T(\mathcal{T}_{\mathbf{J}}^-)}\right),$$

which is clearly 0 unless $\mathbf{I} = \mathbf{J}$. If $\mathbf{I} = \mathbf{J}$, then by the projection formula the above is equal to

$$(-1)^{rn} p_*\left(\frac{1_{\mathbf{I}}}{e_T(\mathcal{T}_{\mathbf{I}}^-)} \cup \left(\frac{i^* i_*(1_{\mathbf{I}})}{e_T(\mathcal{T}_{\mathbf{I}}^-)}\right)\right) = (-1)^{rn} \frac{e_T(\mathcal{T}_{\mathbf{I}})}{e_T(\mathcal{T}_{\mathbf{I}}^-) e_T(\mathcal{T}_{\mathbf{I}}^-)} = 1.$$

Note that the last equality in the above line follows from our splitting of $\mathcal{T}_{\mathbf{I}}$, which gave

$$e_T(\mathcal{T}_{\mathbf{I}}) = e_T(\mathcal{T}_{\mathbf{I}}^-) e_T(\mathcal{T}_{\mathbf{I}}^+) = (-1)^{rn} e_T(\mathcal{T}_{\mathbf{I}}^-) e_T(\mathcal{T}_{\mathbf{I}}^-).$$

□

A priori, the classes $[\mathbf{I}]$ are elements of the localized equivariant cohomology $\mathcal{H}_T^{2rn}(\mathcal{M}_{\mathbf{c}}(r, n))$. However, despite the division by the equivariant Euler class in the definition, we expect that they lie in the non-localized equivariant cohomology.

Let

$$A_{\mathbf{c}}(r, n) = \mathrm{Span}_{\mathbb{C}}\{[\mathbf{I}] \mid \mathbf{I} \in \mathcal{M}_{\mathbf{c}}(r, n)^T\} \subset \mathcal{H}_T^{2rn}(\mathcal{M}_{\mathbf{c}}(r, n)),$$
$$\mathbf{A} = \bigoplus_{\mathbf{c} \in \mathbb{Z}^r,\, n \in \mathbb{N}} A_{\mathbf{c}}(r, n).$$

Recall that the set $\{[\mathbf{I}] \mid \mathbf{I} \in \mathcal{M}_{\mathbf{c}}(r, n)^T\}$ is a $\mathbb{C}(b_1, \ldots, b_r, \epsilon)$-basis of

$$\bigoplus_{n,\mathbf{c}} \mathcal{H}_T^*(\mathcal{M}_{\mathbf{c}}(r, n)).$$

Thus $\mathbf{A}$ is a full $\mathbb{C}$-lattice in this space.

Corollary 2.5 *The restriction of* $\langle \cdot, \cdot \rangle$ *to* $\mathbf{A}$ *is non-degenerate and* $\mathbb{C}$*-valued.*

2.4 Operators on equivariant cohomology

We define a bilinear form on the localized equivariant cohomology of a product of moduli spaces $\mathcal{M}_{\mathbf{c}}(r, n_1) \times \mathcal{M}_{\mathbf{d}}(r, n_2)$ in a matter analogous to that introduced in Sect. 2.3. Namely,

$$\langle a, b\rangle_{n_1,\mathbf{c},n_2,\mathbf{d}} = (-1)^{rn_2} p_*((i_1 \times i_2)_*)^{-1}(a \cup b),$$

where i_1 and i_2 are the inclusions of the T-fixed points into the first and second factors, respectively. We extend this to a bilinear form

$$\langle \cdot, \cdot \rangle = \bigoplus_{n_1,\mathbf{c},n_2,\mathbf{d}} \langle \cdot, \cdot \rangle_{n_1,\mathbf{c},n_2,\mathbf{d}}$$

on $\bigoplus_{n_1,\mathbf{c},n_2,\mathbf{d}} \mathcal{H}_T^{2r(n_1+n_2)}(\mathcal{M}_{\mathbf{c}}(r, n_1) \times \mathcal{M}_{\mathbf{d}}(r, n_2))$ by declaring elements in different summands to be orthogonal.

If $\alpha \in \mathcal{H}_T^{2r(n_1+n_2)}(\mathcal{M}_{\mathbf{c}}(r, n_1) \times \mathcal{M}_{\mathbf{d}}(r, n_2))$, then α defines an operator

$$\alpha : \mathcal{H}_T^{2rn_1}(\mathcal{M}_{\mathbf{c}}(r, n_1)) \longrightarrow \mathcal{H}_T^{2rn_2}(\mathcal{M}_{\mathbf{d}}(r, n_2))$$

by using the bilinear form to define structure constants:

$$\langle \alpha x, y\rangle_{n_2,\mathbf{d}} \stackrel{\text{def}}{=} \langle x \otimes y, \alpha\rangle_{n_1,\mathbf{c},n_2,\mathbf{d}}.$$

In particular, if E is a T-equivariant vector bundle on $\mathcal{M}_{\mathbf{c}}(r, n_1) \times \mathcal{M}_{\mathbf{d}}(r, n_2)$ and $\beta \in \mathcal{H}_T^{2l}(\mathcal{M}_{\mathbf{c}}(r, n_1) \times \mathcal{M}_{\mathbf{d}}(r, n_2))$ then $\beta \cup c_{r(n_1+n_2)-l}(E)$ defines such an operator. We note that in the sequel, our choices will yield operators that restrict to non-localized equivariant cohomology.

The following lemma is the basic localization tool we will use to compute the action of our geometric Heisenberg and Clifford operators.

Lemma 2.6 *Suppose* $\mathbf{I} \in \mathcal{M}_{\mathbf{c}}(r, n_1)^T$ *and* $\mathbf{J} \in \mathcal{M}_{\mathbf{d}}(r, n_2)^T$. *Let E be a T-equivariant vector bundle on* $\mathcal{M}_{\mathbf{c}}(r, n_1) \times \mathcal{M}_{\mathbf{d}}(r, n_2)$ *and* $\beta \in \mathcal{H}_T^{2l}(\mathcal{M}_{\mathbf{c}}(r, n_1) \times \mathcal{M}_{\mathbf{d}}(r, n_2))$ *for some* $l \in \mathbb{N}$. *Then*

$$\langle \beta \cup c_{r(n_1+n_2)-l}(E)[\mathbf{I}], [\mathbf{J}]\rangle = \frac{\beta_{\mathbf{I},\mathbf{J}} \cup c_{r(n_1+n_2)-l}(E_{(\mathbf{I},\mathbf{J})})}{e_T(\mathcal{T}_{\mathbf{I}}^-)e_T(\mathcal{T}_{\mathbf{J}}^+)},$$

where $c_{r(n_1+n_2)-l}(E_{(\mathbf{I},\mathbf{J})}) \in H_T^*(\mathrm{pt}) = \mathbb{C}[b_1, \ldots, b_r, \epsilon]$ *is the polynomial given by the equivariant Chern class of the fiber of E over the point* $(\mathbf{I}, \mathbf{J})$ *and* $\beta_{\mathbf{I},\mathbf{J}} = i_{\mathbf{I},\mathbf{J}}^*(\beta)$ *where* $i_{\mathbf{I},\mathbf{J}} : (\mathbf{I}, \mathbf{J}) \hookrightarrow \mathcal{M}_{\mathbf{c}}(r, n_1) \times \mathcal{M}_{\mathbf{d}}(r, n_2)$ *is the inclusion of the fixed point.*

Proof

$$\begin{aligned}&\langle\beta\cup c_{r(n_1+n_2)-l}(E)[\mathbf{I}],[\mathbf{J}]\rangle\\&=(-1)^{rn_2}p_*((i_1\times i_2)_*)^{-1}\left([\mathbf{I}]\otimes[\mathbf{J}]\cup\beta\cup c_{r(n_1+n_2)-l}(E)\right)\\&=(-1)^{rn_2}p_*((i_1\times i_2)_*)^{-1}\left((i_1\times i_2)_*(1_{(\mathbf{I},\mathbf{J})})\cup\frac{\beta\cup c_{r(n_1+n_2)-l}(E)}{e_T(\mathcal{T}_{\mathbf{I}}^-)e_T(\mathcal{T}_{\mathbf{J}}^-)}\right),\end{aligned}$$

where the last line just used the definition of [**I**] and [**J**]. By the projection formula and the functoriality of Chern classes, this is equal to

$$\begin{aligned}&(-1)^{rn_2}p_*\left(1_{(\mathbf{I},\mathbf{J})}\cup\frac{(i_1\times i_2)^*(\beta\cup c_{r(n_1+n_2)-l}(E))}{e_T(\mathcal{T}_{\mathbf{I}}^-)e_T(\mathcal{T}_{\mathbf{J}}^-)}\right)\\&=(-1)^{rn_2}p_*\left(1_{(\mathbf{I},\mathbf{J})}\cup\frac{(i_1\times i_2)^*(\beta\cup c_{r(n_1+n_2)-l}((i_1\times i_2)^*E))}{e_T(\mathcal{T}_{\mathbf{I}}^-)e_T(\mathcal{T}_{\mathbf{J}}^-)}\right)\\&=\frac{\beta_{\mathbf{I},\mathbf{J}}\cup c_{r(n_1+n_2)-l}(E_{(\mathbf{I},\mathbf{J})})}{e_T(\mathcal{T}_{\mathbf{I}}^-)e_T(\mathcal{T}_{\mathbf{J}}^+)}.\end{aligned}$$

□

3 Geometric Heisenberg and Clifford operators

3.1 A complex of tautological bundles

One has the *tautological bundles*

$$V\times_{GL(V)}M(r,n)\to\mathcal{M}(r,n),\quad W\times\mathcal{M}(r,n)\to\mathcal{M}(r,n),$$

where $M(r,n)=\{(A,B,i,j)\mid[A,B]+ij=0,\ (A,B,i,j)\text{ is stable}\}$. We denote these vector bundles by V and W, respectively. They are T-equivariant via the natural action of T on $M(r,n)$ and W. We can then consider A and B to be sections of the bundle $\mathrm{Hom}(V,V)$ and i and j to be sections of the bundles $\mathrm{Hom}(W,V)$ and $\mathrm{Hom}(V,W)$, respectively. Over the product $\mathcal{M}_{\mathbf{c}}(r,n_1)\times\mathcal{M}_{\mathbf{d}}(r,n_2)$, we then have the tautological bundles V_k and W_k, and sections A_k, B_k, i_k and j_k, where $k=1,2$, coming from the tautological bundles and sections on the kth factor.

We define a T-equivariant complex of vector bundles on $\mathcal{M}_{\mathbf{c}}(r,n_1)\times\mathcal{M}_{\mathbf{d}}(r,n_2)$ by

$$\mathrm{Hom}(V_1,V_2)\xrightarrow{\sigma}\begin{matrix}t\,\mathrm{Hom}(V_1,V_2)\oplus t^{-1}\,\mathrm{Hom}(V_1,V_2)\\\oplus\\\mathrm{Hom}(W_1,V_2)\oplus\mathrm{Hom}(V_1,W_2)\end{matrix}\xrightarrow{\tau}\mathrm{Hom}(V_1,V_2),\qquad(3.1)$$

where σ and τ are defined by

$$\sigma(\xi) = \begin{pmatrix} \xi A_1 - A_2\xi \\ \xi B_1 - B_2\xi \\ \xi i_1 \\ -j_2\xi \end{pmatrix}, \quad \tau \begin{pmatrix} C \\ D \\ I \\ J \end{pmatrix} = ([A, D] + [C, B] + i_2 J + I j_1),$$

where

$$[A, D] = A_2 D - D A_1, \quad [C, B] = C B_1 - B_2 C.$$

One easily checks that $\tau\sigma = 0$.

Remark 3.1 In the case when $\mathbf{c} = \mathbf{d}$ and $n_1 = n_2$, the cohomology of this complex is the tangent bundle to $\mathcal{M}_{\mathbf{c}}(r, n_1)$ (more precisely, the tangent bundle to the diagonal copy of $\mathcal{M}_{\mathbf{c}}(r, n_1)$ inside $\mathcal{M}_{\mathbf{c}}(r, n_1) \times \mathcal{M}_{\mathbf{c}}(r, n_2)$). See [20], in particular, the proof of Theorem 2.11. Thus, (3.1) can be seen be as a generalization of the complex computing the tangent bundle. It is also related to the complex [18, Equation (5.1)] used by Nakajima to define Hecke correspondences yielding the action of Kac–Moody algebras on the homology of quiver varieties. Nakajima identified a section of the cohomology of that complex whose zero set defined the correspondences used in his construction.

Lemma 3.2 *The cohomology* $\ker\tau/\operatorname{im}\sigma$ *is a vector bundle on* $\mathcal{M}_{\mathbf{c}}(r, n_1) \times \mathcal{M}_{\mathbf{d}}(r, n_2)$.

Proof We must show that τ is surjective and that σ is injective. The argument is analogous to the proof of [18, Lemma 3.10] but since our notation and stability condition differ from those used in [18], we include the proof here.

To show that σ is injective, suppose $\xi \in \ker\sigma$. Then $\ker\xi \subseteq V_1$ satisfies

$$\operatorname{im} i_1 \subseteq \ker\xi, \quad A_1(\ker\xi) \subseteq \ker\xi, \quad B_1(\ker\xi) \subseteq \ker\xi.$$

Thus, stability of the point (A_1, B_1, i_1, j_1) implies $\ker\xi = V_1$ and so $\xi = 0$. Therefore, σ is injective.

To show that τ is surjective, suppose that $\zeta \in \operatorname{Hom}(V_2, V_1)$ is orthogonal to $\operatorname{im}\tau$ with respect to the (non-degenerate) trace pairing

$$\langle\cdot, \cdot\rangle : \operatorname{Hom}(V_1, V_2) \times \operatorname{Hom}(V_2, V_1) \longrightarrow \mathbb{C}.$$

Then we have

$$A_1\zeta = \zeta A_2, \quad B_1\zeta = \zeta B_2, \quad \zeta i_2 = 0.$$

Thus $\ker\zeta \subseteq V_2$ satisfies

$$\operatorname{im} i_2 \subseteq \ker\zeta, \quad A_2(\ker\zeta) \subseteq \ker\zeta, \quad B_2(\ker\zeta) \subseteq \ker\zeta,$$

and so the stability condition for (A_2, B_2, i_2, j_2) implies that $\ker \zeta = V_2$. Thus $\zeta = 0$ and therefore τ is surjective. □

Denote the vector bundle $\ker \tau / \operatorname{im} \sigma$ by $\mathcal{K}_{\mathbf{c},\mathbf{d}}(r, n_1, n_2)$. The rank of $\mathcal{K}_{\mathbf{c},\mathbf{d}}(r, n_1, n_2)$ is equal to $r(n_1 + n_2)$ as can be seen from the following lemma.

Lemma 3.3 *Let* $(\mathbf{I}, \mathbf{J}) \in \mathcal{M}_{\mathbf{c}}(r, n_1)^T \times \mathcal{M}_{\mathbf{d}}(r, n_2)^T$. *The equivariant Euler class of the restriction of* $\mathcal{K}_{\mathbf{c},\mathbf{d}}(r, n_1, n_2)$ *to the point* $(\mathbf{I}, \mathbf{J})$ *is given as follows*:

$$e_T(\mathcal{K}_{\mathbf{c},\mathbf{d}}(r, n_1, n_2)_{(\mathbf{I},\mathbf{J})}) = \prod_{\alpha,\beta=1}^{r} \prod_{s \in \lambda(I^\alpha)} (b_\beta - b_\alpha + (d^\beta - c^\alpha - h_{I^\alpha, J^\beta}(s))\epsilon) \\ \times \prod_{s \in \lambda(J^\alpha)} (b_\alpha - b_\beta + (d^\alpha - c^\beta + h_{J^\alpha, I^\beta}(s))\epsilon). \quad (3.2)$$

Proof The T-module decomposition of $\mathcal{K}_{\mathbf{c},\mathbf{d}}(r, n_1, n_2)$ for the case $\mathbf{c} = \mathbf{d} = \mathbf{0}$ is computed in [20, Theorem 2.11] (one must set $t_1 = t$ and $t_2 = t^{-1}$ there). The general result follows after replacing b_α by $b_\alpha + c^\alpha \epsilon$ or $b_\alpha + d^\alpha \epsilon$ as appropriate for all $\alpha \in \{1, \ldots, r\}$. □

3.2 Geometric Clifford operators

Fix $l \in \{1, \ldots, r\}$. We will see in Corollary 3.7 that

$$c_{r(n_1+n_2)}(\mathcal{K}_{\mathbf{c},\mathbf{c}\pm 1_l}(r, n_1, n_2)) \neq 0, \quad n_1, n_2 \in \mathbb{N},\ l \in \{1, \ldots, r\},\ c \in \mathbb{Z}^r,$$

and so we define

$$c_{\text{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}\pm 1_l}(r, n_1, n_2)) \stackrel{\text{def}}{=} c_{r(n_1+n_2)}(\mathcal{K}_{\mathbf{c},\mathbf{c}\pm 1_l}(r, n_1, n_2)),$$

the top non-vanishing equivariant Chern class of $\mathcal{K}_{\mathbf{c},\mathbf{c}\pm 1_l}(r, n_1, n_2)$.

Definition 3.4 For $l \in \{1, \ldots, r\}$ and $n \in \mathbb{Z}$, define operators

$$\Psi^l(n), \Psi^l(n)^* : \bigoplus_{\mathbf{c},k} \mathcal{H}_T^{2rk}(\mathcal{M}_{\mathbf{c}}(r, k)) \to \bigoplus_{\mathbf{c},k} \mathcal{H}_T^{2rk}(\mathcal{M}_{\mathbf{c}}(r, k))$$

by

$$\begin{aligned} \Psi^l(n)|_{\mathcal{H}_T^{2rk}(\mathcal{M}_{\mathbf{c}}(r,k))} &= c_{\text{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}+1_l}(r, k, k+n-c^l-1)) \\ &\in H_T^{2r(2k+n-c^l-1)}(\mathcal{M}_{\mathbf{c}}(r, k) \times \mathcal{M}_{\mathbf{c}+1_l}(r, k+n-c^l-1)), \\ \Psi^l(n)^*|_{\mathcal{H}_T^{2rk}(\mathcal{M}_{\mathbf{c}}(r,k))} &= c_{\text{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}-1_l}(r, k, k-n+c^l)) \\ &\in H^{2r(2k-n+c^l)}(\mathcal{M}_{\mathbf{c}}(r, k) \times \mathcal{M}_{\mathbf{c}-1_l}(r, k-n+c^l)). \end{aligned}$$

These operators will be called *geometric Clifford operators* (or *geometric fermions*).

Lemma 3.5 *For $l \in \{1, \dots, r\}$ and $n \in \mathbb{Z}$, the operators $\Psi^l(n)$ and $\Psi^l(n)^*$ are adjoint.*

Proof For $\mathbf{I} \in \mathcal{M}_{\mathbf{c}}(r,k)^T$, $\mathbf{J} \in \mathcal{M}_{\mathbf{c}+1^l}(r, k+n-c^l-1)^T$, we have

$$\begin{aligned}
&c_{\mathrm{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}+1_l}(r,k,k+n-c^l-1)_{(\mathbf{I},\mathbf{J})}) \\
&\quad = (-1)^{r(2k+n-c^l-1)} c_{\mathrm{tnv}}(\mathcal{K}_{\mathbf{c}+1_l,\mathbf{c}}(r,k+n-c^l-1,k)_{(\mathbf{J},\mathbf{I})})
\end{aligned}$$

by Lemma 3.3. Thus, by Lemmas 2.3 and 2.6, we have

$$\begin{aligned}
\langle \Psi^l(n)[\mathbf{I}], [\mathbf{J}]\rangle &= \frac{c_{\mathrm{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}+1_l}(r,k,k+n-c^l-1)_{(\mathbf{I},\mathbf{J})})}{e_T(\mathcal{T}_{\mathbf{I}}^-)e_T(\mathcal{T}_{\mathbf{J}}^+)} \\
&= \frac{(-1)^{r(2k+n-c^l-1)} c_{\mathrm{tnv}}(\mathcal{K}_{\mathbf{c}+1_l,\mathbf{c}}(r,k+n-c^l-1,k)_{(\mathbf{J},\mathbf{I})})}{(-1)^{rk} e_T(\mathcal{T}_{\mathbf{I}}^+)(-1)^{r(k+n-c^l-1)} e_T(\mathcal{T}_{\mathbf{J}}^-)} \\
&= \frac{c_{\mathrm{tnv}}(\mathcal{K}_{\mathbf{c}+1_l,\mathbf{c}}(r,k+n-c^l-1,k)_{(\mathbf{J},\mathbf{I})})}{e_T(\mathcal{T}_{\mathbf{I}}^+)e_T(\mathcal{T}_{\mathbf{J}}^-)} \\
&= \langle \Psi^l(n)^*[\mathbf{J}], [\mathbf{I}]\rangle.
\end{aligned}$$

□

Theorem 3.6 *The geometric Clifford operators $\Psi^l(n)$, $\Psi^l(n)^*$ preserve* **A** *and satisfy the relations*

$$\begin{aligned}
&\{\Psi^l(n), \Psi^l(m)^*\} = \delta_{nm}\,\mathrm{id}, \quad \{\Psi^l(n), \Psi^l(m)\} = 0 = \{\Psi^l(n)^*, \Psi^l(m)^*\}, \\
&[\Psi^l(n), \Psi^k(m)] = [\Psi^l(n), \Psi^k(m)^*] = [\Psi^l(n)^*, \Psi^k(m)^*] = 0, \quad l \neq k.
\end{aligned}$$

In particular, the maps

$$\psi^l(n) \mapsto \Psi^l(n), \quad \psi^l(n)^* \mapsto \Psi^l(n)^*, \quad n \in \mathbb{Z}, \quad l \in \{1, \dots, r\},$$

define a representation of Cl *on* **A** *and the linear map* $\mathbf{A} \to \mathbf{F}$ *given by* $[\mathbf{I}] \mapsto \mathbf{I}$ *is an isometric isomorphism of* Cl*-modules. This isomorphism sends $A_{\mathbf{c}}(r,n)$ to $\mathbf{F}_n^{\mathbf{c}}$.*

The proof of this theorem, which involves a series of technical combinatorial computations, will be given in Sect. 4.

Corollary 3.7 *For $n_1, n_2 \in \mathbb{N}$, $l \in \{1, \dots, r\}$, and $\mathbf{c} \in \mathbb{Z}^r$, we have*

$$c_{r(n_1+n_2)}(\mathcal{K}_{\mathbf{c},\mathbf{c}\pm 1_l}(r, n_1, n_2)) \neq 0.$$

Proof Let $n = n_2 - n_1 + c^l + 1$. Then, by Definition 3.4,

$$\Psi^l(n)|_{\mathcal{H}_T^{2rn_1}(\mathcal{M}_{\mathbf{c}}(r,n_1))} = c_{r(n_1+n_2)}(\mathcal{K}_{\mathbf{c},\mathbf{c}+1_l}(r,n_1,n_2)).$$

We claim that $\psi^l(n)$ is a non-zero operator on $\mathbf{F}^{\mathbf{c}}_{n_1}$. If $n_1 = 0$, let $I^l = |c^l\rangle$. Otherwise, if $n_2 > n_1$, let

$$I^l = (c^l + 1) \wedge c^l \wedge (c^l - 1) \wedge \cdots \wedge (c^l - n_1 + 2) \wedge (c^l - n_1) \wedge (c^l - n_1 - 1) \wedge \cdots,$$

and if $n_2 \leq n_1$, let

$$\begin{aligned} I^l &= (c^l + 1 + n_2) \wedge c^l \wedge (c^l - 1) \\ &\quad \wedge \cdots \wedge (c^l - n_1 + n_2 + 2) \wedge (c^l - n_1 + n_2) \wedge (c^l - n_1 + n_2 - 1) \wedge \cdots. \end{aligned}$$

One easily checks that $c(I^l) = c^l$ and $|I^l| = n_1$. Thus if we set

$$\mathbf{I} = (|c^1\rangle, \ldots, |c^{l-1}\rangle, I^l, |c^{l+1}\rangle, \ldots, |c^r\rangle),$$

we have $\mathbf{c}(\mathbf{I}) = \mathbf{c}$ and $|\mathbf{I}| = n_1$. Since $\psi^l(n)(\mathbf{I}) \neq 0$, we see that $\psi^l(n)$ is a non-zero operator on $\mathbf{F}^{\mathbf{c}}_{n_1}$. Therefore, $\Psi^l(n)$ is a non-zero operator on $\mathcal{H}^{2rn_1}_T(\mathcal{M}_{\mathbf{c}}(r, n_1))$ by Theorem 3.6 and the first statement follows. The proof of the second statement is analogous. □

As mentioned above, this corollary justifies our calling

$$c_{r(n_1+n_2)}(\mathcal{K}_{\mathbf{c},\mathbf{c}\pm 1_l}(r, n_1, n_2))$$

the top non-vanishing equivariant Chern class of $\mathcal{K}_{\mathbf{c},\mathbf{c}\pm 1_l}(r, n_1, n_2)$.

3.3 Geometric Heisenberg operators

Let $(\mathbf{I}, \mathbf{J}) \in \mathcal{M}_{\mathbf{c}}(r, n_1)^T \times \mathcal{M}_{\mathbf{c}}(r, n_2)^T$. By Lemma 3.3, we see that

$$\begin{aligned} e_T(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r, n_1, n_2)_{\mathbf{I},\mathbf{J}}) = \prod_{\alpha,\beta=1}^{r} & \prod_{s \in \lambda(I^\alpha)} (b_\beta - b_\alpha + (c^\beta - c^\alpha - h_{I^\alpha, J^\beta}(s))\epsilon) \\ & \times \prod_{s \in \lambda(J^\alpha)} (b_\alpha - b_\beta + (c^\alpha - c^\beta + h_{J^\alpha, I^\beta}(s))\epsilon). \end{aligned}$$

Now, if $n_1 \neq n_2$, then we must have $I^\alpha \neq J^\alpha$ for some α. But then, since $c(I^\alpha) = c(J^\alpha)$, we have $\lambda(I^\alpha) \neq \lambda(J^\alpha)$ and so $h_{I^\alpha, J^\alpha} = 0$ or $h_{J^\alpha, I^\alpha} = 0$ by Lemma 1.1. Therefore,

$$e_T(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r, n_1, n_2)) = c_{r(n_1+n_2)}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r, n_1, n_2)) = 0 \quad \text{for } n_1 \neq n_2$$

by the Localization Theorem. We will see in Corollary 3.15 that

$$c_{r(n_1+n_2)-1}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r, n_1, n_2)) \neq 0$$

and so we define

$$c_{\text{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r,n_1,n_2)) \stackrel{\text{def}}{=} c_{r(n_1+n_2)-1}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r,n_1,n_2)),$$

the top non-vanishing equivariant Chern class of $\mathcal{K}_{\mathbf{c},\mathbf{c}}(r,n_1,n_2)$.

Lemma 3.8 *Let* $(\mathbf{I},\mathbf{J}) \in \mathcal{M}_{\mathbf{c}}(r,n_1)^T \times \mathcal{M}_{\mathbf{c}}(r,n_2)^T$ *such that* $\lambda(I^l) \neq \lambda(J^l)$ *and let* k *be the smallest integer such that* $\lambda(I^l)_k \neq \lambda(J^l)_k$. *We have*

$$c_{\text{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r,n_1,n_2)_{(\mathbf{I},\mathbf{J})}) = \prod_{\alpha,\beta=1}^{r} \prod_{\substack{s\in\lambda(I^\alpha),\\ s\neq(k,\lambda(I^l)_k)\\ \text{if } \alpha=\beta=l}} (b_\beta - b_\alpha + (c^\beta - c^\alpha - h_{I^\alpha,J^\beta}(s))\epsilon) \times \prod_{s\in\lambda(J^\alpha)} (b_\alpha - b_\beta + (c^\alpha - c^\beta + h_{J^\alpha,I^\beta}(s))\epsilon)$$

if $\lambda(I^l)_k > \lambda(J^l)_k$, *and*

$$c_{\text{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r,n_1,n_2)_{(\mathbf{I},\mathbf{J})}) = \prod_{\alpha,\beta=1}^{r} \prod_{s\in\lambda(I^\alpha)} (b_\beta - b_\alpha + (c^\beta - c^\alpha - h_{I^\alpha,J^\beta}(s))\epsilon) \times \prod_{\substack{s\in\lambda(J^\alpha),\\ s\neq(k,\lambda(J^l)_k) \text{ if } \alpha=\beta=l}} (b_\alpha - b_\beta + (c^\alpha - c^\beta + h_{J^\alpha,I^\beta}(s))\epsilon)$$

if $\lambda(J^l)_k > \lambda(I^l)_k$.

Proof By the above comments, $c_{\text{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r,n_1,n_2)_{(\mathbf{I},\mathbf{J})})$ is obtained by removing one factor of zero from the product appearing in (3.2). The result then follows from Lemma 1.1. □

Remark 3.9 If $(\mathbf{I},\mathbf{J}) \in \mathcal{M}_{\mathbf{c}}(r,n_1)^T \times \mathcal{M}_{\mathbf{c}}(r,n_2)^T$ such that $\lambda(I^l) \neq \lambda(J^l)$ for more than one choice of l, then $c_{\text{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r,n_1,n_2)_{\mathbf{I},\mathbf{J}}) = 0$. Thus it does not matter which l we choose in Lemma 3.8.

If $n_1 = n_2$, then we will see in Corollary 3.15 that the $r(n_1+n_2)$-th equivariant Chern class of $\mathcal{K}_{\mathbf{c},\mathbf{c}}(r,n_1,n_2)$ does *not* vanish and so we define

$$c_{\text{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r,n,n)) \stackrel{\text{def}}{=} c_{2rn}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r,n,n)).$$

Lemma 3.10 *Let* $(\mathbf{I},\mathbf{J}) \in \mathcal{M}_{\mathbf{c}}(r,n)^T \times \mathcal{M}_{\mathbf{c}}(r,n)^T$. *Then*

$$c_{\text{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r,n,n)_{\mathbf{I},\mathbf{J}}) = 0$$

if $\mathbf{I} \neq \mathbf{J}$ *and*

$$c_{\mathrm{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r, n, n)_{\mathbf{I},\mathbf{I}}) = e_T(\mathcal{T}_{\mathbf{I}}).$$

Proof The case $\mathbf{I} \neq \mathbf{J}$ follows as above. The case $\mathbf{I} = \mathbf{J}$ follows immediately from Lemmas 2.3 and 3.3. □

Fix $n \neq m$ and consider the action of the subtorus $(\mathbb{C}^*)^r \subset T$ on $\mathcal{M}_{\mathbf{c}}(r, n) \times \mathcal{M}_{\mathbf{c}}(r, m)$, acting on the framing over $l_\infty \subset \mathbb{P}^2$. The connected components of the fixed point set of this action are products of Hilbert schemes

$$\begin{aligned}&(\mathcal{M}_{\mathbf{c}}(r, n) \times \mathcal{M}_{\mathbf{c}}(r, m))^{(\mathbb{C}^*)^r}\\ &\quad= \bigsqcup_{\mathbf{n},\mathbf{m}} \left(\mathbb{C}^{2[n^1]} \times \cdots \times \mathbb{C}^{2[n^r]}\right) \times \left(\mathbb{C}^{2[m^1]} \times \cdots \times \mathbb{C}^{2[m^r]}\right),\end{aligned}$$

where the union is over $\mathbf{n} = (n^1, \ldots, n^r)$, $\mathbf{m} = (m^1, \ldots, m^r)$ with $\sum_i n^i = n$, $\sum_i m^i = m$, and $\mathbb{C}^{2[k]}$ denotes the Hilbert scheme of k points in $\mathbb{C}^2$. We have the inclusions

$$\begin{aligned}i_{\mathbf{n},\mathbf{m}} : &\left(\mathbb{C}^{2[n^1]} \times \cdots \times \mathbb{C}^{2[n^r]}\right) \times \left(\mathbb{C}^{2[m^1]} \times \cdots \times \mathbb{C}^{2[m^r]}\right)\\ &\hookrightarrow \mathcal{M}_{\mathbf{c}}(r, n) \times \mathcal{M}_{\mathbf{c}}(r, m).\end{aligned}$$

Define disjoint sets A_l, $l \in \{1, \ldots, r\}$, by

$$A_l = \{(\mathbf{n}, \mathbf{m}) \mid n^\alpha = m^\alpha \text{ for } \alpha < l,\ n^l \neq m^l\}.$$

Then, for $l \in \{1, \ldots, r\}$, $n \neq m$, we have a partition of the $(\mathbb{C}^*)^r$ fixed-point components

$$\begin{aligned}(\mathcal{M}_{\mathbf{c}}(r, n) \times \mathcal{M}_{\mathbf{c}}(r, m))^{(\mathbb{C}^*)^r} &= \bigsqcup_{l=1}^{r} \mathcal{A}_l,\\ \mathcal{A}_l = \bigsqcup_{(\mathbf{n},\mathbf{m}) \in A_l} &\left(\mathbb{C}^{2[n^1]} \times \cdots \times \mathbb{C}^{2[n^r]}\right) \times \left(\mathbb{C}^{2[m^1]} \times \cdots \times \mathbb{C}^{2[m^r]}\right),\end{aligned}$$

and the associated inclusions $i_l : \mathcal{A}_l \hookrightarrow \mathcal{M}_{\mathbf{c}}(r, n) \times \mathcal{M}_{\mathbf{c}}(r, m)$. By the Localization Theorem, the restriction

$$i^* = \sum_{\mathbf{n},\mathbf{m}} i^*_{\mathbf{n},\mathbf{m}} : \mathcal{H}^*_T(\mathcal{M}_{\mathbf{c}}(r, n) \times \mathcal{M}_{\mathbf{c}}(r, m)) \longrightarrow \mathcal{H}^*_T\left((\mathcal{M}_{\mathbf{c}}(r, n) \times \mathcal{M}_{\mathbf{c}}(r, m))^{(\mathbb{C}^*)^r}\right)$$

is an isomorphism. By construction, the classes

$$\tilde{\gamma}^l \stackrel{\text{def}}{=} i_l^*(1) = \sum_{(\mathbf{n},\mathbf{m})\in A_l} i_{\mathbf{n},\mathbf{m}}^*(1) \in \mathcal{H}_T^0(\mathcal{A}_l) \subseteq \mathcal{H}_T^0\left((\mathcal{M}_{\mathbf{c}}(r,n)\times\mathcal{M}_{\mathbf{c}}(r,m))^{(\mathbb{C}^*)^r}\right)$$

are orthogonal idempotents ($\tilde{\gamma}^k \cup \tilde{\gamma}^l = \delta_{k,l}\tilde{\gamma}^k$), and they decompose the identity:

$$1 = \sum_{l=1}^r \tilde{\gamma}^l \in \mathcal{H}_T^0\left((\mathcal{M}_{\mathbf{c}}(r,n)\times\mathcal{M}_{\mathbf{c}}(r,m))^{(\mathbb{C}^*)^r}\right).$$

For $l \in \{1,\dots,r\}$, let

$$\gamma^l = \epsilon \cup (i^*)^{-1}(\tilde{\gamma}^l) \in \mathcal{H}_T^2((\mathcal{M}_{\mathbf{c}}(r,n)\times\mathcal{M}_{\mathbf{c}}(r,m)).$$

It follows from the definitions that

$$\gamma^l_{\mathbf{I},\mathbf{J}} = \begin{cases} \epsilon & \text{if } |I^\alpha| = |J^\alpha| \text{ for } \alpha < l \text{ and } |I^l| \neq |J^l|, \\ 0 & \text{otherwise,} \end{cases}$$

where $\gamma^l_{\mathbf{I},\mathbf{J}} = i^*_{\mathbf{I},\mathbf{J}}(\gamma^l)$ and $i_{\mathbf{I},\mathbf{J}} : (\mathbf{I},\mathbf{J}) \hookrightarrow \mathcal{M}_{\mathbf{c}}(r,n)\times\mathcal{M}_{\mathbf{c}}(r,m)$ is the inclusion of the fixed point.

Definition 3.11 For $l \in \{1,\dots,r\}$ and $n \in \mathbb{Z}$, define an operator

$$P^l(n) : \bigoplus_{\mathbf{c},k} \mathcal{H}_T^{2rk}(\mathcal{M}_{\mathbf{c}}(r,k)) \to \bigoplus_{\mathbf{c},k} \mathcal{H}_T^{2rk}(\mathcal{M}_{\mathbf{c}}(r,k))$$

by

$$\begin{aligned} P^l(n)|_{\mathcal{H}_T^{2rk}(\mathcal{M}_{\mathbf{c}}(r,k))} &= \begin{cases} \gamma^l \cup c_{\text{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r,k,k-n)) & n<0, \\ -\gamma^l \cup c_{\text{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r,k,k-n)) & n>0, \end{cases} \\ &\in \mathcal{H}_T^{2r(2k-n)}(\mathcal{M}_{\mathbf{c}}(r,k)\times\mathcal{M}_{\mathbf{c}}(r,k-n)), \\ P^l(0)|_{\mathcal{H}_T^{2rk}(\mathcal{M}_{\mathbf{c}}(r,k))} &= c^l c_{\text{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r,k,k)) = c^l \,\text{id} \end{aligned}$$

(the last equality follows from Lemmas 2.6 and 3.10). These operators will be called *geometric Heisenberg operators* (or *geometric bosons*).

Remark 3.12 We motivate the presence of the classes γ^l in Definition 3.11. Note that the decomposition $\epsilon = \sum_{l=1}^r \gamma^l$ implies

$$\epsilon \cup c_{\text{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r,k,k-n)) = \sum_{l=1}^r P^l(n)|_{\mathcal{H}_T^{2rk}(\mathcal{M}_{\mathbf{c}}(r,k))}.$$

Thus, we have decomposed the operator $\epsilon \cup c_{\mathrm{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r,k,k-n))$ as a sum of r different operators. In particular, for $r=1$, the Heisenberg operators are simply given by the classes $\epsilon \cup c_{\mathrm{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r,k,k-n))$.

Lemma 3.13 *For $n \in \mathbb{Z}$, $l \in \{1,\dots,n\}$, the operators $P^l(n)$ and $P^l(-n)$ are adjoint.*

Proof For $n=0$, the statement is obvious. Assume $n>0$. For $\mathbf{I} \in \mathcal{M}_{\mathbf{c}}(r,k)^T$, $\mathbf{J} \in \mathcal{M}_{\mathbf{c}}(r,k-n)^T$, we have

$$c_{\mathrm{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r,k,k-n)_{(\mathbf{I},\mathbf{J})}) = (-1)^{r(2k-n)-1} c_{\mathrm{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r,k-n,k)_{(\mathbf{J},\mathbf{I})})$$

by Lemma 3.8. Thus, by Lemmas 2.3 and 2.6, we have

$$\begin{aligned}
\langle P^l(n)[\mathbf{I}],[\mathbf{J}]\rangle &= \frac{-\gamma^l_{\mathbf{I},\mathbf{J}} \cup c_{\mathrm{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r,k,k-n)_{(\mathbf{I},\mathbf{J})})}{e_T(\mathcal{T}^-_{\mathbf{I}})e_T(\mathcal{T}^+_{\mathbf{J}})} \\
&= \frac{-(-1)^{r(2k-n)-1}\gamma^l_{\mathbf{I},\mathbf{J}} \cup c_{\mathrm{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r,k-n,k)_{(\mathbf{J},\mathbf{I})})}{(-1)^{rk}e_T(\mathcal{T}^+_{\mathbf{I}})(-1)^{r(k-n)}e_T(\mathcal{T}^-_{\mathbf{J}})} \\
&= \frac{\gamma^l_{\mathbf{I},\mathbf{J}} \cup c_{\mathrm{tnv}}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r,k-n,k)_{(\mathbf{J},\mathbf{I})})}{e_T(\mathcal{T}^+_{\mathbf{I}})e_T(\mathcal{T}^-_{\mathbf{J}})} \\
&= \langle P^l(-n)[\mathbf{J}],[\mathbf{I}]\rangle.
\end{aligned}$$

□

Theorem 3.14 *The geometric Heisenberg operators $P^l(n)$ preserve $\mathbf{A}$ and satisfy the relations*

$$[P^k(n),P^l(0)]=0, \quad [P^k(m),P^l(n)] = \frac{1}{m}\delta_{m,-n}\delta_{k,l}\,\mathrm{id}, \ m \neq 0.$$

In particular, the maps

$$p^l(n) \mapsto P^l(n), \quad n \in \mathbb{Z}, \ l \in \{1,\dots,r\}, \quad K \mapsto \mathrm{id},$$

define a representation of $\mathfrak{s}$ on $\mathbf{A}$ and the linear map $\mathbf{A} \to \mathbf{B}$ given by

$$[\mathbf{I}] \mapsto (q^{c(I^1)}s_{\lambda(I^1)},\dots,q^{c(I^r)}s_{\lambda(I^r)})$$

is an isometric isomorphism of $\mathfrak{s}$-modules. This isomorphism identifies $A_{\mathbf{c}}(r,n)$ with $\mathbf{B}^{\mathbf{c}}_n$.

The proof of this theorem, which involves a series of technical combinatorial computations, will be given in Sect. 5.

Corollary 3.15 *For $n_1,n_2 \in \mathbb{N}$, $n_1 \neq n_2$, and $\mathbf{c} \in \mathbb{Z}^r$, we have*

$$c_{r(n_1+n_2)-1}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r,n_1,n_2)) \neq 0 \quad \textit{and} \quad c_{2rn_1}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r,n_1,n_1)) \neq 0.$$

Proof Let $n = n_1 - n_2$. Then, by Definition 3.11,

$$P^l(n)|_{\mathcal{H}_T^{2rn_1}(\mathcal{M}_{\mathbf{c}}(r,n_1))} = \pm\gamma^l \cup c_{r(n_1+n_2)-1}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r, n_1, n_2)).$$

It is easily seen from the description of the operators $p^l(n)$ in Sect. 1.1 that $p^l(n)$ is a non-zero operator on $B^{\mathbf{c}}_{n_1}$ (note that our assumptions imply $n \le n_1$). Therefore, $P^l(n)$ is a non-zero operator on $\mathcal{H}_T^{2rn_1}(\mathcal{M}_{\mathbf{c}}(r, n_1))$ by Theorem 3.14 and the first result follows. The second follows analogously from the fact that $p^l(0)$ is a non-zero operator. □

As mentioned above, this corollary justifies our calling

$$c_{r(n_1+n_2)-1}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r, n_1, n_2)) \quad \text{and} \quad c_{2rn}(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r, n, n))$$

the top non-vanishing equivariant Chern classes of $\mathcal{K}_{\mathbf{c},\mathbf{c}}(r, n_1, n_2)$, $n_1 \neq n_2$, and $\mathcal{K}_{\mathbf{c},\mathbf{c}}(r, n, n)$, respectively.

3.4 Sheaf theoretic interpretation of the geometric Heisenberg and Clifford operators

Using arguments similar to those appearing in [20, Section 2], one can interpret our geometric Heisenberg and Clifford operators in the language of sheaves. In so doing, we see that in the rank 1 case they are closely related to operators defined by Carsslon and Okouknov [2]. In particular, we have the following.

Proposition 3.16 *Let* $((E_1, \Phi_1), (E_2, \Phi_2)) \in \mathcal{M}_{\mathbf{c}}(r, n_1) \times \mathcal{M}_{\mathbf{d}}(r, n_2)$ *be a pair of framed torsion-free sheaves. Then*

$$\mathcal{K}_{\mathbf{c},\mathbf{d}}(r, n_1, n_2)_{((E_1,\Phi_1),(E_2,\Phi_2))} = \operatorname{Ext}^1(E_1, E_2(-l_\infty)).$$

Proof Let $(E, \Phi) \in \mathcal{M}_{\mathbf{c}}(r, n)$ be a framed torsion-free sheaf with corresponding ADHM data (A, B, i, j) (see Sect. 2.1). The tangent space to $\mathcal{M}_{\mathbf{c}}(r, n)$ at (E, Φ) can be computed in two ways, one using sheaf theory and the other using the ADHM description (see [20], in particular, the proof of Theorem 2.11). In the first case, the tangent space at (E, Φ) is given by $\operatorname{Ext}^1(E, E(-l_\infty))$. In the second, the tangent space at (E, Φ) is the middle cohomology of the complex

$$\operatorname{Hom}(V, V) \xrightarrow{\sigma} \begin{array}{c} t\operatorname{Hom}(V, V) \oplus t^{-1}\operatorname{Hom}(V, V) \\ \oplus \\ \operatorname{Hom}(W, V) \oplus \operatorname{Hom}(V, W) \end{array} \xrightarrow{\tau} \operatorname{Hom}(V, V), \tag{3.3}$$

where σ and τ are defined by

$$\sigma(\xi) = \begin{pmatrix} \xi A - A\xi \\ \xi B - B\xi \\ \xi i \\ -j\xi \end{pmatrix}, \quad \tau\begin{pmatrix} C \\ D \\ I \\ J \end{pmatrix} = [A, D] + [C, B] + iJ + Ij.$$

Now consider the particular case of the framed rank $2r$ torsion-free sheaf

$$(E, \Phi) = (E_1 \oplus E_2, \Phi_1 \oplus \Phi_2),$$

which is a point of $\mathcal{M}_{(\mathbf{c},\mathbf{d})}(2r, n_1 + n_2)$ where $(\mathbf{c}, \mathbf{d}) = (c^1, \ldots, c^r, d^1, \ldots, d^r)$. In the language of sheaves, the tangent space at (E, Φ) is then

$$\mathrm{Ext}^1(E_1 \oplus E_2, E_1(-l_\infty) \oplus E_2(-l_\infty))$$

while in the ADHM description, it is the cohomology of the complex

$$\begin{aligned} &\mathrm{Hom}(V_1 \oplus V_2, V_1 \oplus V_2) \\ &\xrightarrow{\sigma} \begin{matrix} t\,\mathrm{Hom}(V_1 \oplus V_2, V_1 \oplus V_2) \oplus t^{-1}\,\mathrm{Hom}(V_1 \oplus V_2, V_1 \oplus V_2) \\ \oplus \\ \mathrm{Hom}(W_1 \oplus W_2, V_1 \oplus V_2) \oplus \mathrm{Hom}(V_1 \oplus V_2, W_1 \oplus W_2) \end{matrix} \\ &\xrightarrow{\tau} \mathrm{Hom}(V_1 \oplus V_2, V_1 \oplus V_2). \end{aligned} \tag{3.4}$$

Define a $\mathbb{C}^*$-action on $\mathcal{M}_{(\mathbf{c},\mathbf{d})}(2r, n_1 + n_2)$ using the one-parameter subgroup

$$s \mapsto \mathrm{id}_{W_1} \oplus s\,\mathrm{id}_{W_2} \in GL(W_1) \times GL(W_2) \subset GL(W_1 \oplus W_2).$$

The fixed points of this $\mathbb{C}^*$-action are those rank $2r$ torsion-free sheaves which are isomorphic to a direct sum of two rank r torsion-free sheaves (see [20, Proposition 2.9]). In particular, $(E_1 \oplus E_2, \Phi_1 \oplus \Phi_2)$ is fixed by this action and thus the tangent space $\mathrm{Ext}^1(E_1 \oplus E_2, E_1(-l_\infty) \oplus E_2(-l_\infty))$ has an induced $\mathbb{C}^*$-action and decomposes into isotypic components for this action. Over the $\mathbb{C}^*$-fixed point $(E_1 \oplus E_2, \Phi_1 \oplus \Phi_2)$, the complex (3.4) also decomposes into isotypic components. As in the proof of [20, Theorem 2.11], ones see that the complex (3.1) is the isotypic subcomplex of (3.4) of weight 1. Similarly the subbundle $\mathrm{Ext}^1(E_1, E_2(-l_\infty))$ is the isotypic subbundle of $\mathrm{Ext}^1(E_1 \oplus E_2, E_1(-l_\infty) \oplus E_2(-l_\infty))$ of weight 1. □

Consider now the case $r = 1$ and recall that $\mathcal{M}(1, n)$ is the Hilbert scheme of n points in $\mathbb{C}^2$. Note that if one takes $\mathcal{L} = \mathcal{O}(-l_\infty)$ in [2, Section 1.2], then $\chi(\mathcal{L}) = 0$ and thus

$$\mathrm{E}_{(I,J)} = -\chi(I, J(-l_\infty)) = -\sum_{i=0}^{2}(-1)^i\,\mathrm{Ext}^i(I, J(-l_\infty)) = \mathrm{Ext}^1(I, J(-l_\infty))$$

for ideal sheaves I and J. Here we use the fact that $\mathrm{Ext}^0(I, J(-l_\infty)) = \mathrm{Ext}^2(I, J(-l_\infty)) = 0$ (see [20, Proposition 2.1]). Therefore, in the rank one case, our vector bundle $\mathcal{K}_{\mathbf{c},\mathbf{d}}(1, n_1, n_2)$ is an example of the virtual vector bundles considered in [2] with a modified torus action.

4 Proof of Theorem 3.6

4.1 Notation

In order to simplify notation in what follows, for r-colored semi-infinite monomials $\mathbf{I} = (I^1, \dots, I^r)$ and $\mathbf{J} = (J^1, \dots, J^r)$ with $\mathbf{c}(\mathbf{I}) + 1_l = \mathbf{c}(\mathbf{J})$ for some $l \in \{1, \dots, r\}$, define

$$f_{\mathbf{I},\mathbf{J}} = c_{\text{tnv}} \left(\mathcal{K}_{\mathbf{c}(\mathbf{I}),\mathbf{c}(\mathbf{J})}(r, |\mathbf{I}|, |\mathbf{J}|)_{(\mathbf{I},\mathbf{J})} \right).$$

When non-zero, $f_{\mathbf{I},\mathbf{J}}$ is homogeneous of degree $2r(|\mathbf{I}| + |\mathbf{J}|)$. When $r = 1$ and $c(J) = c(I) + 1$,

$$f_{I,J} = (-1)^{|I|} \epsilon^{|I|+|J|} \prod_{s \in \lambda(I)} (a_I(s) + l_J(s)) \prod_{s \in \lambda(J)} (a_J(s) + l_I(s) + 2).$$

For $\mathbf{I} \in \mathcal{M}_{\mathbf{c}}(r, n_1)^T$, $\mathbf{J} \in \mathcal{M}_{\mathbf{d}}(r, n_2)^T$, we also define the polynomial

$$d_{\mathbf{I},\mathbf{J}} = e_T(\mathcal{T}_{\mathbf{I}}^-) e_T(\mathcal{T}_{\mathbf{J}}^+) \in \mathbb{C}[b_1, \dots, b_r, \epsilon].$$

Note that when $r = 1$, we have

$$d_{I,J} = (-1)^{|I|} h_I h_J \epsilon^{|I|+|J|}.$$

Thus

$$f_{I,J} = (-1)^{|I|} \tilde{f}_{I,J} \epsilon^{|I|+|J|}, \quad d_{I,J} = (-1)^{|I|} \tilde{d}_{I,J} \epsilon^{|I|+|J|},$$

where

$$\tilde{f}_{I,J} = \prod_{s \in \lambda(I)} (a_I(s) + l_J(s)) \prod_{s \in \lambda(J)} (a_J(s) + l_I(s) + 2), \quad \tilde{d}_{I,J} = h_I h_J.$$

4.2 Combinatorics of the rank one fermionic Fock space

Let $I = i_1 \wedge i_2 \wedge \dots$ be a semi-infinite monomial of charge c, and let $J = j_1 \wedge j_2 \wedge \dots$ be a semi-infinite monomial of charge $c + 1$.

Lemma 4.1 *Suppose $i_1 = j_1$. Let l be the largest positive integer such that $i_l = i_1 - l + 1$ (in other words, $\lambda(I)_l = \lambda(I)_1$) and let*

$$I' = i_{l+1} \wedge i_{l+2} \wedge \cdots, \quad J' = j_{l+1} \wedge j_{l+2} \wedge \cdots.$$

Then we have

$$\tilde{f}_{I,J} = \begin{cases} (-1)^l \tilde{f}_{I',J'} \prod_{s \in R(I)} h_I(s) \prod_{s \in R(J)} h_J(s) & \text{if } i_k = j_k \ \forall\, 1 \le k \le l, \\ 0 & \text{otherwise}, \end{cases}$$

where

$$R(I) = (\lambda(I)_1, \lambda(I)_2, \ldots, \lambda(I)_l), \quad \textit{and}$$
$$R(J) = (\lambda(J)_1, \lambda(J)_2, \ldots, \lambda(J)_l).$$

Proof Suppose $i_k \neq j_k$ for some $1 \leq k \leq l$. Choose k to be minimal. Then $\lambda(J)_p = \lambda(I)_p - 1 = \lambda(I)_1 - 1$ for all $1 \leq p \leq k-1$ and $\lambda(J)_k < \lambda(I)_k - 1$. Let $s = (k, \lambda(I)_k - 1)$. Then

$$a_I(s) + l_J(s) = 1 + (-1) = 0$$

and the result follows. Therefore, we assume $i_k = j_k$ for all $1 \leq k \leq l$. So for all $1 \leq k \leq l$ we have $\lambda(I)_k = \lambda(I)_1$, $\lambda(J)_k = \lambda(J)_1$ and $\lambda(I)_k = \lambda(J)_k + 1$. Also, by our choice of l, we have $\lambda(I)_{l+1} < \lambda(I)_l = \lambda(I)_1$. Note that

$$\lambda(I') = (\lambda(I)_{l+1}, \lambda(I)_{l+2}, \ldots), \quad \text{and}$$
$$\lambda(J') = (\lambda(J)_{l+1}, \lambda(J)_{l+2}, \ldots).$$

We have

$$\begin{aligned}\tilde{f}_{I,J} &= \prod_{s\in R(I)} (a_I(s) + l_J(s)) \prod_{s\in R(J)} (a_J(s) + l_I(s) + 2) \\ &\quad \times \prod_{s\in\lambda(I')} (a_{I'}(s) + l_{J'}(s)) \prod_{s\in\lambda(J')} (a_{J'}(s) + l_{I'}(s) + 2) \\ &= \tilde{f}_{I',J'} \prod_{s\in R(I)} (a_I(s) + l_J(s)) \prod_{s\in R(J)} (a_J(s) + l_I(s) + 2).\end{aligned}$$

Now, for $s \in R(J)$, we have $a_I(s) = a_J(s) + 1$. Therefore,

$$\prod_{s\in R(I)} (a_I(s) + l_J(s)) = \prod_{s\in R(I)\setminus R(J)} (a_I(s) + l_J(s)) \prod_{s\in R(J)} h_J(s),$$

and

$$\prod_{s\in R(J)} (a_J(s) + l_I(s) + 2) = \prod_{s\in R(J)} h_I(s).$$

Now,

$$\prod_{s\in R(I)\setminus R(J)} (a_I(s) + l_J(s)) = (-1)(-2)\cdots(-l) = (-1)^l \prod_{s\in R(I)\setminus R(J)} h_I(s).$$

Therefore,

$$\tilde{f}_{I,J} = (-1)^l \tilde{f}_{I',J'} \prod_{s\in R(I)} h_I(s) \prod_{s\in R(J)} h_J(s).$$

□

Lemma 4.2 *If $i_k = j_{k+1}$ for all $k \geq 1$, then*

$$\tilde{f}_{I,J} = h_I h_J.$$

Proof The hypotheses imply that the partition $\lambda(I)$ is obtained from the partition $\lambda(J)$ by removing the largest part $\lambda(J)_1$. Therefore,

$$l_J(s) = l_I(s) + 1, \quad s \in \lambda(J),$$

and so

$$\prod_{s\in\lambda(I)} (a_I(s) + l_J(s)) = \prod_{s\in\lambda(I)} (a_I(s) + l_I(s) + 1) = h_I,$$
$$\prod_{s\in\lambda(J)} (a_J(s) + l_I(s) + 2) = \prod_{s\in\lambda(J)} (a_J(s) + l_J(s) + 1) = h_J,$$

and the result follows. □

Lemma 4.3 *If $i_1 \neq j_1$ and $i_k \neq j_{k+1}$ for some $k \geq 1$, then $\tilde{f}_{I,J} = 0$.*

Proof Suppose $i_1 \neq j_1$ and let k be the smallest positive integer such that $i_k \neq j_{k+1}$. Thus k is also the smallest positive integer such that $\lambda(I)_k \neq \lambda(J)_{k+1}$. Suppose $j_{k+1} > i_k$. Then $\lambda(J)_{k+1} > \lambda(I)_k$. Set $s = (k+1, \lambda(J)_{k+1})$. Then $a_J(s) = 0$. Also, since $\lambda(I)_{k-1} = \lambda(J)_k \geq \lambda(J)_{k+1}$, we have $l_I(s) = -2$. Then

$$a_J(s) + l_I(s) + 2 = 0 + (-2) + 2 = 0.$$

Now suppose $j_{k+1} < i_k$. Then $\lambda(J)_{k+1} < \lambda(I)_k$. If $s = (k, \lambda(I)_k)$, then $a_I(s) = 0$. Also, for $k > 1$, since $\lambda(J)_k = \lambda(I)_{k-1} \geq \lambda(I)_k$, we have $l_J(s) = 0$. Therefore,

$$a_I(s) + l_J(s) = 0.$$

If $k = 1$, then either $\lambda(J)_1 \geq \lambda(I)_1$, in which case the above still holds, or $\lambda(J)_1 \leq \lambda(I)_1 - 2$ (we cannot have $\lambda(J)_1 = \lambda(I)_1 - 1$ since this would imply $i_1 = j_1$). In this case, let $s = (1, \lambda(I)_1 - 1)$. Then $a_I(s) = 1$ and $l_J(s) = -1$ and we have

$$a_I(s) + l_J(s) = 0.$$

□

Proposition 4.4 *We have*

$$f_{I,J} = \begin{cases} (-1)^n d_{I,J} & \text{if } J = (-1)^n \psi(k) I \text{ for some } k, n \in \mathbb{Z}, \\ 0 & \text{otherwise}. \end{cases}$$

Proof It suffices to consider only the coefficients $\tilde{f}_{I,J}$ and $\tilde{d}_{I,J}$. We first note that there exists a k such that $J = \pm\psi(k)I$ if and only if there exists an l such that $i_m = j_m$ for all $m < l$, $j_l = k$ and $j_{m+1} = i_m$ for all $m \geq l$. In this case, we have $J = (-1)^{l-1}\psi(k)I$.

Let l be the smallest positive integer such that $i_l \neq j_l$ and let

$$I' = i_l \wedge i_{l+1} \wedge \cdots, \quad J' = j_l \wedge j_{l+1} \wedge \cdots.$$

Repeated application of Lemma 4.1 gives

$$\tilde{f}_{I,J} = (-1)^{l-1} \tilde{f}_{I',J'} \prod_{s \in R(I)} h_I(s) \prod_{s \in R(J)} h_J(s).$$

By Lemmas 4.2 and 4.3, $\tilde{f}_{I',J'} = 0$ unless $i_m = j_{m+1}$ for all $m \geq l$, in which case

$$\tilde{f}_{I',J'} = h_{I'} h_{J'},$$

and so

$$\tilde{f}_{I,J} = (-1)^{l-1} h_I h_J = (-1)^{l-1} \tilde{d}_{I,J}.$$

Thus, we have

$$\tilde{f}_{I,J} = \begin{cases} (-1)^n \tilde{d}_{I,J} & \text{if } J = (-1)^n \psi(k) I \text{ for some } k, n \in \mathbb{Z}, \\ 0 & \text{otherwise} \end{cases}$$

as desired. □

4.3 Combinatorics of the r-colored fermionic Fock space

Suppose $\mathbf{I}$ and $\mathbf{J}$ are r-colored semi-infinite monomials and $\alpha, \beta \in \{1, \ldots, r\}$. In order to simplify notation in the following proofs, we define

$$X_{\mathbf{I},\mathbf{J}}^{\alpha,\beta} = \prod_{s \in \lambda(I^\alpha)} (b_\alpha - b_\beta + (c(I^\alpha) - c(J^\beta) + h_{I^\alpha, J^\beta}(s))\epsilon).$$

Proposition 4.5 *If* $\mathbf{I}$ *and* $\mathbf{J}$ *are semi-infinite monomials with* $\mathbf{c}(\mathbf{I}) + 1_l = \mathbf{c}(\mathbf{J})$, *then*

$$f_{\mathbf{I},\mathbf{J}} = \begin{cases} (-1)^n d_{\mathbf{I},\mathbf{J}} & \text{if } \mathbf{J} = (-1)^n \psi^l(k) \mathbf{I} \text{ for some } k, n \in \mathbb{Z}, \\ 0 & \text{otherwise}. \end{cases}$$

Proof We have

$$f_{\mathbf{I},\mathbf{J}} = \prod_{\alpha,\beta=1}^{r} (-1)^{|I^\alpha|} X^{\alpha,\beta}_{\mathbf{I},\mathbf{J}} X^{\alpha,\beta}_{\mathbf{J},\mathbf{I}} = f_{I^l,J^l} \prod_{\substack{1\le\alpha,\beta\le r \\ \alpha\neq l \text{ or } \beta\neq l}} (-1)^{|I^\alpha|} X^{\alpha,\beta}_{\mathbf{I},\mathbf{J}} X^{\alpha,\beta}_{\mathbf{J},\mathbf{I}}. \tag{4.1}$$

If $\alpha = \beta \neq l$, then

$$\begin{aligned} X^{\alpha,\beta}_{\mathbf{I},\mathbf{J}} X^{\alpha,\beta}_{\mathbf{J},\mathbf{I}} &= X^{\alpha,\alpha}_{\mathbf{I},\mathbf{J}} X^{\alpha,\alpha}_{\mathbf{J},\mathbf{I}} \\ &= \prod_{s\in\lambda(I^\alpha)} h_{I^\alpha,J^\alpha}(s)\epsilon \prod_{s\in\lambda(J^\alpha)} h_{J^\alpha,I^\alpha}(s)\epsilon = h_{I^\alpha,J^\alpha} h_{J^\alpha,I^\alpha} \epsilon^{|I^\alpha|+|J^\alpha|}. \end{aligned}$$

By Lemma 1.1, this is equal to zero unless $I^\alpha = J^\alpha$. We thus need only consider the case where $I^\alpha = J^\alpha$ for all $\alpha \neq l$.

Note that

$$\begin{aligned} X^{\alpha,\beta}_{\mathbf{I},\mathbf{J}} &= X^{\alpha,\beta}_{\mathbf{J},\mathbf{J}}, \quad X^{\alpha,\beta}_{\mathbf{J},\mathbf{I}} = X^{\alpha,\beta}_{\mathbf{I},\mathbf{I}}, \quad \text{if } \alpha \neq l, \quad \text{and} \\ X^{\alpha,\beta}_{\mathbf{I},\mathbf{J}} &= X^{\alpha,\beta}_{\mathbf{I},\mathbf{I}}, \quad X^{\alpha,\beta}_{\mathbf{J},\mathbf{I}} = X^{\alpha,\beta}_{\mathbf{J},\mathbf{J}}, \quad \text{if } \beta \neq l. \end{aligned}$$

Therefore,

$$\prod_{\substack{1\le\alpha,\beta\le r \\ \alpha\neq l \text{ or } \beta\neq l}} (-1)^{|I^\alpha|} X^{\alpha,\beta}_{\mathbf{I},\mathbf{J}} X^{\alpha,\beta}_{\mathbf{J},\mathbf{I}} = \prod_{\substack{1\le\alpha,\beta\le r \\ \alpha\neq l \text{ or } \beta\neq l}} (-1)^{|I^\alpha|} X^{\alpha,\beta}_{\mathbf{I},\mathbf{I}} X^{\alpha,\beta}_{\mathbf{J},\mathbf{J}}.$$

By Proposition 4.4, we have

$$f_{I^l,J^l} = (-1)^n d_{I^l,J^l} = (-1)^n (-1)^{|I^l|} X^{l,l}_{\mathbf{I},\mathbf{I}} X^{l,l}_{\mathbf{J},\mathbf{J}},$$

if $J^l = (-1)^n \psi(k) I^l$ for some $k, n \in \mathbb{Z}$ and $f_{I^l,J^l} = 0$ otherwise. The result now follows from the definition of $\psi^l(k)$ and the fact that

$$d_{\mathbf{I},\mathbf{J}} = \prod_{\alpha,\beta=1}^{r} (-1)^{|I^\alpha|} X^{\alpha,\beta}_{\mathbf{I},\mathbf{I}} X^{\alpha,\beta}_{\mathbf{J},\mathbf{J}}.$$

□

4.4 Proof of the theorem

Proof of Theorem 3.6 Let $n \in \mathbb{Z}$,

$$\mathbf{I} \in \mathcal{M}_{\mathbf{c}}(r,k)^T \quad \text{and} \quad \mathbf{J} \in \mathcal{M}_{\mathbf{c}+1_l}(r, k+n-c^l-1)^T.$$

By Lemma 2.6,

$$\langle \Psi^l(n)[\mathbf{I}], [\mathbf{J}]\rangle = \frac{c_{\text{tnv}}\left(\mathcal{K}_{\mathbf{c}(\mathbf{I}),\mathbf{c}(\mathbf{J})}(r, |\mathbf{I}|, |\mathbf{J}|)_{(\mathbf{I},\mathbf{J})}\right)}{e_T(\mathcal{T}_{\mathbf{I}}^-)e_T(\mathcal{T}_{\mathbf{J}}^+)} = \frac{f_{\mathbf{I},\mathbf{J}}}{d_{\mathbf{I},\mathbf{J}}}.$$

The set $\{[\mathbf{I}] \mid \mathbf{c}(\mathbf{I}) = \mathbf{c},\ |\mathbf{I}| = n\}$ forms a $\mathbb{C}(b_1, \ldots, b_r, \epsilon)$ basis of the localized equivariant cohomology of $\mathcal{M}_{\mathbf{c}}(r, n)$. Since the above structure coefficients are complex numbers by Proposition 4.5, we see immediately that the operators $\Psi^l(n)$ (and hence their adjoints $\Psi^l(n)^*$) preserve the subspace $\mathbf{A}$. Let φ be the vector space isomorphism $\mathbf{A} \cong \mathbf{F}$ given by $\varphi([\mathbf{I}]) = \mathbf{I}$. It follows from Proposition 4.5 that $\varphi(\Psi^l(n)[\mathbf{I}]) = \psi^l(n)\varphi([\mathbf{I}])$ for all $n \in \mathbb{Z}$ and $l \in \{1, \ldots, r\}$. Since $\Psi^l(n)^*$ and $\psi^l(n)^*$ are adjoint to $\Psi^l(n)$ and $\psi^l(n)$, respectively, we see that $\varphi(\Psi^l(n)^*[\mathbf{I}]) = \psi^l(n)^*\varphi([\mathbf{I}])$ for all $n \in \mathbb{Z}$ and $l \in \{1, \ldots, r\}$. The result follows. □

5 Proof of Theorem 3.14

5.1 Notation

In order to simplify notation in what follows, for r-colored semi-infinite monomials $\mathbf{I}$ and $\mathbf{J}$ with $\mathbf{c}(\mathbf{I}) = \mathbf{c}(\mathbf{J}) = \mathbf{c}$, let

$$g^l_{\mathbf{I},\mathbf{J}} = \gamma^l_{\mathbf{I},\mathbf{J}} \cup c_{\text{tnv}}\left(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r, |\mathbf{I}|, |\mathbf{J}|)_{(\mathbf{I},\mathbf{J})}\right)$$

in the T-equivariant cohomology of a point. We define $g^l_{\mathbf{I},\mathbf{J}}$ to be zero when $\mathbf{c}(\mathbf{I}) \neq \mathbf{c}(\mathbf{J})$. When non-zero, $g^l_{\mathbf{I},\mathbf{J}}$ is homogeneous of degree $2r(|\mathbf{I}| + |\mathbf{J}|)$.

Note that when $r = 1$, $\lambda(I)_k \neq \lambda(J)_k$ and $\lambda(I)_i = \lambda(J)_i$ for all $1 \leq i < k$, we have

$$\begin{aligned} g_{I,J} &\overset{\text{def}}{=} g^1_{I,J} \\ &= \begin{cases} (-1)^{|I|-1}\epsilon^{|I|+|J|}h_{J,I}\prod_{s\in\lambda(I),\ s\neq(k,\lambda(I)_k)} h_{I,J}(s) & \text{if } \lambda(I)_k > \lambda(J)_k, \\ (-1)^{|I|}\epsilon^{|I|+|J|}h_{I,J}\prod_{s\in\lambda(J),\ s\neq(k,\lambda(J)_k)} h_{J,I}(s) & \text{if } \lambda(J)_k > \lambda(I)_k. \end{cases} \end{aligned}$$

Thus, by Lemma 1.1, $g_{I,J} = 0$ if and only if 0 occurs more than once as a relative hook length for the partitions $\lambda(I)$ and $\lambda(J)$.

Recall that for $\mathbf{I} \in \mathcal{M}_{\mathbf{c}}(r, n_1)^T$, $\mathbf{J} \in \mathcal{M}_{\mathbf{d}}(r, n_2)^T$, we defined

$$d_{\mathbf{I},\mathbf{J}} = e_T(\mathcal{T}_{\mathbf{I}}^-)e_T(\mathcal{T}_{\mathbf{J}}^+) \in \mathbb{C}[b_1, \ldots, b_r, \epsilon],$$

and when $r = 1$, we have

$$d_{I,J} = (-1)^{|I|}h_I h_J \epsilon^{|I|+|J|}.$$

We will write $g_{\lambda,\mu}$ and $d_{\lambda,\mu}$ to denote $g_{I,J}$ and $d_{I,J}$ (respectively) for some semi-infinite monomials I and J with $\lambda(I) = \lambda$ and $\lambda(J) = \mu$. Since $g_{I,J}$ and $d_{I,J}$ are independent of the charge of I and J, $g_{\lambda,\mu}$ and $d_{\lambda,\mu}$ are well-defined.

A 2×2 *square* is a set of the form

$$\{(i, j), (i+1, j), (i, j+1), (i+1, j+1)\}, \quad i, j \in \mathbb{N}_+.$$

For $\lambda, \mu \in \mathcal{P}$, we say $\mu - \lambda$ is a *border strip* if $\lambda \subseteq \mu$ (that is, $\lambda_i \le \mu_i$ for all i) and the following two conditions hold:

1. $\mu - \lambda$ contains no 2×2 square, and
2. $\mu - \lambda$ is not the disjoint union of two non-empty subsets ν^1 and ν^2 such that no box in ν^1 shares an edge with a box in ν^2.

The *width* of a border strip is defined to be the number of columns it occupies minus one. That is, the width of a border strip $\mu - \lambda$ is $|\{i \mid \lambda_i < \mu_i\}| - 1$. Note that this is often referred to as the *height* when Young diagrams are written in English or French notation.

5.2 Combinatorics of the rank one bosonic Fock space

Proposition 5.1 *If* $|\lambda| < |\mu|$, *then*

$$g_{\lambda,\mu} = \begin{cases} \frac{(-1)^{\text{width}(\mu-\lambda)}}{|\mu-\lambda|} d_{\lambda,\mu} & \text{if } \mu - \lambda \text{ is a border strip}, \\ 0 & \text{otherwise}. \end{cases}$$

Proof We first prove that $g_{\lambda,\mu} = 0$ unless $\lambda \subseteq \mu$. Since $|\lambda| < |\mu|$, we must have $\mu_k > \lambda_k$ for some positive integer k. Assume k is the smallest such positive integer. If $\lambda \not\subseteq \mu$, we must have $\lambda_j > \mu_j$ for some j. Again, we take the smallest such j. Then $h_{\mu,\lambda}((k, \mu_k)) = 0$ and $h_{\lambda,\mu}((j, \lambda_j)) = 0$ and thus $g_{\lambda,\mu} = 0$.

Now suppose $\lambda \subseteq \mu$. Let k be the smallest positive integer such that $\mu_k > \lambda_k$ and let $s^* = (k, \mu_k)$. Suppose $\mu - \lambda$ contains a 2×2 square. Pick one such 2×2 square

$$\{(i, j), (i+1, j), (i, j+1), (i+1, j+1)\}$$

in $\mu - \lambda$ with the following properties:

1. $(i+1, j+2) \notin \mu$,
2. $(i-1, j) \notin \mu - \lambda$.

If property (1) is not satisfied, then

$$\{(i, j+1), (i+1, j+1), (i, j+2), (i+1, j+2)\}$$

is a 2×2 square in $\mu - \lambda$ and if property (2) is not satisfied, then

$$\{(i-1, j), (i, j), (i-1, j+1), (i, j+1)\}$$

is a 2×2 square in $\mu - \lambda$. Thus, we can always choose a 2×2 square in $\mu - \lambda$ satisfying properties (1) and (2) by moving up and left as necessary. For such a square, $(i+1, j) \neq s^*$ but

$$h_{\mu,\lambda}((i+1, j)) = a_\mu((i+1, j)) + l_\lambda((i+1, j)) + 1 = 1 + (-2) + 1 = 0,$$

and so $g_{\lambda,\mu} = 0$.

Now suppose $\mu - \lambda$ contains no 2×2 square. It is a union of subsets $\nu^1, \ldots, \nu^p$ such that ν^i and ν^j have no edges in common for $i \neq j$. By relabeling if necessary, we may assume that all boxes in ν^i are to the left of all boxes of ν^j for $i < j$. By definition s^* is the top left box of ν^1. If $p \geq 2$, let (i, j) be the top left box of ν^2. Then $(i-1, j), (i, j+1) \notin \nu^2$. We also have $(i-1, j), (i, j+1) \notin \nu^i$ for $i \neq 2$ since ν^2 shares no edges with ν^i for $i \neq 2$. Thus

$$h_{\mu,\lambda}((i, j)) = a_\mu((i, j)) + l_\lambda((i, j)) + 1 = 0 + (-1) + 1 = 0,$$

and so $g_{\lambda,\mu} = 0$.

It remains to consider the case when $\mu - \lambda$ is a border strip. We divide the boxes of the partition μ into subsets as follows. Recall that k is the smallest positive integer such that $\mu_k > \lambda_k$. Let m be the largest positive integer such that $\mu_m > \lambda_m$, let l be the smallest positive integer such that $\mu_l^t > \lambda_l^t$, and let n be the largest positive integer such that $\mu_n^t > \lambda_n^t$. Then define (see Fig. 1)

$$\begin{aligned}
A &= \{(i, j) \in \lambda \mid (i < k \text{ or } i > m) \text{ and } (j < l \text{ or } j > n)\},\\
B &= \{(i, j) \in \lambda \mid i < k,\ l \leq j \leq n\},\\
C &= \{(i, j) \in \lambda \mid k \leq i \leq m,\ j < l\}, \text{ and,}\\
D &= \{(i, j) \in \mu \mid i \geq k,\ j \geq l\}.
\end{aligned}$$

We have $\mu = A \sqcup B \sqcup C \sqcup D$ and

$$\begin{aligned}
h_{\lambda,\mu}(s) &= h_{\mu,\lambda}(s) = h_\lambda(s) = h_\mu(s), \quad \text{for } s \in A,\\
h_{\lambda,\mu}(s) &= h_\mu(s), \quad h_{\mu,\lambda}(s) = h_\lambda(s), \quad \text{for } s \in B,\\
h_{\lambda,\mu}(s) &= h_\lambda(s), \quad h_{\mu,\lambda}(s) = h_\mu(s), \quad \text{for } s \in C.
\end{aligned}$$

We now focus our attention on the boxes in D. These boxes form a partition themselves. More precisely, if we define

$$\begin{aligned}
\tilde{\lambda}_i &= \lambda_{i+k-1} - l + 1, \quad 1 \leq i \leq m + 1 - k,\\
\tilde{\mu}_i &= \mu_{i+k-1} - l + 1, \quad 1 \leq i \leq m + 1 - k,
\end{aligned}$$

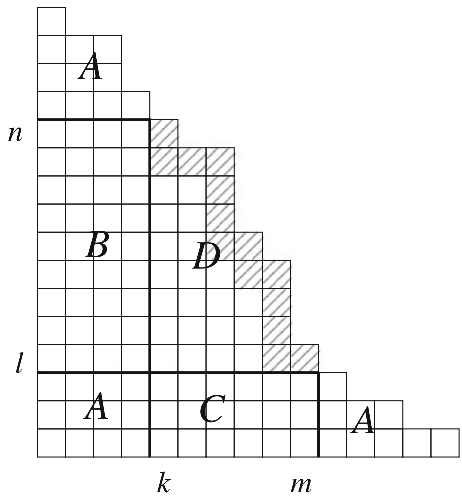

Fig. 1 *Unshaded boxes* correspond to the partition λ. *Shaded boxes* correspond to the border strip $\mu - \lambda$

then $\tilde{\lambda}$ and $\tilde{\mu}$ are partitions and we have

$$\begin{aligned}
(i, j) \in \tilde{\mu} &\Leftrightarrow (i + k - 1, j + l - 1) \in \mu, \\
(i, j) \in \tilde{\lambda} &\Leftrightarrow (i + k - 1, j + l - 1) \in \lambda, \\
h_{\tilde{\lambda},\tilde{\mu}}((i, j)) &= h_{\lambda,\mu}((i + k - 1, j + l - 1)), \\
h_{\tilde{\mu},\tilde{\lambda}}((i, j)) &= h_{\mu,\lambda}((i + k - 1, j + l - 1)), \\
h_{\tilde{\lambda}}((i, j)) &= h_{\lambda}((i + k - 1, j + l - 1)), \\
h_{\tilde{\mu}}((i, j)) &= h_{\mu}((i + k - 1, j + l - 1)).
\end{aligned}$$

Since $\mu - \lambda$ is a border strip, we have

$$\begin{aligned}
\lambda_i &= \mu_{i+1} - 1, \quad k \leq i \leq m - 1, \\
\lambda^t_j &= \mu^t_{j+1} - 1, \quad l \leq j \leq n - 1.
\end{aligned}$$

Let $\tilde{s}^* = (1, \tilde{\mu}_1) = (1, \mu_k - l + 1)$. Let $(i, j) \in \tilde{\lambda}$. Since $\tilde{\mu}^t_j > \tilde{\lambda}^t_j$, we have $(i + 1, j) \in \tilde{\mu}$. Also

$$\begin{aligned}
a_{\tilde{\mu}}((i + 1, j)) &= a_{\tilde{\lambda}}((i, j)) + 1, \\
l_{\tilde{\lambda}}((i + 1, j)) &= l_{\tilde{\lambda}}((i, j)) - 1, \\
l_{\tilde{\mu}}((i + 1, j)) &= l_{\tilde{\mu}}((i, j)) - 1.
\end{aligned}$$

Thus

$$\begin{aligned}
h_{\tilde{\mu},\tilde{\lambda}}((i + 1, j)) &= a_{\tilde{\mu}}((i + 1, j)) + l_{\tilde{\lambda}}((i + 1, j)) + 1 \\
&= a_{\tilde{\lambda}}((i, j)) + l_{\tilde{\lambda}}((i, j)) + 1 = h_{\tilde{\lambda}}((i, j)),
\end{aligned}$$

and

$$h_{\tilde{\lambda},\tilde{\mu}}((i,j)) = a_{\tilde{\lambda}}((i,j)) + l_{\tilde{\mu}}((i,j)) + 1$$
$$= a_{\tilde{\mu}}((i+1,j)) + l_{\tilde{\mu}}((i+1,j)) + 1 = h_{\tilde{\mu}}((i+1,j)).$$

Therefore,

$$\prod_{(i,j)\in\tilde{\lambda}} h_{\tilde{\mu},\tilde{\lambda}}((i+1,j)) = \prod_{(i,j)\in\tilde{\lambda}} h_{\tilde{\lambda}}((i,j)) = h_{\tilde{\lambda}},$$
$$\prod_{(i,j)\in\tilde{\lambda}} h_{\tilde{\lambda},\tilde{\mu}}((i,j)) = \prod_{(i,j)\in\tilde{\lambda}} h_{\tilde{\mu}}((i+1,j)).$$

Now, for $(1,j)\in\tilde{\mu}$ with $1 \le j < \tilde{\mu}_1$ (the second inequality is equivalent to $(1,j) \neq \tilde{s}^*$), we have

$$a_{\tilde{\mu}}((1,j)) = a_{\tilde{\mu}}((1,j+1)) + 1,$$
$$l_{\tilde{\lambda}}((1,j)) = l_{\tilde{\mu}}((1,j+1)) - 1.$$

Thus

$$h_{\tilde{\mu},\tilde{\lambda}}((1,j)) = a_{\tilde{\mu}}((1,j)) + l_{\tilde{\lambda}}((1,j)) + 1$$
$$= a_{\tilde{\mu}}((1,j+1)) + l_{\tilde{\mu}}((1,j+1)) + 1 = h_{\tilde{\mu}}((1,j+1)),$$

and so

$$\prod_{j\,:\,1\le j<\tilde{\mu}_1} h_{\tilde{\mu},\tilde{\lambda}}((1,j)) = \prod_{j\,:\,2\le j\le\tilde{\mu}_1} h_{\tilde{\mu}}((1,j)).$$

For $1 \le j \le \tilde{\mu}_1$,

$$\prod_{i\,:\,\tilde{\lambda}^t_i+1<i\le\tilde{\mu}^t_i} h_{\tilde{\mu},\tilde{\lambda}}((i,j)) = (-1)(-2)\cdots(-(\tilde{\mu}^t_j - \tilde{\lambda}^t_j - 1))$$
$$= (-1)^{\tilde{\mu}^t{}_j - \tilde{\lambda}^t{}_j - 1}(\tilde{\mu}^t_j - \tilde{\lambda}^t_j - 1)!$$

and

$$\prod_{i\,:\,\tilde{\lambda}^t_i+1<i\le\tilde{\mu}^t_i} h_{\tilde{\mu}}((i,j)) = (\tilde{\mu}^t_j - \tilde{\lambda}^t_j - 1)(\tilde{\mu}^t_j - \tilde{\lambda}^t_j - 2)\cdots(2)(1) = (\tilde{\mu}^t_j - \tilde{\lambda}^t_j - 1)!.$$

Note that

$$\prod_{j=1}^{\tilde{\mu}_1}(-1)^{\tilde{\mu}^t{}_j - \tilde{\lambda}^t{}_j - 1} = (-1)^{\tilde{\mu}^t_{\tilde{\mu}_1} - 1}\prod_{j=1}^{\tilde{\mu}_1 - 1}(-1)^{\tilde{\mu}^t{}_j - \tilde{\mu}^t{}_{j+1}} = (-1)^{\tilde{\mu}^t_1 - 1} = (-1)^{\mathrm{width}(\mu-\lambda)},$$

where in the first equality we used the fact that $\tilde{\lambda}^t_j = \tilde{\mu}^t_{j+1} - 1$ for $1 \le j \le \tilde{\mu}_1 - 1$ and $\tilde{\lambda}^t_{\tilde{\mu}_1} = 0$. Combining the above results, we have

$$\prod_{s\in\tilde{\lambda}} h_{\tilde{\lambda},\tilde{\mu}}(s) \prod_{s\in\tilde{\mu},\ s\neq\tilde{s}^*} h_{\tilde{\mu},\tilde{\lambda}}(s) = (-1)^{\mathrm{width}(\mu-\lambda)} \prod_{s\in\tilde{\lambda}} h_{\tilde{\lambda}}(s) \prod_{s\in\tilde{\mu},\ s\neq(1,1)} h_{\tilde{\mu}}(s).$$

Note that

$$h_{\tilde{\mu}}((1,1)) = |\mu-\lambda|.$$

Thus

$$\begin{aligned}\prod_{s\in D\cap\lambda} h_{\lambda,\mu}(s) \prod_{s\in D, s\neq s^*} h_{\mu,\lambda}(s) &= \prod_{s\in\tilde{\lambda}} h_{\tilde{\lambda},\tilde{\mu}}(s) \prod_{s\in\tilde{\mu},\ s\neq\tilde{s}^*} h_{\tilde{\mu},\tilde{\lambda}}(s) \\ &= (-1)^{\mathrm{width}(\mu-\lambda)} \prod_{s\in\tilde{\lambda}} h_{\tilde{\lambda}}(s) \prod_{s\in\tilde{\mu},\ s\neq(1,1)} h_{\tilde{\mu}}(s) \\ &= (-1)^{\mathrm{width}(\mu-\lambda)} \frac{1}{|\mu-\lambda|} \prod_{s\in D\cap\lambda} h_\lambda(s) \prod_{s\in D} h_\mu(s).\end{aligned}$$

Therefore,

$$\begin{aligned} g_{\lambda,\mu} &= (-1)^{|\lambda|}\epsilon^{|\lambda|+|\mu|} \prod_{s\in A\cup B\cup C} h_{\lambda,\mu}(s)h_{\mu,\lambda}(s) \prod_{s\in D\cap\lambda} h_{\lambda,\mu}(s) \prod_{s\in D,\ s\neq s^*} h_{\mu,\lambda}(s) \\ &= (-1)^{|\lambda|}\frac{(-1)^{\mathrm{width}(\mu-\lambda)}}{|\mu-\lambda|}\epsilon^{|\lambda|+|\mu|} \prod_{s\in A\cup B\cup C} h_\lambda(s)h_\mu(s) \prod_{s\in D\cap\lambda} h_\lambda(s) \prod_{s\in D} h_\mu(s) \\ &= (-1)^{|\lambda|}\frac{(-1)^{\mathrm{width}(\mu-\lambda)}}{|\mu-\lambda|} h_\lambda h_\mu \epsilon^{|\lambda|+|\mu|} \\ &= \frac{(-1)^{\mathrm{width}(\mu-\lambda)}}{|\mu-\lambda|} d_{\lambda,\mu} \end{aligned}$$

as desired. □

Let $p(-n), n \in \mathbb{N}_+$, be the operator on symmetric functions given by multiplication by $\frac{1}{n} p_n$, where p_n is the power-sum symmetric function.

Lemma 5.2 *With respect to the basis of Schur functions, the structure constants of the operator* $p(-n)$ *are given by*

$$\langle p(-n)s_\lambda, s_\mu\rangle = \begin{cases} \frac{g_{\lambda,\mu}}{d_{\lambda,\mu}} & \text{if } |\mu-\lambda| = n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof By [16, Example I.3.11],

$$p_n s_\lambda = \sum_{\mu} (-1)^{\mathrm{width}(\mu-\lambda)} s_\mu,$$

where the sum is over all partitions $\mu \supset \lambda$ such that $\mu - \lambda$ is a border strip of size n. Therefore, since the Schur functions form an orthonormal basis for the symmetric functions,

$$\begin{aligned}
\left\langle p(-n)s_\lambda, s_\mu \right\rangle &= \left\langle \frac{1}{n} p_n s_\lambda, s_\mu \right\rangle \\
&= \begin{cases} \frac{1}{n}(-1)^{\mathrm{width}(\mu-\lambda)} & \text{if } \mu-\lambda \text{ is a border strip of size } n, \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{g_{\lambda,\mu}}{d_{\lambda,\mu}} & \text{if } |\mu| - |\lambda| = n, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

□

5.3 Combinatorics of the r-colored bosonic Fock space

Proposition 5.3 *If* $\mathbf{I}$ *and* $\mathbf{J}$ *are semi-infinite monomials with* $\mathbf{c}(\mathbf{I}) = \mathbf{c}(\mathbf{J})$ *and* $|\lambda(I^l)| < |\lambda(J^l)|$*, then*

$$g^l_{\mathbf{I},\mathbf{J}} = \frac{(-1)^{\mathrm{width}\left(\lambda(J^l)-\lambda(I^l)\right)}}{\left|\lambda\left(J^l\right) - \lambda\left(I^l\right)\right|} d_{\mathbf{I},\mathbf{J}}$$

if $\lambda(I^\alpha) = \lambda(J^\alpha)$ *for all* $\alpha \neq l$ *and* $\lambda(J^l) - \lambda(I^l)$ *is a border strip. Otherwise* $g^l_{\mathbf{I},\mathbf{J}} = 0$.

Proof Note that $g^l_{\mathbf{I},\mathbf{J}} = 0$ unless $|I^\alpha| = |J^\alpha|$ for $\alpha < l$ and $|I^l| \neq |J^l|$ (due to the presence of the factor $\gamma^l_{\mathbf{I},\mathbf{J}}$) in which case

$$g^l_{\mathbf{I},\mathbf{J}} = g_{I^l,J^l} \prod_{\substack{1\le\alpha,\beta\le r \\ \alpha\neq l \text{ or } \beta\neq l}} (-1)^{|I^\alpha|} X^{\alpha,\beta}_{\mathbf{I},\mathbf{J}} X^{\alpha,\beta}_{\mathbf{J},\mathbf{I}}, \tag{5.1}$$

where $X^{\alpha,\beta}_{\mathbf{I},\mathbf{J}}$ is defined in Sect. 4.3. As in the proof of Proposition 4.5, for all $\alpha \neq l$ we have $X^{\alpha,\alpha}_{\mathbf{I},\mathbf{J}} = 0$ unless $I^\alpha = J^\alpha$. If $I^\alpha = J^\alpha$ for all $\alpha \neq l$, then

$$g^l_{\mathbf{I},\mathbf{J}} = g_{I^l,J^l} \prod_{\substack{1\le\alpha,\beta\le r \\ \alpha\neq l \text{ or } \beta\neq l}} (-1)^{|I^\alpha|} X^{\alpha,\beta}_{\mathbf{I},\mathbf{I}} \prod_{\substack{1\le\alpha,\beta\le r \\ \alpha\neq l \text{ or } \beta\neq l}} X^{\alpha,\beta}_{\mathbf{J},\mathbf{J}}.$$

By Proposition 5.1,

$$g_{I^l,J^l} = \frac{(-1)^{\mathrm{width}(\lambda(J^l)-\lambda(I^l))}}{|\lambda(J^l)-\lambda(I^l)|} d_{I^l,J^l} = \frac{(-1)^{\mathrm{width}(\lambda(J^l)-\lambda(I^l))}}{|\lambda(J^l)-\lambda(I^l)|} (-1)^{|I^l|} X^{l,l}_{\mathbf{I},\mathbf{I}} X^{l,l}_{\mathbf{J},\mathbf{J}}$$

if $\lambda(J^l)-\lambda(I^l)$ is a border strip and is equal to zero otherwise. Thus, when $\lambda(J^l)-\lambda(I^l)$ is a border strip, we have

$$g^l_{\mathbf{I},\mathbf{J}} = \frac{(-1)^{\mathrm{width}(\lambda(J^l)-\lambda(I^l))}}{|\lambda(J^l)-\lambda(I^l)|} \prod_{\alpha,\beta=1}^{r} (-1)^{|I^\alpha|} X^{\alpha,\beta}_{\mathbf{I},\mathbf{I}} X^{\alpha,\beta}_{\mathbf{J},\mathbf{J}}.$$

The result follows. □

Corollary 5.4 *For $n \in \mathbb{N}_+$, we have*

$$\left\langle p^l(-n)\left(q^{c(I^1)}s_{\lambda(I^1)},\dots,q^{c(I^r)}s_{\lambda(I^r)}\right), \left(q^{c(J^1)}s_{\lambda(J^1)},\dots,q^{c(J^r)}s_{\lambda(J^r)}\right)\right\rangle$$
$$= \begin{cases} \frac{g^l_{\mathbf{I},\mathbf{J}}}{d_{\mathbf{I},\mathbf{J}}} & \text{if } \mathbf{c}(\mathbf{I}) = \mathbf{c}(\mathbf{J}),\ |\lambda(J^l)| - |\lambda(I^l)| = n, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

Proof It follows easily from the definition of the bilinear form and the operators $p^l(-n)$ that the left side of (5.2) is equal to zero unless $\mathbf{c}(\mathbf{I}) = \mathbf{c}(\mathbf{J})$ and $I^\alpha = J^\alpha$ for all $\alpha \neq l$. In this case, by Propositions 5.1 and 5.3,

$$\frac{g^l_{\mathbf{I},\mathbf{J}}}{d_{\mathbf{I},\mathbf{J}}} = \frac{g_{I^l,J^l}}{d_{I^l,J^l}}$$

and the result follows from Lemma 5.2. □

5.4 Proof of the theorem

Proof of Theorem 3.14 Let $n \in \mathbb{N}_+$, $\mathbf{I} \in \mathcal{M}_{\mathbf{c}}(r,k)^T$ and $\mathbf{J} \in \mathcal{M}_{\mathbf{c}}(r,k+n)^T$. By Lemma 2.6,

$$\langle P^l(-n)[\mathbf{I}],[\mathbf{J}]\rangle = \frac{\gamma^l_{\mathbf{I},\mathbf{J}} \cup c_{\mathrm{tnv}}\left(\mathcal{K}_{\mathbf{c},\mathbf{c}}(r,|\mathbf{I}|,|\mathbf{J}|)_{(\mathbf{I},\mathbf{J})}\right)}{e_T(\mathcal{T}^-_{\mathbf{I}})e_T(\mathcal{T}^+_{\mathbf{J}})} = \frac{g^l_{\mathbf{I},\mathbf{J}}}{d_{\mathbf{I},\mathbf{J}}}.$$

As in Sect. 4.4, we see that the operators $P^l(-n)$ (and hence their adjoints $P^l(n)$) preserve the subspace $\mathbf{A}$. Let φ be the vector space isomorphism $\mathbf{A} \cong \mathbf{B}$ given by $\varphi([\mathbf{I}]) = (q^{c^1}s_{\lambda(I^1)},\dots,q^{c^r}s_{\lambda(I^r)})$. It follows from Corollary 5.4 that $\varphi(P^l(-n)[\mathbf{I}]) = p^l(-n)\varphi([\mathbf{I}])$. Since $P^l(n)$ and $p^l(n)$ are adjoint to $P^l(-n)$ and $p^l(-n)$, respectively, we see that $\varphi(P^l(n)[\mathbf{I}]) = p^l(n)\varphi([\mathbf{I}])$ for all $n \in \mathbb{Z}$ (the case $n = 0$ can be seen directly). The result follows. □

6 Vertex operators and geometry

6.1 A new geometric realization of the boson-fermion correspondence

From Theorems 3.6 and 3.14, we see that we have defined actions of the r-colored Heisenberg and Clifford algebras on $\mathbf{A} \subset \bigoplus_{\mathbf{c},n} \mathcal{H}_T^{2rn}(\mathcal{M}_{\mathbf{c}}(r,n))$. In so doing, we obtain a natural geometrically defined isomorphism between bosonic and fermionic Fock spaces. Under this isomorphism, the semi-infinite monomial $\mathbf{I}$ corresponds to the element $(q^{c(I^1)} s_{\lambda(I^1)}, \dots, q^{c(I^r)} s_{\lambda(I^r)})$ of bosonic Fock space since both correspond to the element $[\mathbf{I}] \in A_{\mathbf{c}}(r, |\mathbf{I}|)$. This matches up precisely with the classical (algebraic) boson-fermion correspondence.

The complexes defined in this paper also yield geometric analogs of the vertex operators appearing in the boson-fermion correspondence as follows. Let

$$\mathbf{H} = \prod_{\mathbf{c},\mathbf{d},n_1,n_2} \mathcal{H}_T^{2r(n_1+n_2)}(\mathcal{M}_{\mathbf{c}}(r,n_1) \times \mathcal{M}_{\mathbf{d}}(r,n_2)).$$

We can think of elements of $\mathbf{H}$ as formal (infinite) linear combinations of elements of $\mathcal{H}_T^{2r(n_1+n_2)}(\mathcal{M}_{\mathbf{c}}(r,n_1) \times \mathcal{M}_{\mathbf{d}}(r,n_2))$ for $\mathbf{c}, \mathbf{d} \in \mathbb{Z}^r$, $n_1, n_2 \in \mathbb{N}$. For $l \in \{1, \dots, r\}$, let N^l be the operator on $\prod_{\mathbf{c},\mathbf{d},n_1,n_2} \mathcal{H}_T^*(\mathcal{M}_{\mathbf{c}}(r,n_1) \times \mathcal{M}_{\mathbf{d}}(r,n_2))$ that acts on $\mathcal{H}_T^*(\mathcal{M}_{\mathbf{c}}(r,n_1) \times \mathcal{M}_{\mathbf{c}}(r,n_2))$ as $\gamma^l(n_2 - n_1)\,\mathrm{id}$ for $n_1 \neq n_2$ and as $c^l\,\mathrm{id}$ for $n_1 = n_2$. It acts on $\mathcal{H}_T^*(\mathcal{M}_{\mathbf{c}}(r,n_1) \times \mathcal{M}_{\mathbf{d}}(r,n_2))$, $\mathbf{c} \neq \mathbf{d}$, as the identity (its action on these pieces is actually irrelevant).

Now, the complex (3.1) can be considered as a complex over

$$\mathcal{M} \stackrel{\text{def}}{=} \bigsqcup_{\mathbf{c},\mathbf{d},n_1,n_2} \mathcal{M}_{\mathbf{c}}(r,n_1) \times \mathcal{M}_{\mathbf{d}}(r,n_2).$$

Let $\mathcal{K}$ denote the vector bundle $\ker \tau / \operatorname{im} \sigma$ on $\mathcal{M}$ and let $\mathcal{B}$, $\mathcal{B}_-$, and $\mathcal{B}_+$ be its restrictions to

$$\bigsqcup_{\mathbf{c},n_1,n_2} \mathcal{M}_{\mathbf{c}}(r,n_1) \times \mathcal{M}_{\mathbf{c}}(r,n_2),$$

$$\bigsqcup_{\mathbf{c},n_1,n_2 : n_1 < n_2} \mathcal{M}_{\mathbf{c}}(r,n_1) \times \mathcal{M}_{\mathbf{c}}(r,n_2), \quad \text{and}$$

$$\bigsqcup_{\mathbf{c},n_1,n_2 : n_1 > n_2} \mathcal{M}_{\mathbf{c}}(r,n_1) \times \mathcal{M}_{\mathbf{c}}(r,n_2),$$

respectively. Then

$$N^l c_{\mathrm{tnv}}(\mathcal{B}) = P^l(0) + \sum_{n \in \mathbb{Z} \setminus \{0\}} |n| P^l(n), \tag{6.1}$$

$$\gamma^l c_{\mathrm{tnv}}(\mathcal{B}_-) = \sum_{n>0} P^l(-n), \qquad -\gamma^l c_{\mathrm{tnv}}(\mathcal{B}_+) = \sum_{n>0} P^l(n) \tag{6.2}$$

are elements of $\mathbf{H}$ and are geometric versions of the usual bosonic vertex operators. Note that while vertex operators usually involve a formal variable, this can always be recovered by degree considerations. Also, when comparing to the presentations in [8,23], one must remember that the variables x_i used in the bosonic Fock space there correspond to p_i/i.

Now, let $\mathcal{F}^l$ and $\mathcal{F}^{l,*}$ denote the restriction of $\mathcal{K}$ to

$$\bigsqcup_{\mathbf{c},n_1,n_2} \mathcal{M}_{\mathbf{c}}(r,n_1) \times \mathcal{M}_{\mathbf{c}+1_l}(r,n_2) \quad \text{and} \quad \bigsqcup_{\mathbf{c},n_1,n_2} \mathcal{M}_{\mathbf{c}}(r,n_1) \times \mathcal{M}_{\mathbf{c}-1_l}(r,n_2),$$

respectively. Then

$$c_{\text{tnv}}(\mathcal{F}^l) = \sum_{n\in\mathbb{Z}} \Psi^l(n) \quad \text{and} \quad c_{\text{tnv}}(\mathcal{F}^{l,*}) = \sum_{n\in\mathbb{Z}} \Psi^l(n)^* \tag{6.3}$$

are elements of $\mathbf{H}$ and are geometric versions of the usual fermionic vertex operators.

We introduce the usual *normal ordering operator*

$$:\Psi^l(k)\Psi^l(j)^* : \stackrel{\text{def}}{=} \begin{cases} \Psi^l(k)\Psi^l(j)^* & \text{if } j > 0, \\ -\Psi^l(j)^*\Psi^l(k) & \text{if } j \le 0. \end{cases}$$

Also, for $l \in \{1, \dots, r\}$, define an operator $Q_l : \mathbf{A} \to \mathbf{A}$ by

$$Q_l : A_{\mathbf{c}}(r,n) \to A_{\mathbf{c}+1_l}(r,n), \quad Q_l([\mathbf{I}]) = [\mathbf{J}],$$

where $\mathbf{J}$ is the semi-infinite monomial of charge $\mathbf{c} + 1_l = \mathbf{c}(\mathbf{I}) + 1_l$ with $\boldsymbol{\lambda}(\mathbf{J}) = \boldsymbol{\lambda}(\mathbf{I})$.

Proposition 6.1 *We have*

$$N^l c_{\text{tnv}}(\mathcal{B}) = \; : c_{\text{tnv}}(\mathcal{F}^l) c_{\text{tnv}}(\mathcal{F}^{l,*}), \tag{6.4}$$

$$c_{\text{tnv}}(\mathcal{F}^l) = Q_l \exp(\gamma^l c_{\text{tnv}}(\mathcal{B}_-)) \exp(\gamma^l c_{\text{tnv}}(\mathcal{B}_+)), \quad \textit{and} \tag{6.5}$$

$$c_{\text{tnv}}(\mathcal{F}^{l,*}) = Q_l^{-1} \exp(-\gamma^l c_{\text{tnv}}(\mathcal{B}_-)) \exp(-\gamma^l c_{\text{tnv}}(\mathcal{B}_+)) \tag{6.6}$$

as formal operators on $\mathbf{A}$. *More precisely, if we decompose into blocks according to the decomposition* $\mathbf{A} = \bigoplus_{\mathbf{c},n} A_{\mathbf{c}}(r,n)$*, the above equalities signify equality of blocks* $A_{\mathbf{c}}(r,n_1) \to A_{\mathbf{d}}(r,n_2)$ *for all* $\mathbf{c}, \mathbf{d} \in \mathbb{Z}^r$ *and* $n_1, n_2 \in \mathbb{N}$.

Proof This follows from the (algebraic) boson-fermion correspondence. See, for example, [23, Theorem 1.2] and note that our generators $p^l(m)$ of the oscillator algebra are equal to $\frac{1}{|m|}\alpha_l(m)$ for $1 \le l \le r$, $m \in \mathbb{Z}\backslash\{0\}$, where the $\alpha_l(m)$ are the generators used in [23]. One could also produce a geometric proof of this result in the language of the current paper that mimics the algebraic proof. □

Remark 6.2 We note the operators $N^l c_{\text{tnv}}(\mathcal{B})$, $c_{\text{tnv}}(\mathcal{F}^l)$, and $c_{\text{tnv}}(\mathcal{F}^{l,*})$ (more precisely, their homogeneous components) preserve the integral form $\mathbf{A}_{\mathbb{Z}} = \text{Span}_{\mathbb{Z}}\{[\mathbf{I}]\}$ of the Fock space $\mathbf{A}$.

We refer to Eq. (6.4) as *geometric bosonization* and to Eqs. (6.5) and (6.6) as *geometric fermionization*. Note that the algebraic boson-fermion correspondence involves the expression $z^{p^l(0)}$ (or $z^{\alpha_l(0)}$) where z is the formal variable appearing in the vertex operators. However, such factors are unnecessary in the geometric formulation. Their role is played by the relative grading shift inherent in the definitions of our geometric operators. More precisely, it arises from the fact that

$$P^l(n)|_{A_{\mathbf{c}}(r,k)} \in \mathcal{H}_T^{2r(2k-n)}(\mathcal{M}_{\mathbf{c}}(r,k) \times \mathcal{M}_{\mathbf{c}}(r,k-n))$$

while

$$\Psi^l(n)|_{A_{\mathbf{c}}(r,k)} \in H_T^{2r(2k+n-c^l-1)}(\mathcal{M}_{\mathbf{c}}(r,k) \times \mathcal{M}_{\mathbf{c}+1_l}(r,k+n-c^l-1)), \quad \text{and}$$
$$\Psi^l(n)^*|_{A_{\mathbf{c}}(r,k)} \in H_T^{2r(2k-n+c^l)}(\mathcal{M}_{\mathbf{c}}(r,k) \times \mathcal{M}_{\mathbf{c}-1_l}(r,k-n+c^l)).$$

6.2 Additional vertex operator constructions and future directions

Vertex operators entered the mathematical literature as a method of providing explicit constructions of integrable modules for affine Lie algebras. In fact, the same integrable representation usually has several different vertex operator realizations, all of which should acquire some geometric interpretation using moduli spaces of sheaves on surfaces. For the basic level one representation of an affine Lie algebra $\widehat{\mathfrak{g}}$, vertex operator constructions are parameterized by conjugacy classes of Heisenberg subalgebras $\widehat{\mathfrak{h}} \subset \widehat{\mathfrak{g}}$, which correspond bijectively to conjugacy classes in the finite Weyl group. In the case of the affine Lie algebra $\widehat{\mathfrak{gl}}_r$, this means that there is one vertex operator construction of the basic representation for each conjugacy class in the symmetric group S_r, and all of these vertex operators have been worked out explicitly in [23] (see also [9,11]). For the conjugacy class of the identity element, the corresponding vertex operator construction of the basic representation is known as the homogeneous realization [6,22]. This construction acquires a geometric interpretation in the context of the current paper using an embedding of $\widehat{\mathfrak{gl}}_r$ into a completion of the r-colored Clifford algebra. We expect that all of the vertex operators of the homogeneous realization have a natural geometric interpretation using the moduli spaces $\mathcal{M}_{\mathbf{c}}(r,n)$ and complexes involving tautological bundles. Indeed, the r-colored Heisenberg algebra action constructed in this paper defines the action of the homogeneous Heisenberg subalgebra $\widehat{\mathfrak{h}}_{\text{hom}} \subset \widehat{\mathfrak{gl}}_r$.

It would be interesting to give geometric realizations of other vertex operator constructions of the basic representation of $\widehat{\mathfrak{gl}}_r$, using cyclic group fixed point components of moduli spaces of framed sheaves on $\mathbb{P}^2$. For example, the vertex operators of the principal realization corresponding to the conjugacy class of the Coxeter element should acquire a geometric interpretation using $\mathbb{Z}_r$-fixed point components of the moduli space of framed rank 1 sheaves on $\mathbb{P}^2$, which are Nakajima quiver varieties of type $\widehat{A}_{r-1}$. In this context, Nakajima's original Hecke operators will be identified with homogeneous components of vertex operators in the principal realization. Geometric constructions of other realizations of the basic representation would yield, as a corollary, a geometric interpretation of the vertex operators needed for level-rank duality. We hope to say more about this in a future paper.

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