Multistability and Delayed Recurrent Loops

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Multistable dynamical systems have important applications as pattern recognition and memory storage devices. Conditions under which time-delayed recurrent loops of spiking neurons exhibit multistability are presented. Our results are illustrated on both a simple integrate-and-fire neuron and a Hodgkin-Huxley-type neuron, whose recurrent inputs are delayed versions of their output spike trains. Two kinds of multistability with respect to initial spiking functions are found, depending on whether the neuron is excitable or repetitively firing in the absence of feedback.

Multistability in a dynamical system is the coexistence of multiple attractors. In such a system, qualitative changes in dynamics can result from changes in the initial conditions. A well studied case is the bistability associated with a subcritical Hopf bifurcation [1]. Multistable modes of oscillation can arise in delayed feedback systems when the delay is larger than the response time of the system [2]. Multistability of this type has been demonstrated in experiments involving electronic circuits [3] and laser optical devices [4].

Multistability is a mechanism for memory storage and temporal pattern recognition in both artificial [5] and living neural [6] networks. In a living nervous system, recurrent loops involving two or more neurons are ubiquitous and are particularly prevalent in cortical regions important for memory, e.g., the hippocampal-mesial temporal lobe complex [7]. Here we show that time-delayed recurrent loops have a potentially large capacity to encode information in the form of temporally patterned spike trains. The possibility that some forms of memory in the living nervous system may be encoded into the temporal patterning of neural spike trains has been well recognized [8]. The mechanism we propose can be readily realized in simple to construct electronic circuits and thus may find applications in the design of artificial computing and encoding devices.

We consider the dynamics of a recurrent inhibitory loop composed of two neurons: an excitatory neuron E gives off a collateral branch which excites an inhibitory interneuron I which in turns inhibits the firing of E. The inhibitory influence of I on E depends on the activity of E at a time τ in the past. This time delay represents the sum of the conduction time along the axon and dendrites, the time required for quantal release, processing times in the interneuron(s), and the rise time of the inhibitory potential. Thus, mathematical models of recurrent loops are most naturally formulated in terms of delay differential equations (DDE) [9,10], e.g.,

$$C\dot{V}(t) = -I_{\text{ion}}(V, W_1, W_2, \ldots, W_n) + I_0 + F(V(t-\tau)),$$

$$\dot{W}_i(t) = \beta \frac{[\hat{W}_i(V) - W_i]}{\Gamma(V)}, \tag{1}$$

where $V(t)$, $V(t-\tau)$ is the membrane potential at, respectively, times τ and $t-\tau$, Γ is a time constant, C is the membrane capacitance, I_{ion} is the sum of V- and τ-dependent currents through the various ionic channel types, I_0 is the applied current, W_i describe the fractions of channels of a given type that are in various conducting states (e.g., open versus closed), $\hat{W}_i(V)$ describe the equilibrium functions, and β is a temperaturelike time scale factor. The function F describes the effects of the inhibitory neuron on the membrane potential of the excitatory neuron. In order to obtain the solution of Eq. (1) it is necessary to specify an initial function ϕ on the interval $[-\tau, 0]$. The functions ϕ have the form of neural spike trains.

When $F(V(t-\tau)) = 0$, there is no recurrent input, and the choice of I_0 determines whether the neuron is excitable (firing only when stimulated) or periodically firing [11]. This observation leads to two mechanisms for multistability in a delayed recurrent loop. These mechanisms can be most easily illustrated by considering an integrate-and-fire model representation of Eq. (1). The membrane potential V of the neuron increases linearly at a rate A until it reaches the firing threshold θ. When $V = \theta$, the neuron fires and V is reset to its resting membrane potential V_0. The period is $T = \theta/A$. The firing of the neuron excites the inhibitory interneuron, which in turn at a time τ later, delivers an inhibitory postsynaptic potential (IPSP) to the excitatory neuron. The effect of the IPSP will be to change the timing of the next neuronal firing by an amount $\delta(\varphi)$, where $\delta(\varphi)$ is a function of the phase φ at which the IPSP arrives after the neuron has fired [12–14]. For illustrative purposes we take $\delta(\varphi)$ to be independent of φ. In this simple model, when $A > 0$, changing the period by δ is equivalent to changing V by δ. By convention, when $\delta > 0$, the timing of the next spike is prolonged by δ, and vice versa.

Case 1: Excitable regime.—In the excitable regime both excitatory and inhibitory pulses can cause a neuron to produce an action potential. The former effect is obvious.

The latter effect, known as anode break excitation or "rebound," is paradoxical, although well documented and thought to be important in a firing pattern generation [15]. In the Hodgkin-Huxley model (below) it arises because the IPSP brings the system state across the threshold separatrix, and an action potential follows upon release of the inhibition. Assume that in the absence of input the neuron is at rest (i.e., $A = 0$). If the neuron fires at time t_0 then it must, as a consequence of rebound ($\delta = -1$), also fire at $t_0 + \tau$. The condition for a permissible, periodic spike train pattern containing n spikes occurring at times t_i is simply $t_{i+1} - t_i \geq t_r$, $i = 1, \ldots, n$, where t_1 is the time the first spike occurs, $t_{i+1} \equiv t_1 + \tau$, and t_r is the absolute refractory time of the neuron. The condition that qualitatively different, temporally patterned spike trains can occur is $\tau > 2t_r$. Under this condition an infinite number of patterns can be stored; however, these are neutrally stable since a small perturbation in the timing of a spike is perpetuated. This behavior is similar to that of singularly perturbed DDEs [16].

Case 2: Periodic regime.—Without loss of generality we define the following dimensionless variables: $\tau \equiv \tau/T$, $t \equiv t/T$, $v = V/\theta$, $\Delta = \delta/\theta$. The excitatory neuron fires periodically when $A > 0$. We take $\Delta(\varphi) = \Delta$, where $0 < \Delta < 1$ [17]. The output of the recurrent inhibitory loop depends on the value of τ [18]. When $\tau < 1$, all solutions are periodic with period $1 + \Delta$. When $\tau > 1$ the behavior becomes more complex since the inhibitory pulses are not necessarily the result of the immediately preceding excitatory spike. The complete description of these solutions will be presented elsewhere.
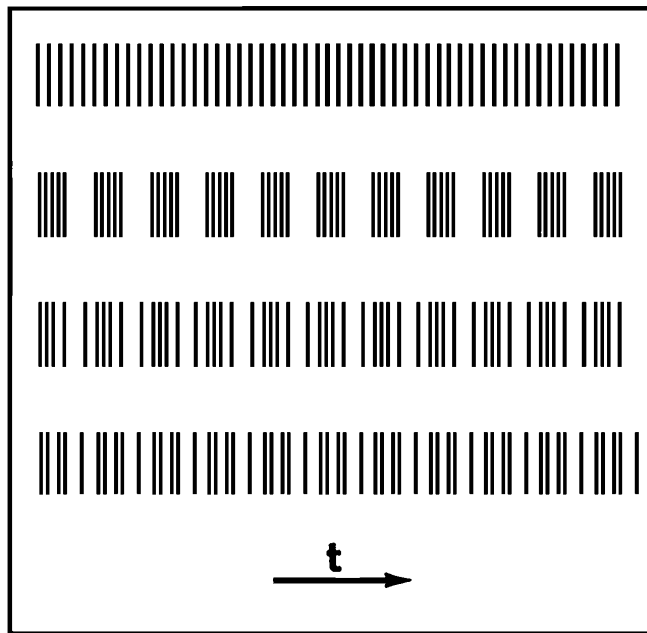
In summary, depending on the choice of τ and Δ, a variety of periodic, qualitatively different spike trains exist with period $T_s = S(1 + \Delta)$, where S is the number of spikes per period (Fig. 1). For all solutions, except one, $T_s > \tau$. The most remarkable property of these solutions is that in contrast to case 1, small perturbations in spike timing are not perpetuated.

We illustrate our findings with a numerical simulation of Eq. (1) in which the excitatory neuron is described by the Hodgkin-Huxley equations [19] and $f(V(t - \tau)) = \mu V(t - \tau)$, i.e.,

$$C\dot{V}(t) = -g_{Na}m^3h(V - V_{na}) - g_K n^4(V - V_K)$$
$$- g_L(V - V_L) - \mu V(t - \tau) + I_0,$$
$$\dot{m}(t) = \alpha_m(V)(1 - m) - \beta_m(V)m,$$
$$\dot{n}(t) = \alpha_n(V)(1 - n) - \beta_n(V)n, \quad (2)$$
$$\dot{h}(t) = \alpha_h(V)(1 - h) - \beta_h(V)h,$$

where the functions α, β are given in [19] and the constants g, I_0, μ in [20]. We choose values of I_0 for the excitable and periodic case that are outside the range over which these equations are known to exhibit bistability [11].

Figure 2 shows the effect of changing the initial function ϕ for Eq. (2) when $I_0 = 0$ (excitable regime). Qualitatively different solutions occur as ϕ is varied. These
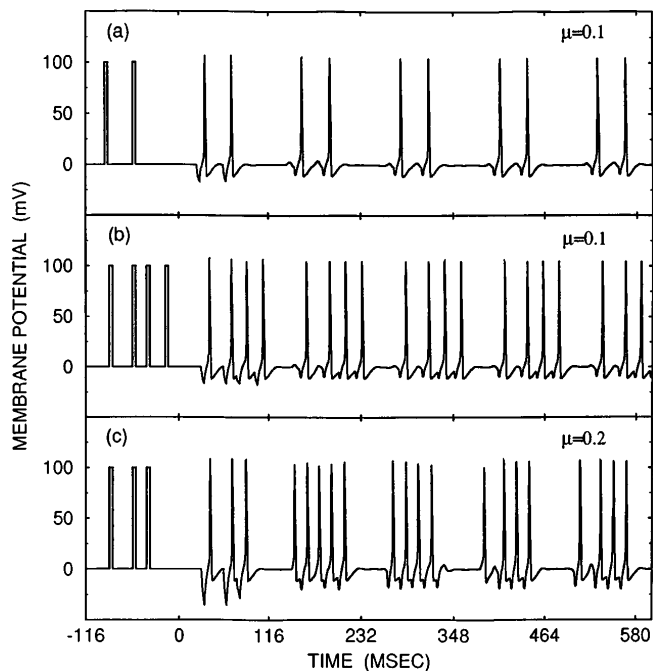

FIG. 2. Three spike train patterns generated by Eq. (2) for three choices of ϕ (left-hand side) when $I_0 = 0.0$ (excitable regime) and $\tau = 116$ msec. The spikes in the initial function on $(-\tau, 0)$ are 4 msec in duration. With $\mu = 0.1$ (a),(b), spikes initially separated by ≈ 17 msec will propagate from one delay interval to the next. For $\mu = 0.2$ (c), other extra spikes can be added or deleted from the initial spike function. The initial conditions for all variables [including $V(0)$ between the spikes] correspond to the fixed point obtained when $\mu = 0$.

FIG. 1. Solutions of the integrate-and-fire model when $\tau = 4.1$ and $\Delta = 0.8$. The initial function ϕ is consisted of three spikes in an interval of length τ; one of these was fixed to occur at time equal to zero (not shown).

solutions can be seen to be formed from repeating identical segments of length τ. Provided that ϕ satisfies the conditions given in case 1, there is a 1:1 correspondence between ϕ and the solution of Eq. (2): If the spikes in ϕ are sufficiently separated, they will propagate without merging. However, if two spikes are too close, their relative separation can change, or merging can occur after which a stable pattern is obtained. Thus, in contrast to the infinitely sharp spikes of the integrate-and-fire model, spike train perturbations can propagate, provided spikes are close [21]. For the excitatory case, spikes propagate independently if separated by 11.6 msec for $\mu = -0.1$ or by 7.5 msec for $\mu = -0.2$. In the inhibitory case, the corresponding values are 17 msec for $\mu = 0.1$ and 4.5 msec for $\mu = 0.2$. These values are governed by the phase resetting properties of Eq. (2) as a function of amplitude and duration of the IPSP's. When $\mu = 0.2$, there is not necessarily a one-to-one mapping between spike positions from one delay to the next [Fig. 2(c)]. Many different asymptotic patterns are then possible (as in the case below), depending on the number and position of the initial spikes in the loop.

Figure 3 shows the effect of changing the initial function ϕ when $I_0 = -10 \ \mu A/cm^2$ (periodic regime). The remaining parameters in Eq. (2) are identical to those used in Fig. 2, and we have taken $\tau/T = 7.9$ [18]. Qualitatively different neural spike train solutions of Eq. (2) result from different choices of ϕ. When multiple attractors coexist for a DDE, there are multiple basins of attraction in a functional space. In Fig. 4 we show a 2D secti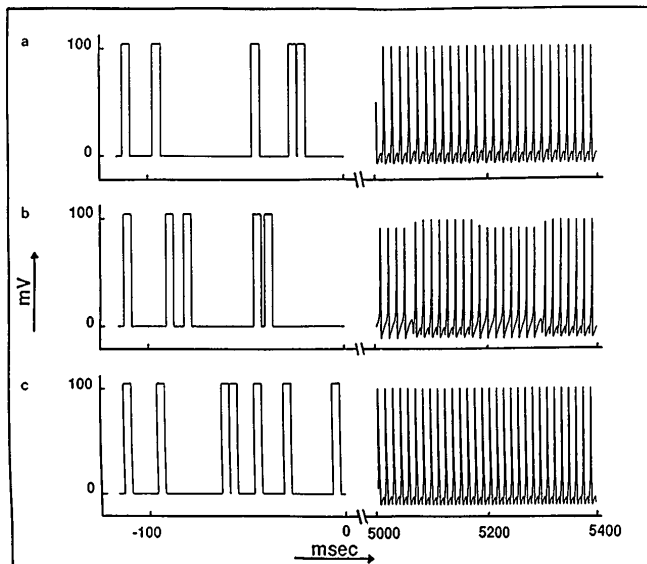on through a portion of this functional space obtained by restricting ϕ to those initial functions composed of exactly five spikes in an interval of length τ for which the timing of three spikes is fixed and that of the remaining two is varied. For these ϕ only the three solutions of Eq. (2) shown in Fig. 3 occur. The ϕ which asymptotically yield the solutions in Figs. 3(a), 3(b), and 3(c) are colored, respectively, white, gray, and black. the union of all areas of a given color corresponds to the basin of attraction of one of the three solutions shown in Fig. 3. Within the limits of our numerical simulations, these basins of attraction are not "riddled" [22]; i.e., except on a boundary, two ϕ within a given color always yield the same solution of Eq. (2). The intricate nature of the initial function space revealed in Fig. 4 stresses the importance of controlling ϕ when studying the dynamics of delayed recurrent loops. Indeed, it is quite possible that much of the variability observed experimentally in delayed feedback dynamical systems [23] may result from the fact that ϕ is typically not controlled between successive observations or simulations.

The extent to which multistability in delayed recurrent loops can be utilized in information processing is a function of the noise level. The observations in Fig. 4 suggest that the basins of attraction have a positive measure implying that the different solutions will be observed experimentally by changing ϕ is the noise level is not too high. As the noise level increases, much of the fine structure seen in Fig. 4 will gradually be washed out. However, there may still be sufficient variability in the transient and asymptotic patterns as a function of

FIG. 3. Three spikes train patterns generated by Eq. (2) for three choices of ϕ (left-hand side) when $I_0 = -10 \ \mu A/cm^2$ (periodic regime) and $\tau = 116$ msec (after 5000 msec of transients). The initial values of $m = 0.0818$, $n = 0.66342$, $h = 0.15005$, and $v = 9.802$ were taken from the limit cycle solution of Eq. (2) when $\mu = 0$.
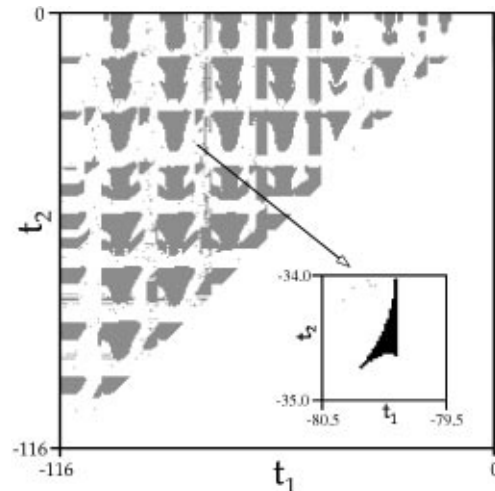

FIG. 4. Initial function space of Eq. (2) in periodic regime (see text for discussion). Three of the spikes in the initial function were fixed at $t_3 = -111$ msec, $t_4 = -44.14$ msec, $t_5 = -4$ msec, and the remaining two, t_1 and t_2, were varied from 0 to -116 msec. This function space is symmetric about the 45° line and only the upper portion has been plotted. The "black" region is quite small, but contains structure (see inset). The resolution for t_1 and t_2 was 0.5 msec (0.01 msec in the inset).

initial spike timing that the capacity to encode remains significant.

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